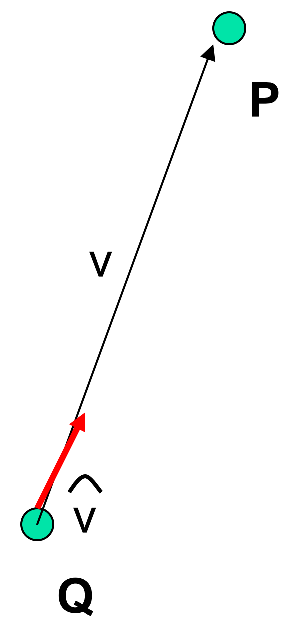
CS 112 – Transformations I

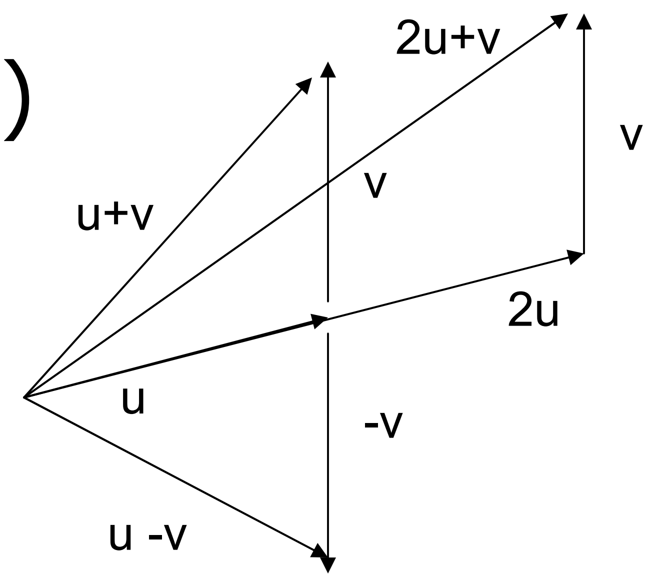
Geometric Entities

- Points – P = (x,y,z)
- Vectors
  - Two points : v = P-Q
    - $(x_P - x_Q, y_P - y_Q, z_P - z_Q) = (v_x, v_y, v_z)$
  - Magnitude - | v |
    - $\mathrm{Sqrt}((x_P - x_Q)^2 + (y_P - y_Q)^2 + (z_P - z_Q)^2)$
  - Direction
    - $v = |v|\,\hat{v}$
    - $\hat{v}$ - Unit Vector



Vector Operations

- u (u_x, u_y, u_z) and v (v_x, v_y, v_z)
- Addition
  - $(u_x+v_x, u_y+v_y, u_z+v_z)$
- Subtraction
- Scaling
  - Direction does not change
- Linear dependency
  - $v = a_1 v_1 + a_2 v_2 + \ldots a_n v_n$



Vector Operations

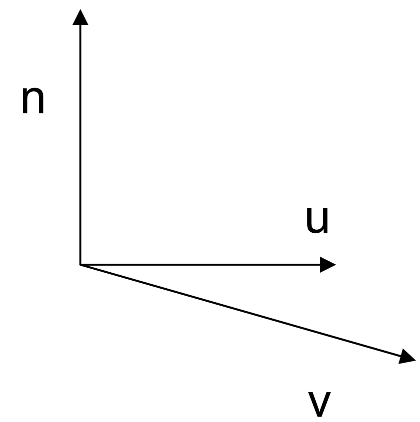
- Dot Products - scalar
  - $(u_x v_x + u_y v_y + u_z v_z)$
  - Magnitude – Sqrt(u.u)
  - Angle between two vectors, $\cos\theta = \dfrac{u.v}{|u||v|}$

  Hence zero for orthogonal vectors

  $$[v_x \ v_y \ v_z] \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = v.u$$

Vector Operations

- Cross Product
  - $n = u \times v$
  - Orthogonal of plane of u and v
  - $(u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x)$
    - $\sin\theta = \dfrac{|u \times v|}{|u|\,|v|}$
  - Hence, zero for parallel vectors



