ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

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Set 4 – Ch. 2.2, 2.3, 8.1

Announcements

- Quiz schedule online *
  - Will allow you to drop 1 quiz
  - * Subject to change
- Homework is online

Today's Lecture

- Chapter 2 (2.2 & 2.3)
  - Set Operations (2.2)
  - Functions (2.3)
- Chapter 8 (8.1)
  - Relations and their properties (8.1)

Chapter 2: Section 2.2 (con't)

Set Operations

Proofs

There are several ways to construct proofs for sets

- Using Cases
- Logical equivalences
  - Set builder notation
- Direct Proof
- Membership tables
  - check out the book for an example
- When proving equality you can show that the two sets are subsets of each other
  - To prove $A=B$ show that $A \subseteq B$ and $B \subseteq A$

Proving Equality – Using Cases

Prove that the following is true for all sets *A, B, and C:*

If $A \cap C = B \cap C$ & $A \cup C = B \cup C$, then $A=B$.

First we will show $A \subseteq B$

Then we will show $B \subseteq A$

We know that $A \cap C = B \cap C$ & $A \cup C = B \cup C$

Proof that $A \subseteq B$:

Let $x \in A$. We need to show that $x \in B$.

We will give a proof by cases,

depending on whether or not $x \in C$.

- Case 1: $x \in C$
  In this case $x \in A \cap C$.
  Because $A \cap C = B \cap C$, we have $x \in B \cap C$, and hence $x \in B$.
- Case 2: $x \notin C$
  In this case $x \in A \cup C$ (because $x \in A$).
  Because $A \cup C = B \cup C$, we have $x \in B \cup C$.
  But $x \notin C$. Therefore we must have $x \in B$.
- Cases 1 and 2 show that
  If $x \in A$, then $x \in B$, or $A \subseteq B$.
- A similar proof can be given to show that $B \subseteq A$.
- Because $A \subseteq B$ and $B \subseteq A$, $A = B$.

Proof using Logical Equivalence

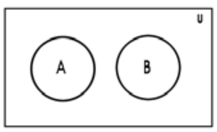
Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution:

We begin with $A \cap (B \cup C)$ and show that this is the same as $(A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) = \{ x \mid x \in A \wedge x \in B \cup C \}$	definition of intersection
$= \{ x \mid x \in A \wedge (x \in B \vee x \in C) \}$	definition of union
$= \{ x \mid (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \}$	distributive law
$= \{ x \mid (x \in A \cap B) \vee (x \in A \cap C) \}$	definition of intersection
$= (A \cap B) \cup (A \cap C)$	definition of union

Direct Proof



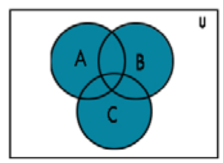
Prove: If $A \subseteq B^c$, then $B \subseteq A^c$.

Solution:

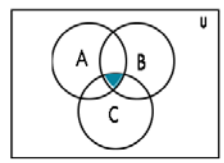
Suppose $A \subseteq B^c$.
We must show that $B \subseteq A^c$.

To show that $B \subseteq A^c$, assume that $x \in B$ and show that $x \in A^c$.
Suppose $x \in B$.
Therefore $x \notin B^c$.
Therefore $x \notin A$ (because $A \subseteq B^c$).
Therefore $x \in A^c$.

Generalized Unions & Intersections



$A \cup B \cup C$



$A \cap B \cap C$

Because of the associative laws we don't need parenthesis

Generalized Unions & Intersections

The ***union*** of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

- Notation: $A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{i=1}^{n} A_i$

The ***intersection*** of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

- Notation: $A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{i=1}^{n} A_i$

Homework for Section 2.2

- 1,3,7,9,15,21,27,31,33,37

Chapter 2: Section 2.3

Functions

What is a function?

Let A & B be sets (non-empty).
A ***function f*** from A to B is an assignment of exactly one element of B to each element of a.

- Notations:

f :A⟶B — F is a function from A to B or F maps A to B

Domain (A), Codomain (B)

More Notations

f :A⟶B
Can also be defined a s a relation from A to B.

Remember: A *relation* is a subset of A x B where there is one order pair for every element a ∈ A and b ∈ B.

f(a)=b — b is the unique element of B assigned by the function f to A

a is the preimage of b; b is the image of a

(a,b) is the unique ordered pair

The ***range*** of f is the set of all images of points in A under f. We denote it by f(A).

