Curved Surfaces Modeling

Note Title 3/10/2006

In CAD design, for eg. designing an airplane or a fighter tank, you encounter several surfaces, not many planar regions. The question is how do you represent, process & render these surfaces efficiently.

I. Piecewise Linear Representation.

E.g. Curve represented by lines
(surfaces by triangles)

a. Simplest representation.

Problems

a. APPROXIMATION :- Can never represent the curve accurately, unless no. of linear segments is close to ∞.

b. NUMBER OF PRIMITIVES :- A very large number of primitives even to represent something reasonably.

c. PROCESSING IS SLOW:- Due to the huge number, processing is slow.

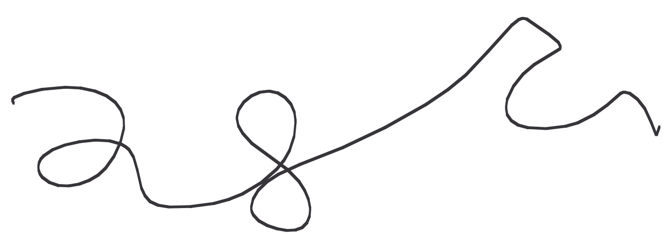
d. NO CONTINUITY AT BOUNDARIES:- Always some discontinuity at the boundaries.

II. Use one parametric, implicit or explicit representation for the whole surface.

Problems

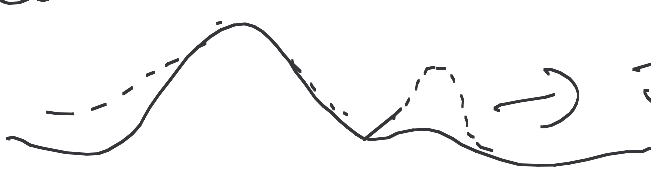
a. May not be mathematically feasible.

Eg.:



b. No local control. Changing one part will change the entire surface/curve.

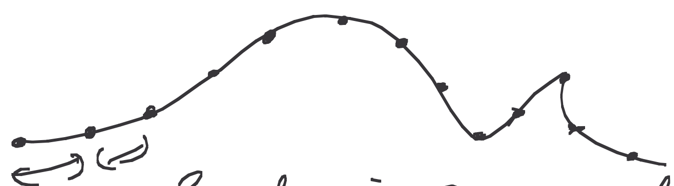
Eg.



Just this change is difficult to make

So we make a tradeoff between the two.

III Piecewise polynomial curves/surfaces.



Each is a curve/surface

Advantage

a) Accurate representation
b) Local control is available to change some of the pieces.
c) We can control the continuity at the boundary.
d) Since curves we do not need to have a huge numbers as lines.
e) Processing is faster PIECES

Parametric form to define the curves.

$C(t) = (x(t), y(t), z(t))$

Just as in straight line.

$x(t) = x_1 + t(x_2 - x_1) \quad y(t) = y_1 + t(y_2 - y_1)$

$z(t) = z_1 + t(z_2 - z_1) \Rightarrow P = P_1 + t(P_2 - P_1)$

controlled by one parameter, plug in different values of t to get points on the curve.

For surfaces,

$$s(u,v) = (x(u,v),\ y(u,v),\ z(u,v))$$

Now, what kind of functions are these x, y, z?

usually they are single or multivariate cubic polynomials.

$$x(t) = at^3 + bt^2 + ct + d$$

$$\begin{aligned} x(u,v) = {} & a_0 \quad \text{— (degree 0 terms).} \\ & + a_1 u + a_2 v \quad \text{— (d-1)} \\ & + a_3 u^2 + a_4 v^2 + a_5 uv \quad \text{(d-2)} \\ & + a_6 u^3 + a_7 v^3 + a_8 uv^2 + a_9 u^2 v \\ & \quad \text{(d-3)} \end{aligned}$$

why?

a) Anything of lower degree does not give enough flexibility.

b) Higher degree curves result in too many wiggles.

c) Provide minimum curvature interpolant n+3 pts. smoothest curve is a cubic from n.

d) No lower degree curve can maintain continuity at the boundary

$$Q(t) = [x(t), y(t), z(t)]$$

$$x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

Represented as a matrix in a <u>compact</u> manner.