Lines

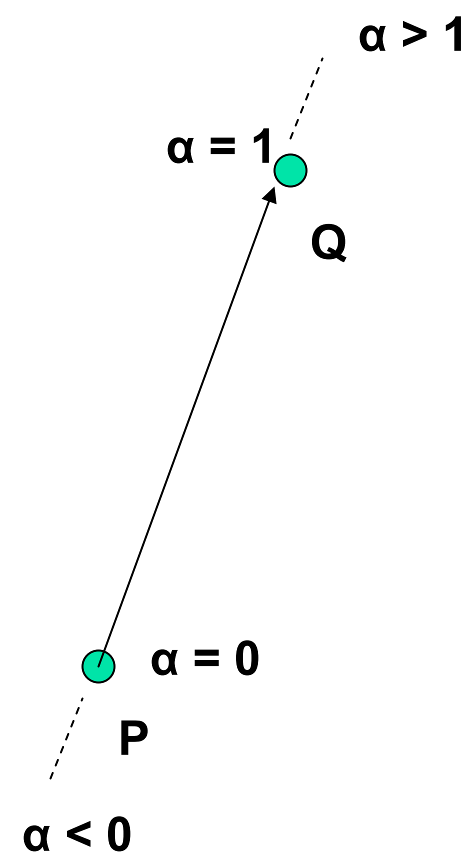
- Defined by a point and a vector
  - $L(\alpha) = P + \alpha(Q-P)$
  - Parametric equation
  - $L(\alpha) = P + \alpha(Q-P)$

$$= \alpha Q + (1-\alpha)P$$

$$= \alpha_1 Q + \alpha_2 P$$

where $\alpha_1 + \alpha_2 = 1$

Line is an *affine sum* of two points

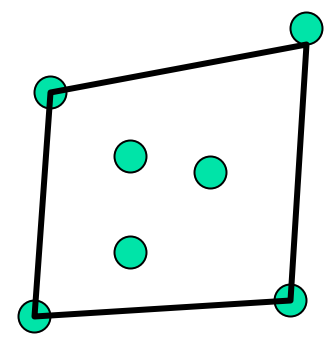


Convexity

- Affine sum
  - Points defined by affine sum lie on the line between the two points
- Generalized for *n* points
  - $L = \alpha_1 P_1 + \alpha_2 P_2 + \dots \alpha_n P_n$

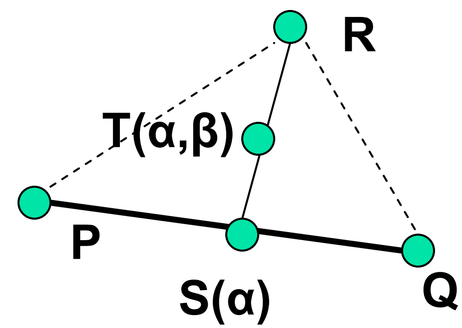
    where $\alpha_1 + \alpha_2 + \dots \alpha_n = 1$, $0 \le a_i \le 1$

