ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

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Set 5 – Ch. 8.1, 8.2, 8.3

Announcements

- Quiz #3 Thursday
  - Will cover 2.2, 2.3, 8.1, 8.2, 8.3
- Homework is online
- Regrades
  - Please look over your quizzes/homeworks carefully
  - Quizzes 1 & 2 & Homeworks 1,2,3 need to be turned in by Thursday.
  - If for some reason you can't turn them in by thurs -- let me know no later than thurs

Today's Lecture

- Chapter 8 (8.1, 8.2, 8.3)
  - Relations and their properties (8.1)
  - n-ary relations and their applications (8.2)

Chapter 8: Section 8.1

Relations and their properties

Transitive Property

Let A be a set and R be a relation on set A.
R is ***transitive*** if whenever (x, y) ∈ R and (y, z) ∈ R
then (x,z) ∈ R for all x, y, z ∈ A

- In other words: R is transitive iff

$\forall x \forall y \forall z[(x, y) \in R \land (y, z) \in R \rightarrow (x, z) \in R]$

Note: If there is an arc from x to y (x,y)
and one from y to z
then there must be one from x to z

This is the most difficult one to check. Lets

Transitive Examples

Eg. >

If $x > y$ & $y > z$ then is $x > z$?

$3 > 2 \land 2 > 1 \rightarrow 3 > 1$

Yes → Transitive

Eg. ≠

If $x \neq y$ & $y \neq z$ then is $x \neq z$?

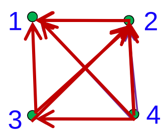
$1 \neq 2 \land 2 \neq 1 \rightarrow 1 \neq 1$

No→ Not Transitive

Transitive Example (2)

Consider the relations on {1,2,3,4}

- R_5={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

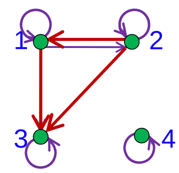


x⟶y	y⟶z	x⟶z
(3, 2)	(2, 1)	(3, 1)
(4, 2)	(2, 1)	(4, 1)
(4, 3)	(3, 2)	(4, 2)

Transitive Example (3)

Consider the relations on {1,2,3,4}

- R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}



x⟶y	y⟶z	x⟶z
(2, 1)	(1, 3)	(2, 3)

Combining Relations

The relations R_1 & R_2 from A to B are subset s of A x B

→ 2 relations R_1 & R_2 , can be combined the same way any sets can be combined

For Example

A= {1, 2, 3, 4}, B = {4,5,6}

R_1 = {(1,4), (2,5), (3,6)}

R_2 = {(1,4),(1,5),(1,6),(2,1)}

$R_1 \cup R_2$ = {(1,4),(1,5),(1,6),(2,1), (2,5),(3,6)}

$R_1 \cap R_2$ = {(1,4)}

$R_1 - R_2$ = {(2,5),(3,6)}

$R_2 - R_1$ = {(1,5),(1,6),(2,1)}

$R_2 \oplus R_1$ = {(1,5),(1,6),(2,1), (2,5),(3,6)}

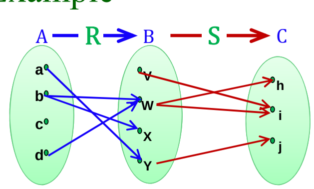
Composite

Let R be a relation from set A to B &
S be a relation from B to C. The ***composite*** is the relation consisting of ordered pairs (a, c) where a ∈ A, c ∈ C, where there exists a b ∈ B such that (a,b) ∈ R and (b,c) ∈ S.

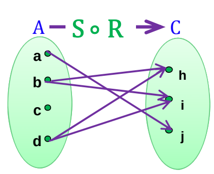
- Notations: $S \circ R$
- In other words

Let R be a relation from set A to B &
S be a relation from set B to C then
$S \circ R$ is the relation from A to C
(provided b exists , (a,b) ∈ R and (b,c) ∈ S)

Composite Example



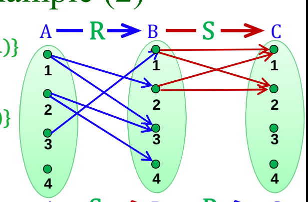
R={(a, y), (b, w),(b, x), (d, w)}
S={(v, i), (w, h),(w, i),(y, j)}
$S \circ R$={(a, j),(b, h),(b,i),(d, h),(d, i)}



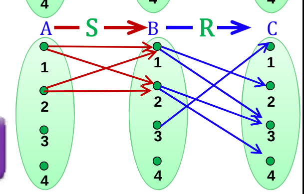
Composite Example (2)

R={(1, 2), (1, 3),(2, 3), (2, 4),(3, 1)}
S={(1,1), (1, 2),(2, 1),(2, 2)}

$S \circ R$ = {(1, 1), (1, 2), (3, 1), (3, 2)}



$R \circ S$= {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)}



Note: This is similar to composition of functions.

