ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

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Set 7 – Ch. 8.4, 8.5, 8.6

Announcements

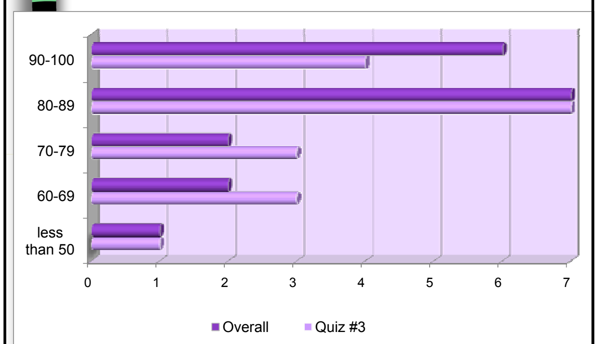
- Regrades for Quiz #3 and Homeworks #4 & 5 are due Thursday

Grades

Quiz #3
- Max: 100%
- Min: 42%
- Median: 83%

Overall
- Max: 99%
- Min 45%
- Median: 84%

Overall Grades & Quiz #3…



Today's Lecture

- Chapter 8 (8.4, 8.5, 8.6)
  - Closures of Relations (8.4)
  - Equivalence Relations (8.5)
  - Partial Orderings (8.6)

Chapter 8: Section 8.4

Closures of Relations

Closure of Relations

Let R be a relation on set A.
Let P be a property (reflexive, symmetric, etc.)
The ***closure*** of R with respect to the property P is the smallest relation containing R which has this property.

- In other words, add the minimum number of pairs to obtain property P.
- Note: This may not be possible.

Example:

A={1,2,3,4}, R={(1,1),(1,3),(14)}

P is being "irreflexive"

- If the closure S of R w.r.t. P exists,
  - Then the relations S is the intersection of all the relations R which satisfy property p.

Reflexive Closure

Example

A={1,2,3}

R={(1,1),(1,2),(1,3)}

P="being reflexive"

R is not reflexive, b/c its missing (2,2), (3,3)

The smallest reflexive relation containing R is

S={(1,1),(1,2),(1,3),(2,2),(3,3)}

This is the reflexive closure of R & it's the intersection of all of the reflexive relations that contain R

Any relation on A which is reflexive and contains R must include:

(1,1),(1,2),(1,3) and (1,1), (2,2), (3,3)

R — The diagonal pairs in AxA

Reflexive Closure (2)

- Let R be a relation on set A.
- Then the reflexive closure of R always exists: we just need to add all the elements of the form (a,a) with a ∈ A.
- In other words the "diagonal Δ in AxA"

Theorem:
If R is a relation on A, denote by Δ ={(a,a): a∈A} the diagonal in AxA. Then the ***reflexive closure*** of R exists and is equal to

$S_{reflexive} = R \cup \Delta$

Symmetric Closure

Example

A={1,2,3}

R={(1,1),(1,2),(1,3)}

P="being symmetric"

R is not symmetric , b/c it's missing (2,1), (3,1)

The smallest symmetric relation containing R is

S={(1,1),(1,2),(1,3),(2,1),(3,1)} ← Note: we are adding R^{-1}

This is the symmetric closure of R.

Generalized:
If R is a relation on A. Then the symmetric closure of R exists and is equal to

$S_{sym} = R \cup R^{-1}$

Symmetric Closure (2)

Example

A = {1,2,3,4}

R = {(1,3),(2,2), (2,4), (3,3), (3,4), (4,3)}

R^{-1}={(3,1), (2,2), (4,2), (3,3), (4,3), (3,4)}

Then

$R \cup R^{-1}$ ={(1,3),(3,1),(2,2),(2,4),(4,2),(3,3),(3,4),(4,3)}

This is the symmetric closure of R

$R \cup R^{-1}$ is the smallest symm-relation containing R, basically we are adding (3,1) & (4,2) which is what R needed to become symmetric

Irreflexive, AntiSymmetric & Asymmetric Closures

Assume P = "being irreflexive"

A={1,2,3,4}, R={(1,1),(1,3),(14)}

- Shows that if R is not irreflexive we can't make it irreflexive.
  →Thus the irreflexive closure of R does not exist
- When R is irreflexive
  → the irreflexive closure of R exists – it is R itself.
- The relation R then is the smallest irreflexive relation containing R
- This also applies to:
  →Antisymmetric & Asymmetric closures.

In terms of a Digraph

- To find the reflexive closure
  - add loops.
- To find the symmetric closure
  - add arcs in the opposite direction.
- To find the transitive closure - if there is a path from a to b
  - add a direct arc from a to b.