    Lies in the *convex hull* of the points

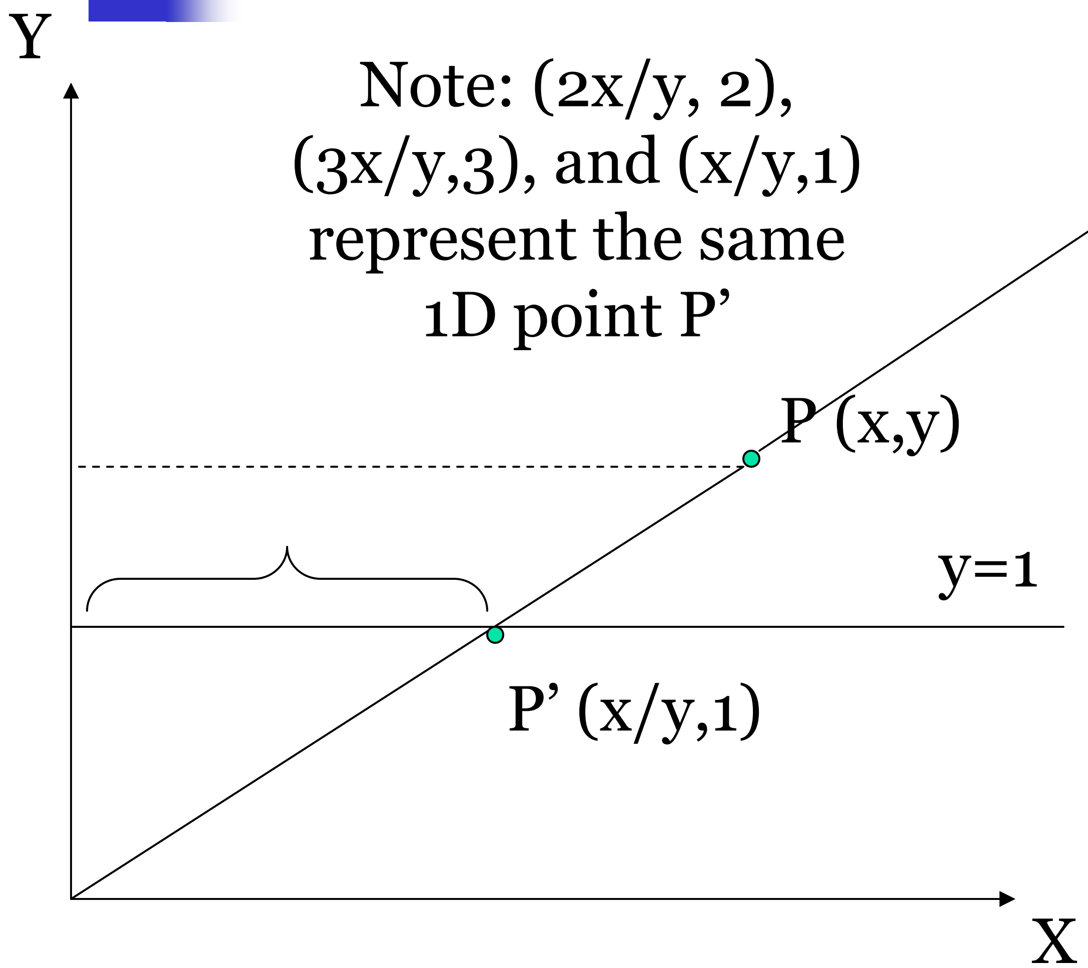


Planes

- Three non-collinear points define a plane
- $S(\alpha) = \alpha P + (1-\alpha)Q$
- $T(\beta) = \beta S + (1-\beta)R = \alpha\beta P + \beta(1-\alpha)Q + (1-\beta)R$
- $T(\beta) = P + \beta(1-\alpha)(Q-P) + (1-\beta)(R-P)$
  - Defined by a point and two vectors
- $n = (Q-P) \times (R-P)$
  - n is the normal to the plane
- $n.(T - P) = 0$