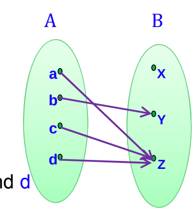
Functions - Examples

If S is a subset of A then

f(S) = {f(s) | s in S}

f :A⟶B

- f(a) = Z
- The image of d is Z
- Domain of f is A = {a, b, c, d}
- The codomain is B={X, Y, Z}
- f(A) = {Y, Z}
- The preimage of Y is B
- The preimages of Z are a, c and d
- f({c,d}) = {Z}



Definitions

Let f be a function from A to B. (f :A⟶B)

- f is ***one-to-one*** or ***injective*** if preimages are unique
  - **Note:** this means that if a ≠b then f(a) ≠ f(b).
  - **Notation:** 1-1
- f is ***onto*** or ***surjective*** if every y in B has a preimage
  - **Note:** this means that for every y in B there must be an x in A such that f(x) = y.
- f is ***bijective*** or ***one-to-one correspondence*** if it is surjective and injective

One to One or Injection Example

- f is ***one-to-one*** or ***injective*** if preimages are unique
- Functions that never assign the same value to two different domain elements

Which are 1-1?

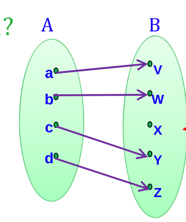
Domain: Z

f(x)=x

f(x)=x^2

- f(-1)≠ f(1)

f(x)=|x|

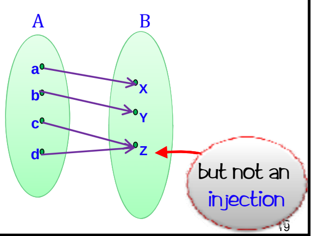


but not a surgection

Onto or Surjection Example

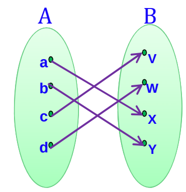
- f is ***onto*** or ***surjective*** if every y in B has a preimage.
- Every element in the codomain is the image of some element in the domain.

Which are onto?
Domain: **Z**
f(x)=x
f(x)=x^2
When does x^2=-1?
f(x)=|x|



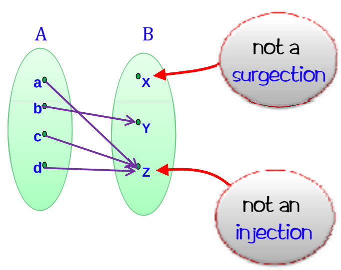
Bijection (one-to one and onto)

- f is ***bijective*** if it is surjective and injective

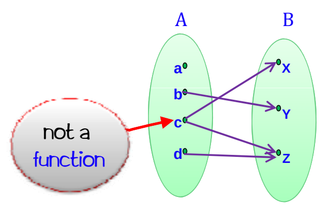


Note: Whenever there is a bijection from A to B, the two sets must have the same number of elements or the same ***cardinality***.

What about this one?



What about this one?



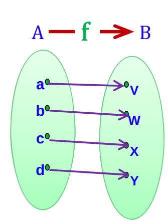
Inverse Functions

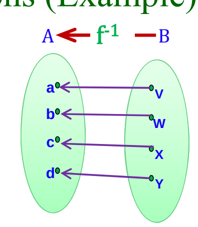
Let f be a bijection from A to B.
Then the ***inverse of f***, is the function from B to A
That assigns to an element b the unique element
Such that f(a)=b.

- Notation: f^{-1}
- In other words...

$f^{-1}(b) = a$ when $f(a) = b$

Inverse Functions (Example)





A bijection is called ***invertible*** because you can define the inverse function. To be invertible it must be a bijection.

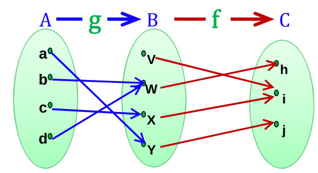
Composition

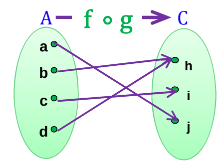
Let g: A ⟶B, f: B ⟶ C.
The ***composition*** of the functions f and g, denoted $f \circ g$, is the function from A to C defined By $f \circ g\ (a)=f(g(a))$

- Apply g(a) then f(g(a))
- The range of g must be a subset of the domain of f.

Composition Example

If $f(x)=x^2$ and
$g(x)=2x+1$
Then
$f(g(x))=f\,(2x+1)$
$=(2x+1)^2$
$= 2x^2+1$





Floors & Ceilings

The ***floor function*** assigns to the real number x the largest integer that is $\leq$ x.

