ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

Michele Rousseau

Set 6 – Ch. 8.4

Announcements

- Don't forget Regrades for Quizzes 1 & 2, and Homeworks 1-3 are due today

Where you stand

- Homeworks

12 – 90-100
4 - 80-89
2 - less than 5

Quizzes

- Quiz #1
  - Max: 97%
  - Min: 28%
  - Avg: 70%
- Quiz #2
  - Max: 97%
  - Min: 46%
  - Avg: 77%

Overall

3 – 90-100
3 – 80-89
3 – 70-79
4 – 60-69
1 – 50-59
4 – less than 50

Some perspective

You can drop your lowest Quiz Score

I suffer from test anxiety – what can I do?

http://www.studygs.net/tstprp8.htm
http://ub-counseling.buffalo.edu/stresstestanxiety.shtml
http://www.sdc.uwo.ca/learning/mcanx.html
http://www.kidshealth.org/teen/school_jobs/school/test_anxiety.html

How do I improve my performance on Quizzes – and the final?

- If you have to miss lecture –
  - get notes from your friends
- Review lecture slides (take notes)
- Do the reading
- Form a study group
- Ask questions
  - In class
  - Email
  - Office hours
- What if I aced them? → WTG!

Today's Lecture

- Chapter 8 (8.3, 8.4)
  - Representing Relations (8.3)
  - Closures of Relations (8.4)

Chapter 8: Section 8.3

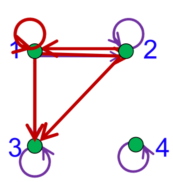
Representing Relations

Property	Definition	Matrix Def	In other words
Reflexive	$\forall i=1\ldots n$ $(a_i, a_i) \in R$	$\forall i=1\ldots n$ $m_{ii}=1$	All the diagonal entries of $M_R=1$
Irreflexive	$\forall i=1\ldots n$ $(a_i, a_i) \in R$	$\forall i=1\ldots n$ $m_{ii}=0$	All the diagonal entries of $M_R=0$
Symmetric	$\forall i,j=1\ldots n, i \neq j$ $(a_i, a_j) \in R$ & $(a_j, a_i) \in R$	$\forall i,j=1\ldots n, i \neq j$ $m_{ij},= 1$ & $m_{ji},= 1$ $m_{ij},= m_{ji}$	$m_{ij},= m_{ji}$ is either (0,0) or (1,1) M_R is symmetric wrt diagonal
Antisymmetric	$\forall i,j=1\ldots n, i \neq j$ $(a_i, a_j) \in R$ & $(a_j, a_i) \notin R$	$\forall i,j=1\ldots n, i \neq j$ $m_{ij},= 1$ & $m_{ji},= 0$ $m_{ij} \neq m_{ji}$	$\{m_{ij},,m_{ji}\} \neq \{1,1\}$ If i=j then {1,1} or {0,0}
Asymmetric	R both irreflexive & Antisymmetric	$\forall I,j=1\ldots n$ $m_{ii}=0$ **$i \neq j$** $m_{ij} \neq m_{ji}$	All the diagonal entries of $M_R=0$ **AND** $m_{ij},,m_{ji}\} \neq \{1,1\}$ If i=j then {1,1} or {0,0}
Transitivity	$\forall i,j,k=1\ldots n$ If $(a_i, a_j) \in R$ **&** $(a_j, a_k) \in R$ **then** $(a_i, a_k) \in R$	Well discuss this later	

Transitivity revisited

Consider the relations on {1,2,3,4}

- R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}



x→y	y→z	x→z
(2, 1)	(1, 3)	(2, 3)
(1, 2)	(2, 3)	(1, 3)

Transitive Property Example

Consider the relations on {1,2,3,4}

Which are Transitive?