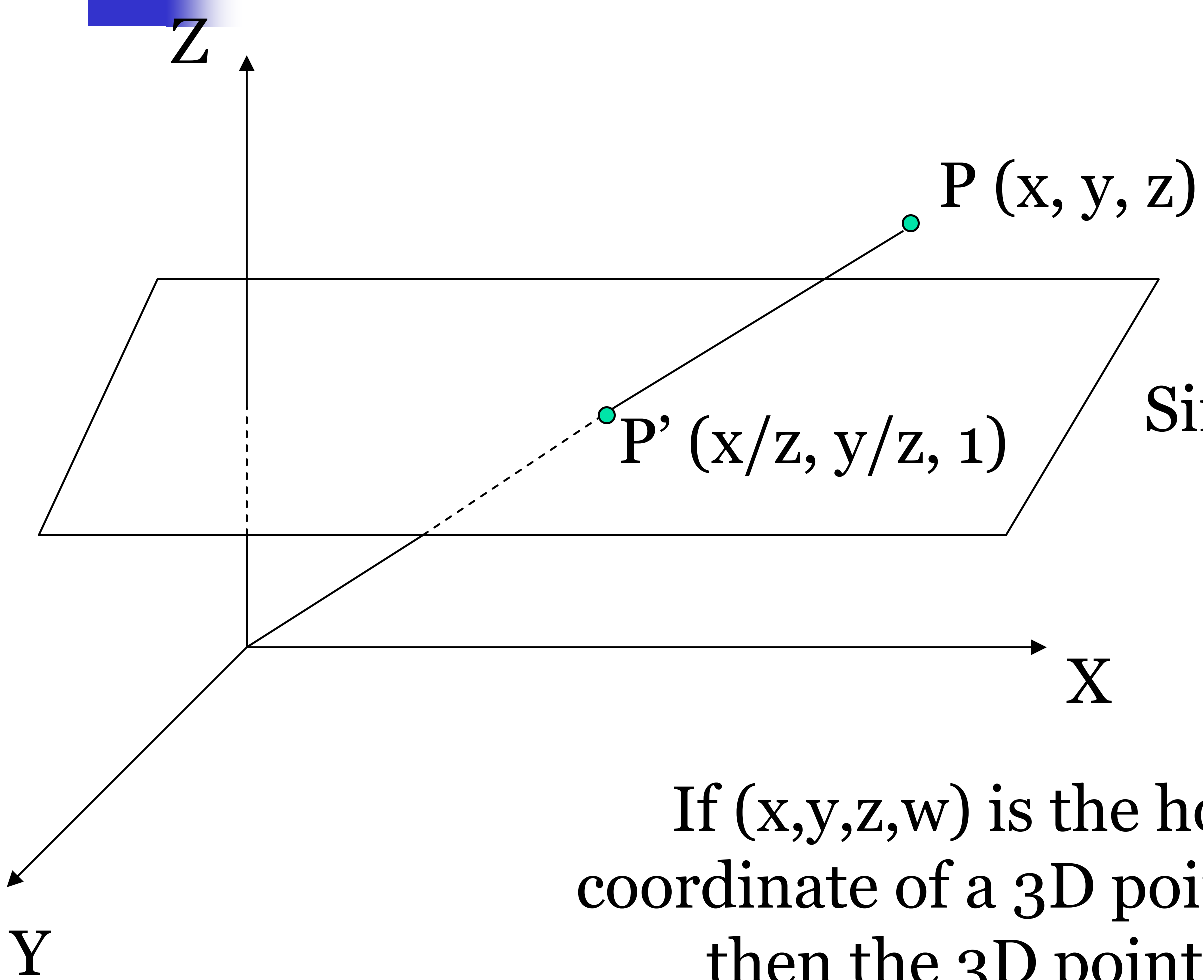
Homogeneous Coordinates



Any point on the same vector has the same homogeneous coordinates

1D points on the line is represented by 2D array, called homogeneous coordinates

Generalize to Higher Dimensions



2D points represented by homogeneous coordinates

Similarly, 3D points are represented by homogeneous coordinates

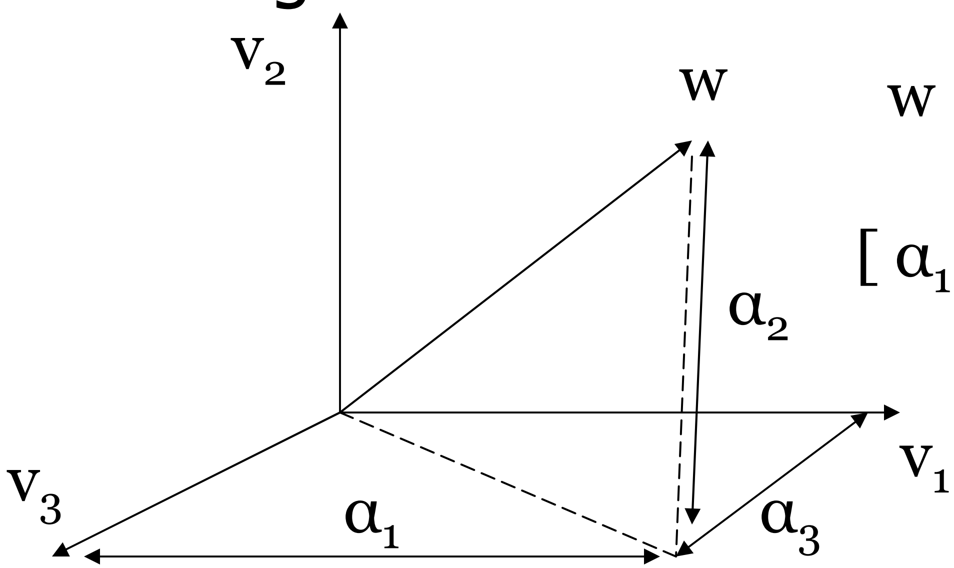
If (x,y,z,w) is the homogeneous coordinate of a 3D point, where $w \neq 1$, then the 3D point is given by $(x/w,y/w,z/w,1)$