- Notation: $\lfloor x \rfloor$ or floor(x)

The ***ceiling function*** assigns to the real number x the smallest integer that is $\geq$ x.

- Notation: $\lceil x \rceil$ or ceiling(x)

Examples:

$\lfloor 3.5 \rfloor = 3$, $\lceil 3.5 \rceil = 4$

Homework for Section 2.3

- 3, 5, 11, 13, 15, 19

Chapter 8: Section 8.1

Relations and their properties

Binary Relations

Let A and B be sets.
A ***binary relation*** from A to B is a subset of A x B.

In other words:
It is the ordered pair of elements in two sets.

A binary relation then from A to B is a set R of ordered pairs (a,b) such that $a \in A$ and $b \in B$.

$R \subseteq A \times B$

There are no constraints on ordered pairs (like there are on functions)

Binary Relation - Example

- Let A and B be sets:

A={ 1, 2, 3 }

B={ a, b, c}

A x B = {(1,a),(1,b),(1,c), (2,a),(2,b),(2,c), (3,a), (3,b), (3,c)}

Set A

Set B

Binary Relation R

- Let A anb B be sets:

A = { a, b, c}

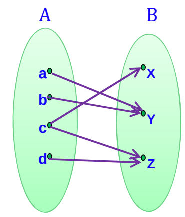
B = {1, 2, 3, 4}

R is defined by the ordered pairs or edges

{(a, 1), (a, 2), (c, 4)}

Binary Relation

- Binary Relations don't adhere to the same constraints as functions



Reflexive Property

Let A be a set and R be a relation on set A.
R is ***reflexive*** if (a, a) ∈ R for every element a ∈ A.

- In other words

∀x[x ∈U⟶(x, x) ∈ R]

Reflexive Property Example

Consider the relations on {1,2,3,4}

Which are reflexive?

R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

R_2={(1,1),(1,2),(2,1)}

R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}

R_6={(3,4)}

3 & 5

3 & 5 because they contain (1,1),(2,2),(3,3),& (4,4)

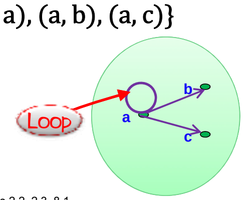
Relations on a Set to itself

Let A be a set.
A ***binary relation*** R on a set A is a subset of A x A or a relation from A to A.

Example:

A = {a, b, c}

R = {(a, a), (a, b), (a, c)}



Symmetric Property

Let A be a set and R be a relation on set A.
R is ***symmetric*** if (b, a) ∈ R whenever (a, b) ∈ R.
for all (a,b) ∈ A.

- In other words: R is ***symmetric*** iff

∀x ∀y[(x, y) ∈ R⟶ (y, x) ∈ R]

Note: If there is an arc (x, y) there must be an arc (y, x).

Antisymmetric Property

Let A be a set and R be a relation on set A.
For all (a,b) ∈ A, if (a, b) ∈ R and (b, a) ∈ R then a=b is called ***antisymmetric.***

- In other words: R is ***antisymmetric*** iff
  $\forall x \forall y\,[(x, y) \in R \land (y, x) \in R \rightarrow x=y]$

Note: If there is an arc from x to y (x,y), then there can't be one from y to x (y, x) if $x \neq y$

You should be able to show that logically:
If (x, y) is in R and $x \neq y$,
then (y, x) is not in R.

Symmetric/Antisymmetric Example

Consider the relations on {1,2,3,4}
Which are symmetric? Which are antiymmetric?

R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}
R_2={(1,1),(1,2),(2,1)} ← 2 & 3
R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}
R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}
R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}
R_6={(3,4)}

2 & 3 because for every pair that is not the same they have the reverse pair only need to check (1,2),(2,2),(1,4),& (4,1)

The rest are antisymmetric

Transitive Property

Let A be a set and R be a relation on set A.
R is ***transitive*** if whenever (x, y) ∈ R and (y, z) ∈ R then (x,z) ∈ R for all x, y, z ∈ A

- In other words: R is transitive iff
  $\forall x \forall y \forall z[(x, y) \in R \land (y, z) \in R \rightarrow (x, z) \in R]$

Note: If there is an arc from x to y (x,y)
and one from y to z
then there must be one from x to z

This is the most difficult one to check.