ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

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Set 3 – Ch. 1.4, 2.1, 2.2

Announcements

- **Quiz schedule online ***
  - Will allow you to drop 1 quiz
  - Next Quiz is on Thursday
  - * Subject to change
- **Homework is online**

Today's Lecture

- Chapter 1 (Section 1.4)
  - Nested Quantifiers(1.4)
- Chapter 2 (Sections 2.1 & 2.2)
  - Sets(2.1)
  - Set Operations (2.2)

Chapter 1: Section 1.4

Nested Quantifiers

What are Nested quantifiers?

If one quantifier is within the scope of the other.

- Eg.

  U:R

  $\forall x \exists y (x + y = 0)$

  This is the same as $\forall x\, Q(x)$,
  where $Q(x)$ is $\exists y\, P(x, y)$,
  where $P(x, y)$ is $(x + y = 0)$

Translating to English

Translate:

U: R

$\forall x \forall y ((x>0) \land (y<0) \rightarrow (xy<0)$

"For every real number x and every real number y, if $x > 0$ and $y < 0$, then $xy < 0$"

Switching order

- If the quantifiers are the same switching order doesn't matter
  - (ie. All ∀'s or all ∃'s)
  - $\forall x \forall y\, P(x,y) \equiv \forall y \forall x\, P(x,y)$
  - $\exists x \exists y\, P(x,y) \equiv \exists y \exists x\, P(x,y)$
- If the quantifiers are different then order matters
  - $\forall x \exists y\, P(x,y) \not\equiv \exists y \forall x\, P(x,y)$

NOT Equivalent

Thinking of Quantification as Loops

To prove or disprove nested quantifications
→think in terms of nested loops

$\forall x \forall y\, P(x, y)$
Loop through x values
For each x value loop through the y values
If we find that P(x, y) is true for all values of y
for every x,
then $\forall x \forall y\, P(x, y)$ is True
If we find one y for any x such that P(x,y) is False
then $\forall x \forall y\, P(x, y)$ is False

Thinking of Quant. as Loops (2)

$\forall x \exists y\, P(x, y)$
Loop through x values
For each x value loop through the y values
If we find one y for each x such that P(x, y) is true
then $\forall x \exists y\, P(x, y)$ is True
If for any one x we can't find a y such that P(x,y) is true
then $\forall x \exists y\, P(x, y)$ is False

$\exists x \forall y\, P(x, y)$
Loop through x values
For each x value loop through the y values
If we find an x such P(x, y) is true for all y's
then $\exists x \forall y\, P(x, y)$ is True
If we can't find such an x
then $\exists x \forall y\, P(x, y)$ is False

Thinking of Quant. as Loops (3)

$\exists x \exists y\, P(x, y)$
Loop through x values
For each x value loop through the y values
If we find one y for one x such that P(x, y) is true
then $\exists x \exists y\, P(x, y)$ is True
If we can't fine one x and one y such that P(x,y) is true
then $\exists x \exists y\, P(x, y)$ is False

Quantification of Two Variables

State-ment	When True?	When False?
∀x ∀y	P(x, y) is true for every x, y pair	There is an x, y pair for which P(x,y) is false
∀x ∃y	For every x, there is at least one y for which P(x, y) is true	There is an x such that P(x, y) is false for every y
∃x ∀y	There is an x for which P(x, y) is true for every y	For every x there is at least one y for which P(x,y) is false
∃x ∃y	There is at least one x, y pair for which P(x, y) is true	P(x, y) is false for every x, y pair

Note: These are not equivalent

Translating from English to Nested Quantifiers

"The product of two positive numbers is positive."

U: R

$\forall x \forall y\, [(x > 0 \land y > 0) \rightarrow (xy > 0)]$

Is there an easier way?

U: R > 0

$\forall x \forall y\, (xy > 0)$

Translating from English to Nested Quantifiers (2)

"Given a number, there is a number greater than it."

In other words:

"For every number x we choose, there is a number y such that y > x."

U: R

$\forall x \, \exists y \, (y > x)$

Homework for Section 1.4

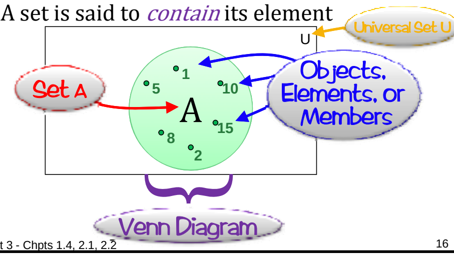
- 3(b,f)
- 5(b,f)
- 9(b,d,h,j)
- 11(b,f,h)
- 15(b,d,f)
- Feel free to do more if you need the practice

Chapter 2: Section 2.1

Sets

What is a Set?