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix}}_{3\times 4} \underbrace{\begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}}_{4\times 1}$$

T

↳ Coefficient Matrix (C)

↳ Parameter Matrix

Now what is $Q'(t) = \frac{d}{dt} Q(t)$.

$$Q'(t) = [x'(t), y'(t), z'(t)]$$

$$x'(t) = 3a_x t^2 + 2b_x t + c_x$$

$$y'(t) = 3a_y t^2 + 2b_y t + c_y$$

$$z'(t) = 3a_z t^2 + 2b_z t + c_z$$

can be again written as a matrix as

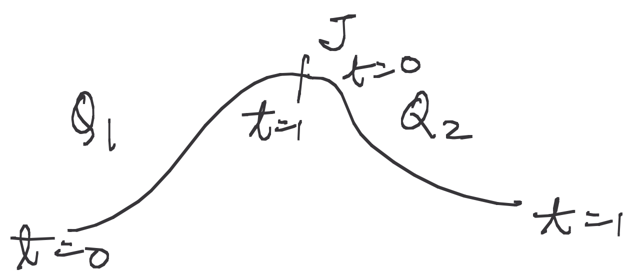
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \end{bmatrix} \begin{bmatrix} 3t^2 \\ 2t \\ 1 \\ 0 \end{bmatrix}$$

C remains unchanged. Derivative of $T = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = C \frac{dT}{dt}$$

CONTINUITY

When designing surfaces it is important to maintain continuity at junction of two surfaces.



we can define different types of continuities depending on behavior of

Q_1 & Q_2 at junction J.

If $Q_1(1) = Q_2(0)$, just the value of the curve is same at J, we call it geometric continuity of order 0; G^0.

If the directions of the derivatives (defines the tangent vector at that point on the curve) has same direction,

$$\frac{Q_1'(1)}{|Q_1'(1)|} = \frac{Q_2'(0)}{|Q_2'(0)|}$$

[defined by normalizing to get unit vectors]

Then called geometric continuity of order 1, G^1

If an additional criteria is added to this, magnitude of tangent vectors should also be same, i.e.

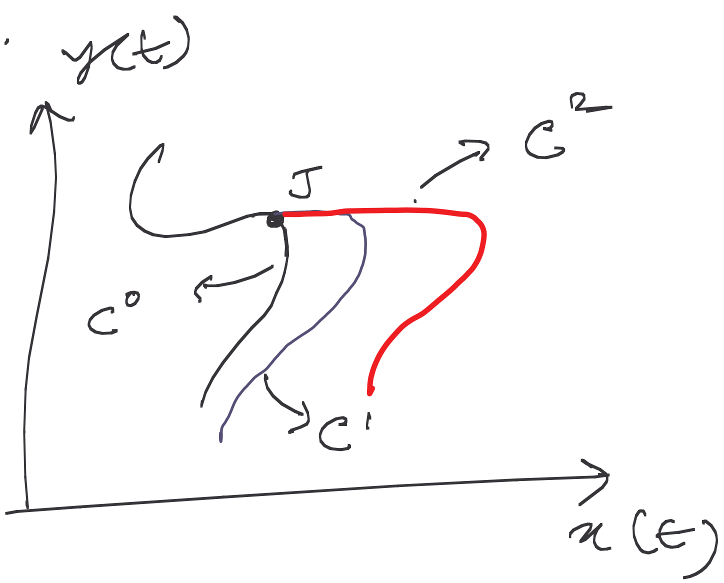
$$|Q_1'(1)| = |Q_2'(0)|$$

then called parametric continuity of order 1, C^1. Thus C^1 is a stricter measure than G^1.

Note that since there is no direction involved with 0th order $C^0 = G^0$.

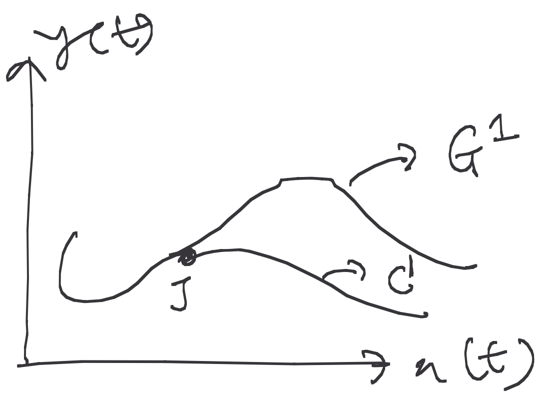
The same concept can be extended to higher orders. For e.g. Order 2 assures continuity in curvature. Will lead to smoother & smoother curves as order is increased.

E.g.



In general C^n continuity $\Rightarrow G^n$ continuity, but not the other way round.

E.g.



If tangent vector is $(0,0,0)$, $C^1 \not\Rightarrow G^1$.
Since nothing cannot be said for a null vector.