Using Relation of Composition to Check if a Relation is Transitive

Definition:

Let R be a relation on the set A.

The powers R^n, n=1,2,3,..., are defined recursively by

$R^1 = R$ and $R^{n+1} = R^n \circ R$.

Theorem:
The relation R on set A is ***transitive*** iff
$R^n \subseteq R$ for $n > 0$

Important Proof:

R transitive iff $R^n \subseteq R$

Proof by induction:

- Assume ***R is transitive.***
- Now show $R^n \subseteq R$ ***by induction.***

Basis: Show it is true for n = 1.

- $R^1 = R$ so $R^n \subseteq R$ – Nothing to prove

Induction:

The induction hypothesis:

- "assume true for n", so assume $R^n \subseteq R$

Show it must be true for n + 1.

- $R^{n+1} = R^n \circ R$
- So if $(a,c) \in R^{n+1}$ then there exist a b such that $(a,b) \in R^n$ and $(b,c) \in R$ (by definition of composite)

We want to prove that $(a,c) \in R$

using the fact that R is *transitive*

- By the inductive hypothesis: $R^n \subseteq R$
  So $(a, b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ (by transitivity).

We just showed that any $(a,c) \in R^{n+1}$ belongs to R,

So $R^{n+1} \subseteq R$.

Q.E.D.

Homework 8.1

- 1(a,b,c), 3 (a-f), 5,7(a,c,e,g),9, 28, 30, 35 (a, c, e, g)

Take a break!

- Stretch, Relax
- Get some water, Use the restroom
- Get to know your classmates…
- Etc…..

When we return…

- 8.2 n-ary relations and their applications

Chapter 8: Section 8.2

n-ary relations and their applications

n-ary relations

Let $A_1, A_2, \ldots, A_n$ be sets.
An ***n-ary relation*** on these sets is a subset of $A_1 x A_2 x \ldots x A_n$.
The sets $A_1, A_2, \ldots, A_n$ are the ***domains*** of the relation, and
n is the ***degree***.

Example:

Let R be the relation on N x N x N consisting of 3-tuples (a,b,c) where a, b, c are integers with a<b<c.

Domains: set of natural #s

(1,2,3) ∈ R

Degree = 3

(2,4,3) ∉ R

Displaying n-ary Relations w/ Tables

Table: *Teaching Assignments*

Professor	Dept	Course No.
Rousseau	In4matX	INF111
Dillencourt	CS	ICS161
Rousseau	Math	Math6B
Suda	CS	ICS151
Ziv	In4matX	INF131

Attribute = index of col

Field = any entry

Record = Row

- ***Degree*** = # of columns = 3 → Columns = ***attributes***
- ***Rows*** = # of n-tuples in R = 5 → ***Record***
- ***Domain***: A1 = Professor, A2 = Dept, A3 = Course No.
- ***Relation*** is {(Rousseau, In4matX,INF111), (Dillencourt,CS,ICS161), (Rousseau,Math,Math6B), (Suda,CS,ICS151), (Ziv, In4matX,INF131)}
- Each entry in the n-tuple is a ***field***

Primary Keys

- Let R be any relation of degree n
  - i.e. it is a subset of $A_1 x A_2 x \ldots x A_n$
  - $A_1, A_2, \ldots, A_n$ are the domains
  - The elements of R are n-tuples $(a_1, a_2, \ldots, a_n)$ with $a_j \in A_j, \forall_j = 1..n$

  The domain A_j is the primary key for R if we can distinguish the n-tuples in R by looking at the jth entry

In other words…

- A primary key essentially is a domain that can act as a unique identifier in the n-tuples
  - This is only possible if no two n-tuples are the same
  - The n-tuple can be identified by that domain

Primary Key Example

Table: *Teaching Assignments*

Professor	Dept	Course No.
Rousseau	In4matX	INF111
Dillencourt	CS	ICS161
Rousseau	Math	Math6B
Suda	CS	ICS151
Ziv	In4matX	INF131

- What are the Domains?
- Which could be a Primary Key?

Composite Key

Let R be an n-ary relation
Let C a condition that the elements in R must satisfy
An ***n-ary relation*** on these sets is a subset of $A_1 x A_2 x \ldots x A_n$.
The sets $A_1, A_2, \ldots, A_n$ are the ***domains*** of the relation, and
n is the ***degree***.