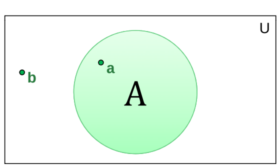
- *Set*: An unordered *collection* of *objects*
  - The point → to group objects together
  - Often objects have some similar properties
- *Objects*: *elements* or *members* of the set
  - A set is said to *contain* its element



Some Notations

a ∈ A → a is an element of the set A

b ∉ A → a is not an element of the set A



Describing sets

- { } denote all the elements in the set

Eg.

V={a,e,i,o,u}

Set V — elements

Sets can also have unrelated objects

O={26, Paul, Pot, a}

... - ellipses denote a pattern

I={2,4,6,...,98}

Set Builders (describing sets)

- You can also use set builders so that you don't have to name every element
  - Just state the properties

I={x| x is a positive even integer less than 100}

Or

I={x ∈ Z^+ | x is even and x<100}

All integers All positive

- You can also use Predicates

I={x | P(x)}

I contains all elements from U which make P true

Common Universal Sets

N = {0,1,2,3,…}, the set of natural numbers

Z = {…,-2,-1,0,1,2,…}, the set of integers

Z^+ = {1,2,3,…}, the set of positive integers

Q = {p/q | p ∈ Z , q ∈ Z, and q ≠ 0}, the set of rational numbers

R, the set of real numbers

Note: Sometimes 0 is not considered a part of the set of natural numbers.

Special Sets

- The *empty set*, *void set*, or *null set*
  - A set with no elements
  - Notation: Ø or { }
  - The assertion x∈ Ø is always false
- The *singleton set*
  - A set with one element

Is this the empty set? {Ø}

No! It is the singleton set with the empty set as its element

Think of the empty set as an empty folder

Think of this {Ø} as a folder with only an empty folder in it

Subsets

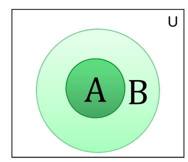
A is a ***subset*** of B iff every element of A is also an element of B.

- In other words A is a subset of B iff

∀x (x ∈ A ⟶ x ∈ B).

- Notation

A ⊆ B



Example:

A ={2,4,6}

B={1,2,3,4,5,6,7,8}

A ⊆ B

Proper Subsets

- A proper subset is a subset in which A ≠ B
- In other words A is a proper subset of B iff
  - ∀x (x ∈ A ⟶ x ∈ B) ⋀ ∃x (x ∈ B ⋀ x ∉ A)
  - Notation: A ⊂ B

Eg.

A={1,2,3}

B={0,1,2,3,4,5,6}

A ⊂ B because B has more elements than A

If A={0,1,2,3,4,5,6}

then A ⊆ B, but A ⊄ B

What if A={1,2,9}?

Theorem 1

For every set S,
(1) Ø ⊆ S and (2) S ⊆ S

Proof

Let S be a set

To show that Ø ⊆ S we must show that

∀x (x ∈ Ø ⟶ x ∈ S) is true

Because Ø contains no elements,

it follows that x ∈ Ø is always false.

It follows that x ∈ Ø ⟶ x ∈ S is always true, because its hypothesis is always false and a conditional statement with a false hypothesis is always true.

Therefore ∀x (x ∈ Ø ⟶ x ∈ S) is true

Equal Sets

Two sets are ***equal*** iff they have the same elements.

- In other words A and B are equal iff

  $\forall x (x \in A \leftrightarrow x \in B)$.
- Notation

  A=B

Eg.

A={x, y, z}

B={z, x, y}

C={z,z,z,z,y,y,y,y,y,y,y,y,x}

A=B=C

Equal Sets (2)

We can prove 2 sets (A & B) are equal if we can show:

$A \subseteq B$ and $B \subseteq A$

Remember :

$A \subseteq B$ is the same as $\forall x (x \in A \rightarrow x \in B)$ and

$B \subseteq A$ is the same as $\forall x (x \in B \rightarrow x \in A)$

Which is the same as saying $\forall x (x \in A \leftrightarrow x \in B)$

Sets as Elements/Members

- Sets may have other sets as elements

Like with the empty set …

A={Ø, {a}, {b}, {a,b}}

Is a ∈ A?