How to find the coefficient matrix?

given two endpts of a curve & the tangent at those end pts.

Say Q, starts at $(1,2,1) \rightarrow Q(0)$ — ①
ends at $(4,5,4) \rightarrow Q(1)$ — ②
Derivative at start is $(1,1,1) \Rightarrow Q'(0)$ — ③
" at end is $(6,6,6) \rightarrow Q'(1)$ — ④

fro ①

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = C \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix} = C \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

from ②

$$\begin{bmatrix} 4 \\ 5 \\ 4 \end{bmatrix} = C \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

From: 3

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = C \begin{bmatrix} 3t^2 \\ 2t \\ 1 \\ 0 \end{bmatrix} = C \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

from 4

$$\begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = C \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 4 & 1 & 6 \\ 2 & 5 & 1 & 6 \\ 1 & 4 & 1 & 6 \end{bmatrix} = C \underbrace{\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{\text{Basis Matrix (B)}}$$

(Remains same as long as the way to define curve is same)

4X4

$$\therefore C = \begin{bmatrix} 1 & 4 & 1 & 6 \\ 2 & 5 & 1 & 6 \\ 1 & 4 & 1 & 6 \end{bmatrix} B^{-1}$$

Inverse exists since square matrix

Adv

For a particular class of curves, Basis matrix is unique $\therefore B^{-1}$ can be precomputed

For example, curves defined by $Q(0), Q(1), Q'(1)$ & $Q'(0)$ are called Hermite Curves.

B for Hermite curves is

$$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\therefore B^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\text{Now } Q = C.T = \underbrace{\begin{bmatrix} P_1 & P_2 & T_1 & T_2 \end{bmatrix}} B^{-1} T.$$

where $P_1 = Q(0)$
$P_2 = Q(1)$
$T_1 = Q'(0)$
$T_2 = Q'(1)$

This is called the Geometry Matrix. This changes from one hermite curve to another and is responsible for defining the Geometry of one particular curve.

$\therefore Q = C\,T$

$$= \begin{bmatrix} P_1 & P_2 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} P_1 & P_2 & T_1 & T_2 \end{bmatrix} \begin{bmatrix} 2t^3 - 3t^2 + 1 \\ -2t^3 + 3t^2 \\ t^3 - 2t^2 + t \\ t^3 - t^2 \end{bmatrix}$$

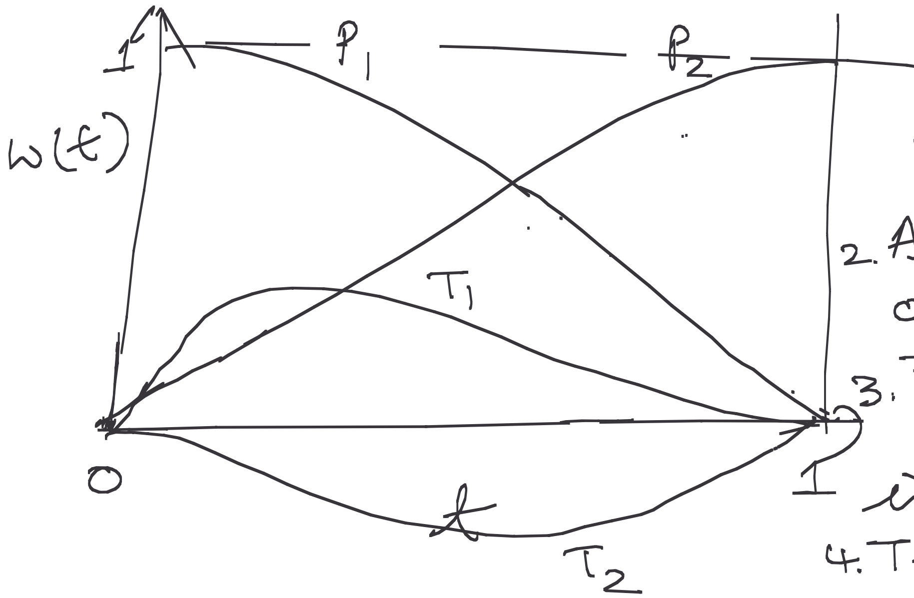
$$= P_1 (2t^3 - 3t^2 + 1) + (-2t^3 + 3t^2) P_2 + (t^3 - 2t^2 + t)\, T_1 + (t^3 - t^2)\, T_2$$

$\therefore$ The curve $Q(t)$ can also be represented as a weighted fn. of its geometric parameters P_1, P_2, T_1, T_2.