Homogeneous Coordinates

- 4 array for describing 3D points and vectors
- Points $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix}$
- Vectors = P2-P1 = $\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ 0 \end{bmatrix}$

Coordinate Systems

- Represent a point as a linear combination of three vectors and the origin
- Linearly independent vectors – *basis*
  - Orthogonal vectors are linearly independent



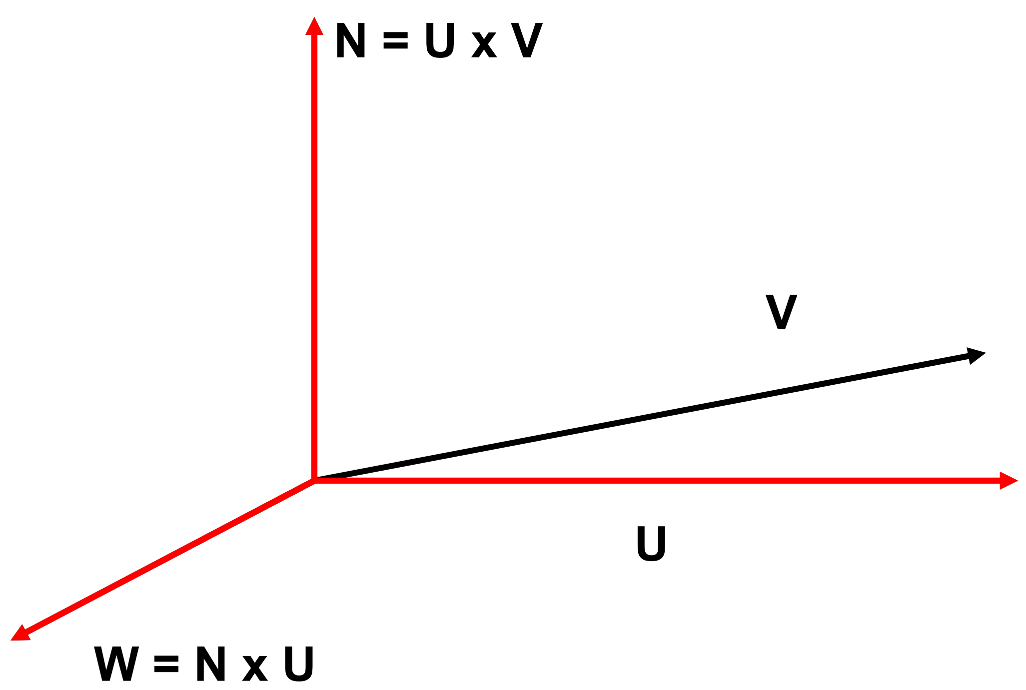
$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + R$$

$$[\, \alpha_1 \; \alpha_2 \; \alpha_3 \; 1 \,] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ R \end{bmatrix} = w$$

Linear Combination

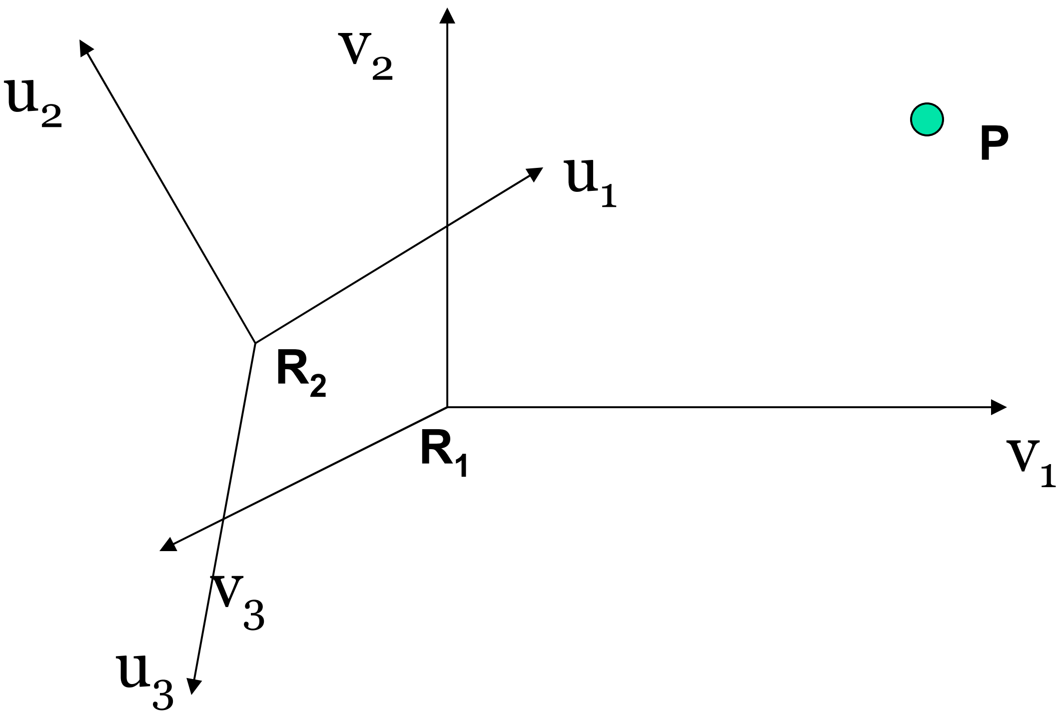
- Points $[\,\alpha_1\ \alpha_2\ \alpha_3\ 1\,]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ R \end{bmatrix} = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + R$
- Vectors $[\,\beta_1\ \beta_2\ \beta_3\ 0\,]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ R \end{bmatrix} = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$