R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

R_2={(1,1),(1,2),(2,1)}

R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

R_5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}

R_6={(3,4)}

4, 5,6

4- shown in slide set #5
5 - shown in previous slide
6 - because (a, b) ∧ (b,c) is false so (a, b) ∧ (b,c) ⟶ (c,d) is true

Using Matrices to represent Relations

Express the relation R_1 as a matrix

A={1.2.3.4}. B={1,2,3,4}

R1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

Think of the (a, b) pairs and remember that A=Rows and B= Cols

A \ B	1	2	3	4
1	(1,1)	(1, 2)	(1,3)	(1,4)
2	(2,1)	(2, 2)	(2,3)	(2,4)
3	(3,1)	(3, 2)	(3,3)	(3,4)
4	(4,1)	(4, 2)	(4,3)	(4,4)

Put a I in each of the Corresponding locations

$$M_{R1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Is R1 Reflexive? No

Is R1 Symmetric? No

Is R1 Antisymmetric? No

Matrices Examples

R2={(1,1),(1,2),(2,1)}

R3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

R4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

R5={(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,3),(4,4)}

$$M_{R2} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M_{R3} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad M_{R4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad M_{R5} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

	M_{R2}	M_{R3}	M_{R4}	M_{R5}
Reflexive?	No	Yes	No	Yes
Symmetric?	Yes	Yes	No	No
Antisymmetric?	No	No	Yes	No
Irreflexive?	No	No	Yes	No

Operations on Matrices/Relations

Let R & S be relations on a set A with corresponding matrices M_r & M_s

How do we find ?

$M_{r \cup s} = M_r \vee M_s$ &

$M_{r \cap s} = M_r \wedge M_s$

It helps if we first introduce a few operations on bits then apply it to matrices....

Join & Meet – bit operations

a "join" b = $a \vee b$ $\begin{cases} 1 & \text{if } a=1 \vee b=1 \\ 0 & \text{otherwise} \end{cases}$ (or)

a "meet" b = $a \wedge b$ $\begin{cases} 1 & \text{if } a=1 \wedge b=1 \\ 0 & \text{otherwise} \end{cases}$ (and)

These are defined in more detail in section 3.8

In other words..

$a \vee b = 0$ if a=b=0, otherwise it is 1

$a \wedge b = 1$ if a=b=1, otherwise it is 0

Now we can extend this to matrices (term by term)

Join and Meet - Matrices

For any two binary (n x n) matrices A & B

We define $A \vee B$ and $A \wedge B$ to be the n x n binary matrices whose ij element is given by:

$(A \text{ join } B)_{ij} = A_{ij} \vee B_{ij}$ &

$(A \text{ meet } B)_{ij} = A_{ij} \wedge B_{ij}$, respectively

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \vee \underset{B}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} = \underset{A \vee B}{\begin{bmatrix} 1 \vee 1 & 1 \vee 0 \\ 0 \vee 1 & 0 \vee 0 \end{bmatrix}} = \underset{A \vee B}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}} \wedge \underset{B}{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}} = \underset{A \vee B}{\begin{bmatrix} 1 \wedge 1 & 1 \wedge 0 \\ 0 \wedge 1 & 0 \wedge 0 \end{bmatrix}} = \underset{A \vee B}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$$

Boolean Product

Let A and B be nxn, 0-1 matrices.
The ***boolean product*** is a nxn, 0-1 matrix whose ij element is
$(a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \ldots \vee (a_{in} \wedge b_{nj})$

Notation: $A \odot B$

In other words:

- $(A \odot B)_{ij} = 1$ iff at least 1 of the terms $a_{ik} \wedge b_{kj} = 1$ or $\exists k(a_{ik} = b_{kj} = 1)$

Boolean Product Example

$(A \odot B)_{23}$ — 3rd Col in B; 2nd Row in A

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \odot \underbrace{\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}}_{B}$$

Check if there is a 1 appearing in the corresponding positions

$$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

gives 0 because $(0 \wedge 1) \vee (1 \wedge 0) \vee (1 \wedge 0)$

Boolean Product Example (2)