No!

Is {a} ∈ A?

Yes!

Is B={x | x is a subset of the set {a,b}} equivalent to A?

Yes!

Cardinality, Finite & Infinite Sets

Let S be a set.
If there is exactly *n* distinct elements in S where *n* is a nonnegative integer, we say that S is a ***finite set*** and that N is the ***cardinality*** of S.

- Notation for cardinality: |S|

Eg.

Let V be the set of vowels in the alphabet

|V| = 5

A set is ***infinite*** if it is not finite.

The Power Set

Given a set S, the ***power set*** is the set of all subsets of S.

- Notation

  P(S)

Eg.

Let S={a,b,c}

What is P(S)?

P(S)={Ø, {a}, {b}, {c},{a,b}, {a,c},{b,c},{a,b,c}}

How many elements does P(S) have if |S| is 6?

2^6 or 64 elements

Ordered n-tuples

- Sometimes order matters
  - Sets are unordered
  - We use ***ordered n-tuples***

The ***ordered n-tuple*** $(a_1, a_2,\ldots, a_n)$ is the ordered collection that has a_1 as its first element, a_2 as its 2nd element ,…, and a_n as its nth element.

- They are equal iff each corresponding pair of elements is equal.
- $\{a_1, a_2,\ldots, a_n\} = \{b_1, b_2,\ldots, b_n\}$ iff $a_i = b_i$

Cartesian Product

Let A & B be sets.
The ***Cartesian product*** of A and B is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$.

- Notation

  A x B

Eg.

Let A={1,2} and B= {a,b,c}

What is A x B

- A x B = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}

Cartesian Product (2)

- Some things to note:

  A x B ≠ B x A **unless**
  - A = Ø or B = Ø (thus A x B = Ø) or
  - A=B

Relations

A subset R of the Cartesian product A x B *is called a* ***relation*** from set A to set B.
The elements of R are ordered pairs where the 1st element belongs to A and the 2nd to B.

Eg.

Let A={1,2} and B= {a,b,c}

R= {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c)}

R is a relations from Set A to Set B

Cartesian Product: More than 2 sets

The ***Cartesian product*** of the sets $A_1, A_2, \ldots, A_n$ is the set of ordered n-tuples $(a_1, a_2, \ldots, a_n)$, where a_i belongs to A_i for i=1,2,…,n

- Notated

  $A_1 \times A_2 \times \ldots \times A_n$

Eg.

Let A={1,2} , B= {a,b,c}, C= {y,z}

What is A x B x C

- A x B x C= {(1,a,y), (1,a,z), (1,b,y), (1,b,z), (1,c,y), (1,c,z), (2,a,y), (2,a,z), (2,b,y), (2,b,z), (2,c,y), (2,c,z)}

Thus A x B x C is all possible ordered tuples (a,b,c) where $a \in A$, $b \in B$, and $c \in C$

Homework for Section 2.1

- 1, 3, 5, 7, 13, 17, 23, 25, 27, 31

Chapter 2: Section 2.2

Set Operations

Union

- Let A & B be sets.

  The ***union*** of the sets A & B is the set that contains those elements in either A or B or both.
- Notation: $A \cup B$
  - In other words: $\{x | x \in A \lor x \in B\}$

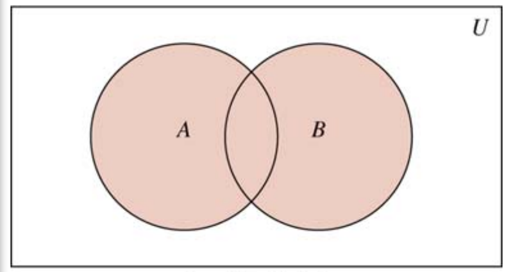
Eg

A={1,2,3}

B={3,4,5}

$A \cup B = \{1,2,3,4,5\}$

Union – Venn Diagram



$A \cup B$ is shaded.

Intersection

- Let A & B be sets.