Generating Coordinate Systems



Change of Coordinates

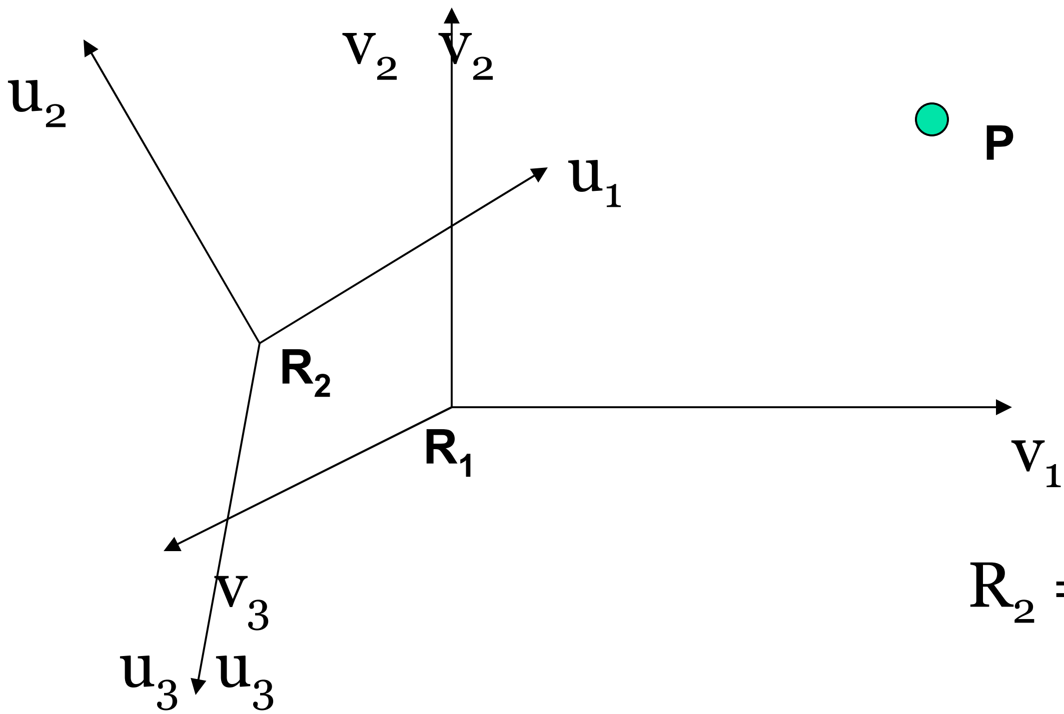
- First coordinate – v_1, v_2, v_3, R_1
- Second coordinate – u_1, u_2, u_3, R_2



$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ 1 \end{bmatrix} = C_2$$

Change of Coordinates

- First coordinate – v_1, v_2, v_3, R_1
- Second coordinate – u_1, u_2, u_3, R_2



$$u_1 = \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3$$

$$u_2 = \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3$$

$$u_3 = \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3$$

$$R_2 = \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + R_1$$

Change of Coordinates

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$

$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$

$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$

$$R_2 = \gamma_{41} v_1 + \gamma_{42} v_2 + \gamma_{43} v_3 + R_1$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ R_2 \end{bmatrix} = \mathbf{M} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ R_1 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}$$