- A ***composite key*** is a set of keys that uniquely identify the n-tuples.
  - It is derived by the Cartesian product of the tuples that uniquely identify the n-tuples

Operations on n-ary Relations

- Selection Operator

  allows you to select n-tuples from a table that satisfy some condition
- Projection

  allows you to form new n-ary relations (ie a new table) from a pre-existing table by deleting the same fields in every record of the relation (removing rows/columns)
- Join
- Allows you to combine two tab;es into one when they share some identical fields

Selection Operator

Let R be an n-ary relation
Let C be a condtion that the elements in R may satisfy.
The ***selection operator*** maps the n-ary relation R to the n-ary relation of all n-tuples from R that satisfy condition C.

Notation: s_C

In other words…

Given a table,

Select only the rows and columns that satisfy the condition

Example – Selection Operator

Table: *Teaching Assignments*

Professor	Dept	Course No.
Rousseau	In4matX	INF111
Dillencourt	CS	ICS161
Rousseau	Math	Math6B
Suda	CS	ICS151
Ziv	In4matX	INF131

S_{C1}, where C1 is the condition In4matX Department

→ The result is

Professor	Dept	Course No.
Rousseau	In4matX	INF111
Ziv	In4matX	INF131

Projection

The ***projection*** $P_{i1\ i2\ldots\ im}$ where $i_1 < i_2 < \ldots < i_m$, maps the n-tuple $(a_1, a_2,\ldots, a_n)$ to the m-tuple $(a_1, a_2,\ldots, a_m)$ where $m \leq n$

- In other words,
  - Delete the columns in the ith position defined by P_i

Projection Example

Table: *Teaching Assignments*

R=

Professor	Dept	Course No.
Rousseau	In4matX	INF111
Dillencourt	CS	ICS161
Rousseau	Math	Math6B
Suda	CS	ICS151
Ziv	In4matX	INF131

$P_{1,2}(R) =$

Professor	Dept
Rousseau	In4matX
Dillencourt	CS
Rousseau	Math
Suda	CS
Ziv	In4matX

Join

Let R be a relation of degree m
Let S be a relation of degree n
The ***join*** $J_p(R,S)$ where $p \leq m$ and $p \leq n$, is a relation of degree m+n-p that consists of all (m+n-p)-tuples where the m-tuple belongs to and the n-tuple belongs to S

In other words

- Suppose that R & S are represent different attributes
  - R & S have the same number of records (rows)
  - R & S have p attributes in common (p identical cols)
  - Without loss of generality we can assume the last p columns of R agree with the first P columns of S

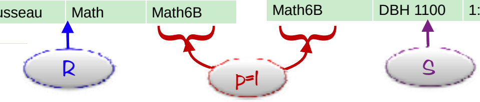
Join Example

Table: *Teaching Assignments*

Professor	Dept	Course No.
Rousseau	In4matX	INF111
Dillencourt	CS	ICS161
Rousseau	Math	Math6B

Table: *Room Assignments*

Course No.	Location	Time
INF111	ELH 100	9:00a
ICS161	DBH 1500	11:00a
Math6B	DBH 1100	1:00p



J(R,S)=

Professor	Dept	Course No.	Location	Time
Rousseau	In4matX	INF111	ELH 100	9:00a
Dillencourt	CS	ICS161	DBH 1500	11:00a
Rousseau	Math	Math6B	DBH 1100	1:00p

Homework 8.2

- 3,5,7 (a-c),9 (a-c),11,17,19

Chapter 8: Section 8.3

Representing Relations

Representing **Binary** Relations

- By Matrices (0,1)
- By Directed Graphs

Using Matrices

Let R be a relation from

$A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$

R is a subset of A x B.

So elements of R are pairs of the form (a_i, b_j) for some i=1..m, j=1..n

We assocciate R to the matrix M_R as follows:

$M_{ij} = 1$ if $(a_i, b_j) \in R$,

0 if $(a_i, b_j) \notin R$.

Entry of the matrix M_R in the ith row and jth column

M_R is a 0-1 matrix

Example Relation as a Matrix

We assume the rows are labeled with the elements of A and the columns are labeled with the elements of B.

Let

A = {a, b, c}

B = {e, f, g, h}

R = {(a, e), (c, g)}

Then the connection matrix M for R is

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Note: the order of the elements of A and B matters

Theorem

Let R be a binary relation on a set A and let M be its connection matrix. Then:

- R is reflexive iff $M_{ii} = 1$ for all i.
- R is symmetric iff M is a symmetric matrix: $M = M^T$
- R is antisymmetric if $M_{ij} = 0$ or $Mji = 0$ for all $i \neq j$.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Reflexive

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Not Reflexive

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Symmetric

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Not Symmetric

Theorem

- R is antisymmetric if $M_{ij} = 0$ or $Mji = 0$ for all $i \neq j$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Not Antisymmetric