  The ***intersection*** of the sets A & B is the set that contains those elements in both A and B.
- Notation: $A \cap B$
  - In other words: $\{x | x \in A \land x \in B\}$

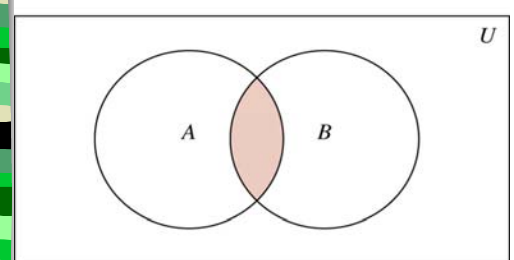
Eg

A={1,2,3}

B={3,4,5}

$A \cap B = \{3\}$

Intersection – Venn Diagram



$A \cap B$ is shaded.

Disjoint

- Let A & B be sets.

  A & B are ***disjoint*** if the intersection of the sets A & B is the empty set.

Eg

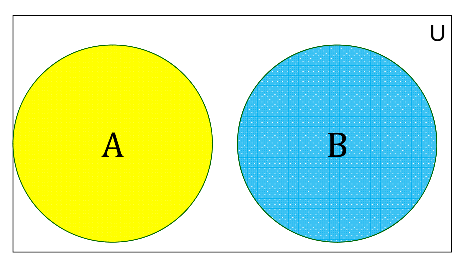
A={1,2,3}

B={4,5,6}

$A \cap B = \emptyset$

A & B are Disjoint

Disjoint – Venn Diagram



Difference / Complement

- Let A & B be sets.

  The *difference* in A & B is the set containing those elements that are in A, but not in B.

  AKA the *complement* of B with respect to A.
- Notation: A - B
  - In other words: $\{x | x \in A \land x \notin B\}$

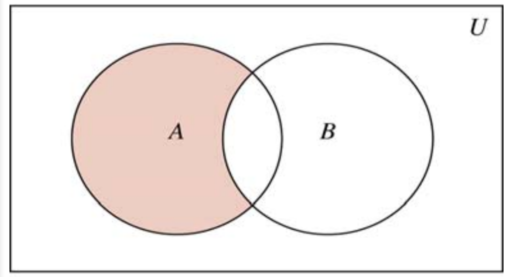
Eg

A={1,2,3}

B={3,4,5,6}

A - B = {1,2}

Difference / Complement – Venn Diagram



$A - B$ is shaded.

(Absolute) Complement

- Once the universal set U can be specified, the complement can be defined

  Let U be the universal set.

  The *(absolute) complement* of set A,

  is the complement of A with respect to U.

  In other words,

  the *(absolute) complement* of set A is U-A
- Notation: $\bar{A}$ or A^c
  - In other words: $\{x | x \notin A\}$ or $\{x | \neg(x \in A)\}$

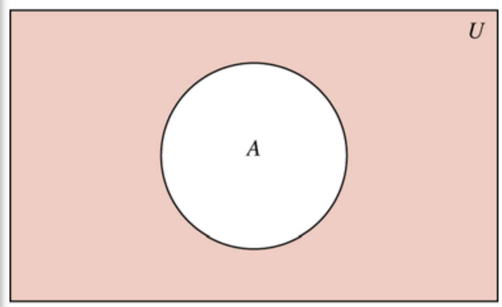
Eg

U={2,4, 6,8,10}

A={2,4,6}

$\bar{A}$ = {8,10}

Complement



$\bar{A}$ is shaded.

Symmetric Difference

- Let A & B be sets.

  The *symmetric difference* in A & B is the set containing those elements that are in A, but not in B and the elements in B, that are not in A.
- Notation: $A \oplus B$
  - In other words: $(A - B) \cup (B - A)$

Eg

A={1,2,3}

B={3,4,5,6}

A - B = {1,2}

B – A = {4,5,6}

$A \oplus B$ = {1,2,4,5,6}

Examples

U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

A= {1, 2, 3, 4, 5}

B = {4, 5, 6, 7, 8}

$A \cup B$ = {1, 2, 3, 4, 5, 6, 7, 8}

$A \cap B$ = {4, 5}

A^c = {0, 6, 7, 8, 9, 10}

B^c = {0, 1, 2, 3, 9, 10}

A - B = {1, 2, 3}

B - A = {6, 7, 8}

$A \oplus B$ = {1, 2, 3, 6, 7, 8}

TABLE 1 Set Identities.	
Identity	*Name*
$A \cup \varnothing = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \varnothing = \varnothing$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws

More Set Identities

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \varnothing$	Complement laws