Change of Coordinates

If $R_1 = R_2$,

then $\gamma_{41} = \gamma_{42} = \gamma_{43} = 0$

$$\mathbf{M} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Change of Coordinates

$$P = C_2^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ R_2 \end{bmatrix} = C_2^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ R_1 \end{bmatrix} = C_1^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ R_1 \end{bmatrix}$$

Hence, $C_1^T = C_2^T M$

Affine Transformations

- Function that maps a point (vector) to another point (vector)
  - $q = T(p)$
- Linear Transformation
  - $T(ap+bq) = aT(p) + bT(q)$
  - If transformation of vertices are known, transformation of linear combination of vertices can be achieved

Matrix Transformations

- $q = \mathbf{T}p$
  - p and q are points or vectors in 4x1 homogeneous coordinates
  - **T** is a 4x4 square matrix (16 degrees of freedom)

$$\mathbf{T} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{bmatrix}$$

Matrix Transformations

- The last element of p and q is either 0 or 1
  - 12 degrees of freedom
- If dealing with vectors
  - Last column does not have an effect (9 degrees of freedom)

$$\mathbf{T} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix Transformations

- Can also be seen as
  - Transforming vertices in the same coordinates
  - Changing the underlying coordinate to change the representation of the vertices

$$\mathbf{T} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Preservation of Lines

- $L(\alpha) = \alpha_1 Q + \alpha_2 P$
- Apply transformation
  - $\mathbf{T}(L(\alpha)) = \mathbf{T}(\alpha_1 Q) + \mathbf{T}(\alpha_2 P)$
    $= \alpha_1 \mathbf{T}(Q) + \alpha_2 \mathbf{T}(P)$
- Practical application of transformations
  - Transforming end points of the lines is sufficient
    - No need to transform all the points in the line
  - Pass vertices through graphics pipeline
  - Render the transformed line during rasterization

Translation

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$P' = \mathbf{T}P$$

$$x' = x + t_x$$
$$y' = y + t_y$$
$$z' = z + t_z$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denoted by $T(t_x, t_y, t_z)$

Translation

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$P = \mathbf{T}^{-1}P'$$

$$x = x' - t_x$$
$$y = y' - t_y$$
$$z = z' - t_z$$

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 & -t_x \\ 0 & 1 & 0 & -t_y \\ 0 & 0 & 1 & -t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = T(-t_x, -t_y, -t_z)$$

Example translation

Scaling

$$P = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \quad P' = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$P' = \mathbf{S}P$$

$$\mathbf{S} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x' = s_x\, x$$
$$y' = s_y\, y$$
$$z' = s_z\, z$$

Denoted by $\mathbf{S}(s_x, s_y, s_z)$

$$\mathbf{S}^{-1} = \mathbf{S}(1/s_x, 1/s_y, 1/s_z)$$

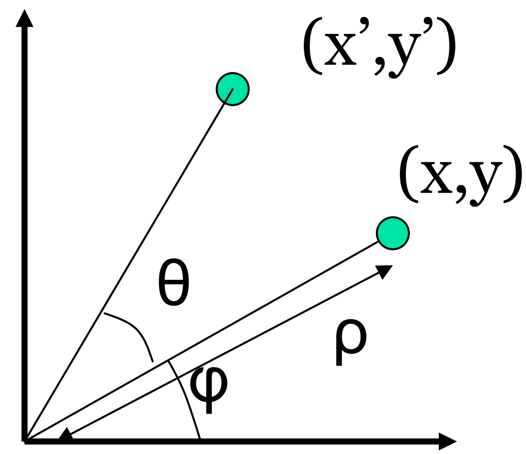
Rotation in 2D

$x = \rho \cos \varphi$

$y = \rho \sin \varphi$

$x' = \rho \cos (\theta + \varphi)$

$y' = \rho \sin (\theta + \varphi)$



$x' = \rho \cos\theta \cos\varphi - \rho \sin\theta \sin\varphi = x \cos\theta - y \sin\theta$