These weights expressed as functions of the parameter t, $w(t)$, are called the blending fns.

$\therefore$ Hermite curves are curves that are blended from their endpts & tangents at the end pts.

Plot of The Blending fn. (Called Interpolating Curves)

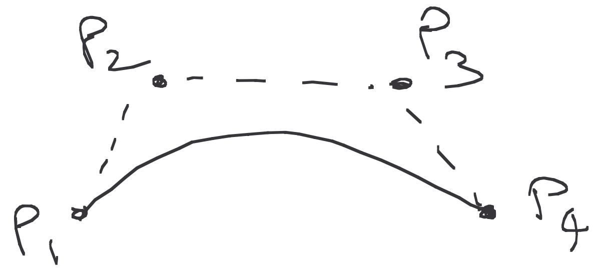


Note: 1. At t=0, P_2 has 0 contribution
2. At t=1, P_1 has 0 contribution.
3. T_1 & T_2 contribute only in middle.
4. T_2 has negative contribution also.

Helps us to decide which pts to manipulate to change the shape of the curve when modeling a curve. C is changed automatically. User friendly. Do not need to think about G when modeling.

Bezier Curves

A different type of curve defined by 4 points.



Four pts to control the shape of the curve.

And the constraint is

$$T_1 = 3(P_2 - P_1)$$
$$T_4 = 3(P_4 - P_3)$$

$P_1 \; P_2 \; P_3 \; P_4$ ↪ Control Polygon

$\therefore$ Geometry Matrix

$$= \begin{bmatrix} P_1 & P_2 & T_1 & T_4 \end{bmatrix}$$

$$= \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Find the Basis Matrix

Same as Hermite Curves since still defining $Q(0)$, $Q(1)$, $Q'(0)$, $Q'(1)$, only that now a special relationship holds.

$$\therefore\ B^{-1} = \begin{bmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$\therefore\ C = G\,B^{-1}$$

$$= \begin{bmatrix} P_1 & P_2 & T_1 & T_4 \end{bmatrix} B^{-1}$$

$$= \underbrace{\begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}}_{\text{Called Control Pts.}} \begin{bmatrix} 1 & 6 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 1 & 0 & 3 \end{bmatrix} B^{-1}$$

$$= \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

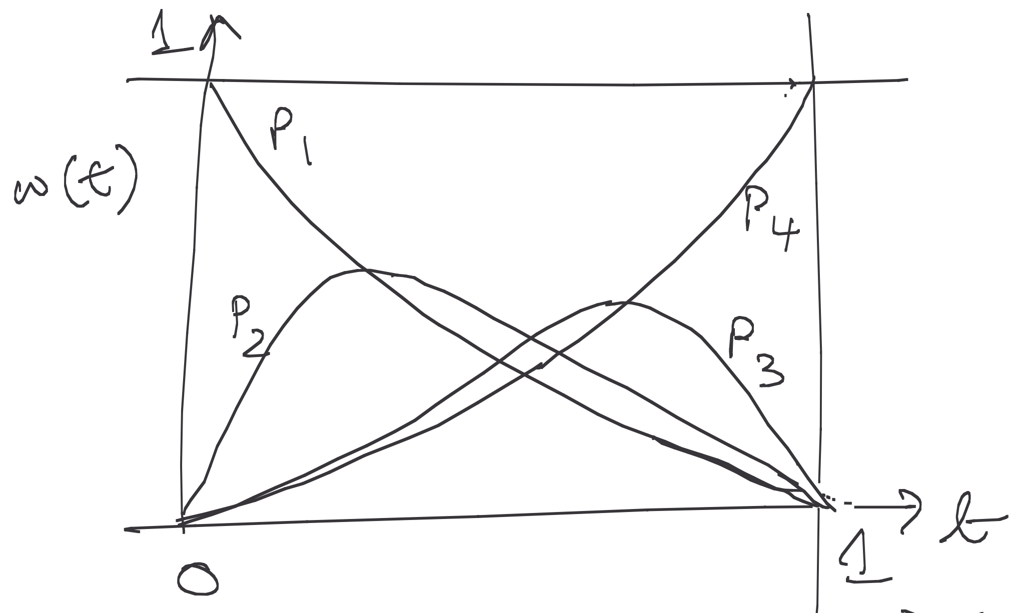
$\therefore$ Blending fn are

$$\begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}$$

$$= P_1 \cdot (-t^3 + 3t^2 - 3t + 1) + P_2 \cdot 3(t^3 - 2t^2 + t) + P_3 \cdot (-3t^3 + 3t^2) + P_4 \cdot t^3$$

$$= P_1 (1-t)^3 + 3t(1-t)^2 P_2 + 3t^2(1-t) P_3 + t^3 P_4$$

These set of Blending fns are Called Bernstein Polynomials.



