ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

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Set 2 – Ch. 1.3, 1.4

Today's Lecture

- Chapter 1 (Sections 1.3 & 1.4)
  - Predicates & Quantifiers (1.3)
  - Nested Quantifiers(1.4)

Chapter 1: Section 1.3

Predicates & Quantifiers

Predicates

- Not *everything* can be expressed as T/F… so we use Predicate Logic
- The term predicate is used similarly in grammar
  - Subject: What we make an assertion about
  - Predicate: What we assert about the subject

For Example

"Socrates is Mortal"

What is the Subject? Socrates

What is the Predicate? being mortal

"X is less than 20"

What is the Subject? X

What is the Predicate? less than 20

Predicates become propositions once we add variables which we then can quantify.

Propositions

"X is less than 20"

Variable: x

Predicate: less than 20

So we say,

Let P(x) denote the statement "x < 20"

or P(x): :=x < 20

P(x) has no truth value until the variable x is bound

For example:

P(3)

Set x=3 so P(3) : 3<20 ➔ True

P (25) is False

P(-32) is True

Another Example

- "The security alarm is beeping in DBH"
- Let A(x) denote the statement

  "The security alarm is beeping in building x"

  So what would the truth value of A(DBH) be?
  How about A(ELH)?

Multiple Variables

- You can also have multi-variable predicates

For example
- Let R be the 3-variable predicate R(x,y,z)::= x+y=z
- Find the truth value of
  - R(2,-1,5)
    - x=1, y=-1, z=5
    - R(2,-1,5) is -2+1=5 which is false
- Now try
  - R(3,4,7) True
  - R(x,4,y) x, y not bound

Quantifiers

- Tell us the range of elements that the proposition is true over
- In other words… over how many objects the predicate is asserted.

Universal Quantifier

The *Universal Quantification* of P(x) is the statement.
"P(x) *for all* values of x in the domain"

- Notation : ∀xP(x)
  - ∀ is the ***universal quantifier***
- In English
  - "for all x P(x) holds"
  - "for every x P(x) holds"
  - "for each x P(x) holds
- A ***counterexample*** of ∀xP(x) is an element for which P(x) is false

The Universe

- Propositions in predicate logic are statements on objects of a universe.
- The universe is thus the domain of the (individual) variables.
- It can be
  - the set of real numbers
  - the set of integers
  - the set of all cars on a parking lot
  - the set of all students in a classroom
  - etc…

For Example:

- "All cars have wheels"

∀xP(x)

U={All cars}

P(x) denotes x has wheels

Universal Quantifier & Conjunction

If you can list all of the elements in the universe of discourse

Then $\forall x P(x)$ is equivalent to the conjunction

$P(x_1) \land P(x_2) \land P(x_3) \land \ldots \land P(x_n)$

If there were only 4 cars (c_1, c_2, c_3, c_4) in our previous example

$U=\{c_1, c_2, c_3, c_4\}$

The we could translate the statement $\forall x P(x)$ to

$P(c_1) \land P(c_2) \land P(c_3) \land P(c_4)$

Truth value of Universal Quantification

- What is the truth value of $\forall x P(x)$
  - $U=\{1,2,3,4\}$
  - $P(x) = x^* 25 < 100$

FALSE – Why?

4*25 is not < 100

- What is the truth value of $\forall x P(x)$
  - $U=\{1,2,3\}$
  - $P(x) = x^2 < 10$

TRUE

Existential Quantification

The *Existential Quantification* of P(x) is the statement.
"There exists an element x in the domain such that P(x)"

- Notation: $\exists x P(x)$
  - ∃ is the ***existential quantifier***
- In English:
  - "There exists an x such that P(x)"
  - "There is an x such that P(x)"
  - "There is at least one x such that P(x) holds"
  - "For some x P(x)"
  - "I can find an x such that P(x)"

Existential Quantifier

- ∃x means at least 1 object in the universe

.. followed by *P(x)* means that *P(x)* is true for at least 1 object in the universe.

- For example:

"Somebody loves you"

$\exists x P(x)$

P(x) is the predicate meaning: "*x* loves you"

U={all living creatures}

Existential Quantifier & Disjunction

If you can list all of the elements in the universe of discourse

Then $\exists x P(x)$ is equivalent to the Disjunction

$P(x_1) \lor P(x_2) \lor P(x_3) \lor \ldots \lor P(x_n)$

If there were only 5 living creatures (me, bear, cat, steve, pete) in our previous example

U={me, bear, cat, steve, pete}

Then we could translate $\exists x P(x)$ to

P(me) V P(bear) V P(girl) V P(steve) V P(pete)

Truth value of Existential Quant.

- What is the truth value of $\exists x P(x)$
  - $U=\{1,2,3\}$
  - $P(x) = x^* 25 < 100$

True

4*25 is not < 100

- What is the truth value of $\exists x P(x)$
  - $U=\{1,8,20\}$
  - $P(x) = x^2 < 10$

True – Why?

Uniqueness Quantifier

The *Uniqueness Quantifier* of P(x) is the statement.
"There exists a unique x in the domain such that P(x)"

- AKA Unique Existential Quantifier
- Notation: $\exists!\, xP(x)$ or $\exists_1\, xP(x)$
- In English:
  - "There is a unique x such that P(x)"
  - "There is one and only one x such that P(x)"
  - "One can find only one x such that P(x)."