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

gives 1 – just need to find one "1"

For More Examples See Section 8.3

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{A} \odot \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

For Matrices: $M_{S \circ R} = M_R \odot M_S$

Note: The Matrix representing the composition $S \circ R = M_R \odot M_S$

Note the Ordering

Some Notations - Composition

For Matrices: $M_{S \circ R} = M_R \odot M_S$

Note: The Matrix representing the composition $S \circ R = M_R \odot M_S$

Note the Ordering

$$M_{R^2} = M_{R \circ R} = M_R \odot M_R = M_R^{[2]}$$

$$M_{R^n} = M_{\underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}} = \underbrace{M_R \odot M_R \odot \ldots \odot M_R}_{n \text{ times}} = M_R^{[n]}$$

Example

- Find the Matrix Representing $M_R^{[2]}$

$$M_R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$M_R^{[2]} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{M_R} \odot \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{M_R} = \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{M_R^{[2]}}$$

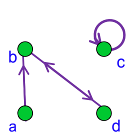
Represent Relations using Digraphs

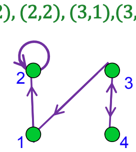
Digraph means "directed graph"

- Means there is an arrow on the arcs connecting the vertices indicating direction

For example:

R={(a,b), (b,d), (c,c), (d,b)} R={(1,2), (2,2), (3,1),(3,4) (4,3)}



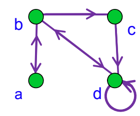


- Every diagraph represents a relation R on a set A

Digraphs make Props easy to see

Property	Definition	Diagraph
Reflexive	∀a∈A (a, a) ∈ R	There is a Loop at every vertex
Irreflexive	∀a∈A (a, a) ∉R	There are no Loops (at any vertex)
Symmetric	∀a≠b in A (a, b) ∈ R ⟷ (b, a) ∈ R	Every edge goes in both directions (can have loops)
Antisymmetric	∀a≠b in A (a, b) ∈ R ⟷ (b, a) ∈ R	Every edge goes one direction (can have loops)
Asymmetric	Irreflexive & Antisymmetric	No loops and All edges go in one direction
Transitive	∀[(a,b) ∧ (b,c)]∈R (a, b) ∈ R ∧ (b,c) ∈ R ⟶(a,c) ∈ R	If you have an edge from a to b, and an edge from b to c, then you also have an edge from a to c (completing the circle)

Example



Reflexive? No - missing loops at a, b, c

Irreflexive? No - it has a loop at d

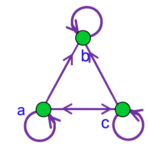
Symmetric? No - single directions at (b,c) and (c,d)

Antisymmetric? No - edges (a,b) and (b,d) go both directions

Asymmectric? Neither Irreflexive or Antisymmectric

Transitive? No – there is an edge (a,b) and (b,c) but no (a,c)

Example



Reflexive? Yes

Irreflexive? No -it has a loops at a,b, & c

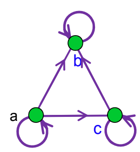
Symmetric? No -No bidirectional edges at (a,b) & (c,b)

Antisymmetric? No edges (a,c) and (c,a) go both directions

Asymmectric? Neither Irreflexive or Antisymmectric

Transitive? No – there is an edge (b,a) and (a,c) but no (b,c)

Example



Reflexive? Yes

Irreflexive? No -it has a loops at a,b, & c

Symmetric? No -No bidirectional edges at (a,b),(a,c) or (c,b)

Antisymmetric? Yes

Asymmectric? Not Irreflexive

Transitive? Yes

What are the ordered Pairs? {(a,a), (a,b), (a,c), (b,b), (c,b) , (c,c))}

Homework for 8.3

- 1 (a-d),3(a-c),5,7(a-c), 14 (a-c), 19 (a-d), 27,31(3 of them)