$y' = \rho \sin\theta \cos\varphi + \rho \cos\theta \sin\varphi = x \sin\theta + y \cos\theta$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \mathbf{R} \begin{bmatrix} x \\ y \end{bmatrix} \qquad \mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotation in 3D about z axis

$x' = x\cos\theta - y\sin\theta$

$y' = x\sin\theta + y\cos\theta$

$z' = z$

$P' = \mathbf{R}_z P$

$$\mathbf{R}_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

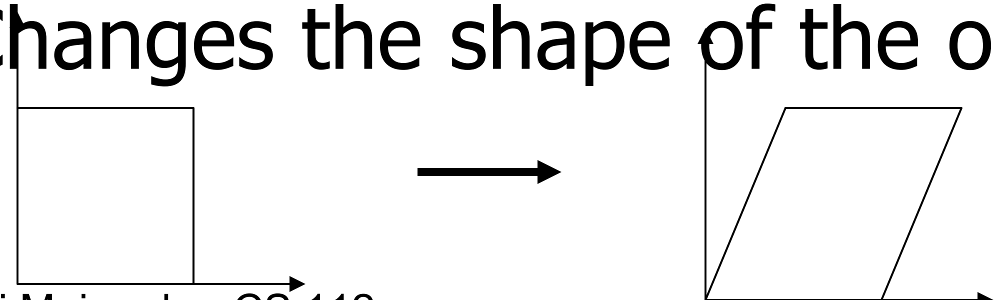
Denoted by $\mathbf{R}(\theta)$

$\mathbf{R}^{-1} = \mathbf{R}(-\theta) = \mathbf{R}^T(\theta)$

Where $\mathbf{R} = \mathbf{R}_x$ or $\mathbf{R}_y$ or $\mathbf{R}_z$

Shear

- Translation of one coordinate of a point is proportional to the 'value' of the other coordinate of the same point.
  - Point : (x,y)
  - After 'y-shear': (x+ay,y)
  - After 'x-shear': (x,y+bx)
- Changes the shape of the object.



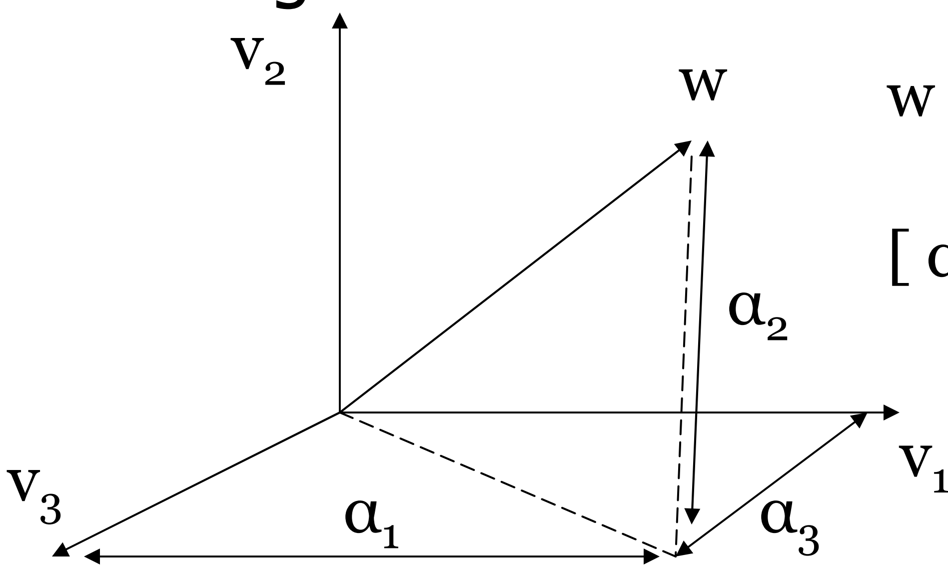
Using matrix for Shear

- Example: Z-shear (Z coordinate does not change)

$$\begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x+az \\ y+bz \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

Coordinate Systems

- Represent a point as a linear combination of three vectors
- Linearly independent vectors – *basis*
  - Orthogonal vectors are linearly independent



$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$[\,\alpha_1\ \alpha_2\ \alpha_3\,] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = w$$

3D Coordinates

- 3 array for describing 3D points
- Points $\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \end{bmatrix}$