See that at no point in t, P_2 & P_3 are the sole contributor. For eg. P_1 is sole contributor at $t=0$. ∴ Curve passes through P_1 since $Q(0) = P_1$. Similarly at $t=1$, $Q(1) = P_4$ ∴ Curve passes through P_4. But cannot pass through P_3 & P_4.

∴ This is a non-interpolating curve. Since does not pass through all the control pts.

These are cubic curves. Since cubic polynomials for defining the curve, blending fn are also cubic.

Properties

a) Curve bounded by the convex hull of control pts. - Called Control Polygon

b) Variation Diminishing Property :- A line intersects the curve no more than its intersection with the control polygon. ∴ No wiggles in control polygon signifies no wiggles in the curve.

c) Symmetric - Reversing control pts order yields same curve with reverse parametrization.

d) Affine Invariant :- Affine Transformations of control pts yields affine transformation of curve itself.

e) Can be easily subdivided for rendering.

More general

Can be of any degree.

For eg. if linear — Need two control pts in Geometry Matrix.

if quadratic - Need three control pts in geometry Matrix

If quadric — Need 5 control pts.

Bezier is most popular for surfaces,

1) only positive blending fn is easy to understand.

2) Non-interpolating gives better control.

3) Only a few control pts & easy to use but has large flexibility to design a varied types of curves.

Higher Order Bezier

Bernstein Polynomials of degree 3.

$$b_{0,3} = (1-u)^3$$

$$b_{1,3} = 3u(1-u)^2$$

$$b_{2,3} = 3u^2(1-u)$$

$$b_{3,3} = u^3$$

They have a more general form for degree n.

$$b_{k,n} = C(n,k)\, u^k (1-u)^{n-k}$$

$$= \frac{n!}{k!\,(n-k)!}\, u^k (1-u)^{n-k}$$

$\therefore$ with n control ~~pts~~ pts,

$$P(u) = \sum_{k=0}^{n} P_k\, C(n,k)\, u^k (1-u)^{n-k}.$$

How do we render?

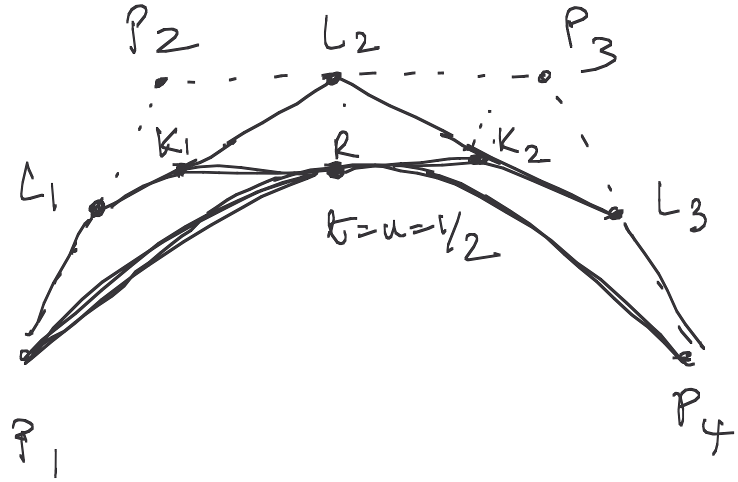
Subdivide the curve to smaller & smaller segments until each segment is close enough to a line & then render the line.

Need an easy subdivision technique.

Say you want to divide the curve at $t=u$, i.e. at $Q(u)$.

De Casteljau Construction

1) Divide each $P_i P_{i+1}$ line segment in u & $1-u$ ratio.



2) This will generate $n-1$ lines.

3) Join the pts in sequence to get $n-2$ lines.

4) Again subdivide these lines in same way. Will give you $n-2$ pts & $n-3$ lines.

5) Continue until you get one pt.

6) This pt will be $Q(u)$.

7) Follow pts of division back to P_1 on left & P_n on right.

Eg. $R\ K_1\ L_1\ P_1 \rightarrow$ on left toward P_1

$R\ K_2\ L_3\ P_4 \rightarrow$ on right towards P_4

8. These will generate the control pts for the subdivided curves.

9. Reparameterize using new control pts.