Truth value of Uniqueness Quant.

- What is the truth value of $\exists!\, xP(x)$
  U={1,2,3}
  $P(x) = x^* 25 < 100$

False

4*25 is not < 100

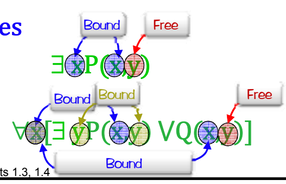
- What is the truth value of $\exists!\, xP(x)$
  U={1,8,20}
  $P(x) = x^2 < 10$

True

Some terms

- Bound – if a specific value is assigned to it or if it is quantified
- Free –if a variable is not bound.
- Scope - the part of the logical expression to which the quantifier is applied

Examples



Reading Quantified Formulas

- Read Left to right

Example

let U={ the set of airplanes}

let F(x, y) denote "x flies faster than y".

∀x∀y F(x, y) can be translated initially as:

"For every airplane x the following holds: x is faster than every (any) airplane y".

More Examples

Translate: ∀ x ∃ y F(x, y)

"For every airplane x the following holds:
for some airplane y, x is faster than y".
or "Every airplane is faster than some airplane".

Note: These are not equivalent

Translate: ∃ x∀y F(x, y)

"There exist an airplane x which satisfies the following:
(or such that) for every airplane y, x is faster than y".
or "There is an airplane which is faster than every airplane"
or "Some airplane is faster than every airplane".

Translate: ∃ x ∃ y F(x, y)

For some airplane x there exists an airplane y such that x is faster than y"
or "Some airplane is faster than some airplane".

More Examples

U=R (all the real numbers)
P(x,y): x,* y=0
∀x∀y P(x, y)

Which of these are True?

∀x∀y P(x, y)	False
∀ x ∃ y P(x, y)	True
∃ x∀y P(x, y)	True
∃ x ∃ y P(x, y)	True

Negation of Quantifiers

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

and

$\neg \exists x P(x) \equiv \forall x \neg P(x)$

For Example

P(x) reprsents “x is happy”

U={people}

"There does not exist a person who is happy"

is equivalent to

"Everyone is not happy".

Negation of Quantifiers

$\neg \forall x P(x) \equiv \exists x \neg P(x)$

and

$\neg \exists x P(x) \equiv \forall x \neg P(x)$

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DeMorgan’s Laws

TABLE 2 De Morgan’s Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

English to Logical Expressions

F(x): x is a fleegle

S(x): x is a snurd

T(x): x is a thingamabob

U={fleegles, snurds, thingamabobs}

“ Everything is a fleegle”

$\forall x F(x) \equiv \neg \exists x \neg F(x)$

“ Nothing is snurd”

$\forall x \neg S(x) \equiv \neg \exists x S(x)$

“ All fleegles are snurds”

$\forall x[F(x) \rightarrow S(x)]$

$\equiv \forall x[\neg F(x) \lor S(x)]$

$\equiv \forall x \neg[F(x) \land \neg S(x)]$

$\equiv \neg \exists x[F(x) \land \neg S(x)]$

More English Translations

“ Some fleegles are thingamabobs”

$\exists x [F(x) \land T(x)] \equiv \neg \forall x [\neg F(x) \lor \neg T(x)$

“ No snurd is a thingamabob”

$\forall x [S(x) \rightarrow \neg T(x)] \equiv \neg \exists x [S(x) \land T(x)]$

“If any fleegle is a snurd then it's also a thingamabob”

$\forall x[(F(x) \land S(x)) \rightarrow T(x)]$

$\equiv \neg \exists x[F(x) \land S(x) \land \neg T(x)]$

HOMEWORK for SECTION 1.3

- 1,3,5,7,19,23

Chapter 1: Section 1.4

Nested Quantifiers

What are Nested quantifiers?

If one quantifier is within the scope of the other.

- Eg.

U:R

$\forall x \exists y(x + y=0)$

$\forall x \forall y(x+y=y+x)$

$\forall x \forall y \forall z(x+(y+z)=(x+y)+z)$

Translating into English

Translate:

U:R

$\forall x \forall y(x+y=y+x)$

$\forall x$ => "For every real number x"

$\forall y$ => "For every real number y"

$x+y=y+x$ => "x+y is equal to y + x"

"For every real number x and for every real number y, x+y is equal to y + x"

Translating to English

Translate:

U: R

$\forall x \forall y((x>0) \land (y<0) \rightarrow (xy<0))$

"For every real number x and every real number y, if x > 0 and y <0, then xy < 0"

Switching order

- If the quantifiers are the same switching order doesn't matter
  - (ie. All $\forall$'s or all $\exists$'s)
  - $\forall x \forall y \, P(x,y) \equiv \forall y \forall x \, P(x,y)$
  - $\exists x \exists y \, P(x,y) \equiv \exists y \exists x \, P(x,y)$
- If the quantifiers are different then order matters
  - $\forall x \exists y \, P(x,y) \not\equiv \exists y \forall x \, P(x,y)$

NOT Equivalent