Note: Reflexive and Symmetric closures are easy
Transitive can be complicated

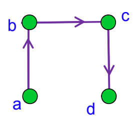
In terms of a Matrix

- To find the reflexive closure
  - Put 1's on the diagonal.
- To find the symmetric closure
  - Take the transpose M^T of the connection matrix M_R

Note: This relation is denoted R^T or R^c and and called the ***converse*** of R

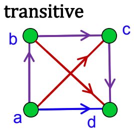
Transitive Closure t(R)

- This is a little more difficult



- •Because (a,b) and (b,c) the transitive closure must contain (a,c)
- •Similarly it must contain (b,d)

- The edges (a,c) and (b,d) seem to be the least amount of edges that need to be added in order to make R transitive



•This is not Transitive – because of (a,c), (b,d) – we need to add (a,d)
Now it is transitive – it may take several iterations
so t(R)=RU{(a,c),(b,d),(a,d)}

Paths

- A path of length n in a diagram G is the sequence of edges:
  - $(x_0, x_1)\ (x_1, x_2)\ldots(x_{n-1}, x_n)$
  - The terminal vertex of the previous arc matches the initial vertex of the following arc
- If $x_0 = x_n$ the path is called a *cycle* or a *circuit*. This is similarly true for relations

Theorem: Let R be a relation on set A.

There is a path of length n from a to b iff $(a,b) \in R^n$

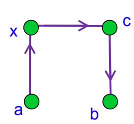
Proof: (by induction)

- Basis:

  An arc from a to b is a path of length 1 which is in R^1=R.
  Hence the assertion is true for n=1

- Induction Hypothesis:

  Assume the assertion is true for n.
  Show it is true for n+1



- There is a path of length n+1 from a to b iff
  there is an $x \in A$ such that
  there is a path of length 1 from a to x and
  a path of length n from x to b.

From the induction Hypothesis

$(a,x) \in R$

And since (x,b) is a path of length n

$(x,b) \in R^n$

if $(a,x) \in R$

and $(x,b) \in R^n$,

then $(a,b) \in R^n \circ R = R^{n+1}$

Q.E.D → quod erat demonstrandum
("that which was to have been demonstrated")

Transitivity Closure (2)

Theorem:
Let R be a relation on set A.
The ***connectivity relation*** or the ***star closure*** is the relation $R^* = \bigcup_{n=1}^{\infty} R^n$

- R^* is the union of all powers of R
- Notice that R^* contains the ordered set (a,b) if there is a path from a to b
- t(R) is the smallest transitive relation containing R
- R is transitive iff R^n is contained in R for all n

Proof

Theorem: R^* is the transitive closure of R.

Proof:

We must show that R^* is transitive

Suppose $(a,b) \in R^*$ and $(b,c) \in R^*$

Show $(a,c) \in R^*$

- By definition of R^*, $(a,b) \in R^m$ for some m and $(b,c) \in R^n$ for some n
- Then $(a,c) \in R^n R^m = R^{m+n}$ which is contained in R^*.

Hence R^* must be transitive

- Notice that R^* contains R
  - Because $R = R^1 \subseteq R^n = R^*$

So R^* is a transitive relation containing R

By definition the transitive closure of R, t(R), is the *smallest* transitive relation containing R.

To prove this lets suppose S is any transitive relation that contains R

We must show S contains R^* to show R^* is the *smallest* such relation.

$R \subset S$, so $R^* \subset S^2 \subset S$ since S is transitive

There fore $R^n \subset S^n \subset S$ for all n.

Hence S must contain R^* since it must also contain the union of all powers of R.

Q.E.D

In fact we only have to consider paths of n or less

Theorem: If $|A| = n$, then any path of *length* $> n$ must contain a cycle

Proof:

If we write down a list of more than n vertices representing a path in R, some vertex must appear at least twice in the list (by the Pigeon Hole Principle).

Thus R^k for $k > n$ doesn't contain any arcs that don't already appear in the first n powers of R.

Corollaries

Corollary: If $|A| = n$, then

$t(R) = R^* = R \cup R^2 \cup \ldots \cup R^n$

Corollary: We can find the connection matrix of t(R) by computing the join of the first n powers of the connection matrix of R.

3 Methods to construct $R^* = R \cup R^2 \cup \ldots \cup R^n$

1. Digraphs
2. Binary Matrices
3. Warshall's Algorithm (detailed in book)

Method 1: Diagraphs

Constructing $R^* = R \cup R^2 \cup \ldots \cup R^n$ using Digraphs:

Given R draw the corresponding diagraph D

Then compute

R=end points of paths of length 1 in D.

R^2=endpoints of paths of length 2 in D

.
.
.

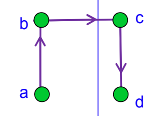
R^n=endpoints of paths of length n in D

Then compute $R^* = R \cup R^2 \cup \ldots \cup R^n$

→ this is the transitive closure of R.

Example: Digraphs

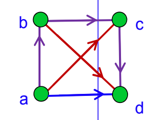
- A={a,b,c,d} ← N=4
- R={(a,b),(b,c),(c,d)}
- The digraph of R is:



We get R^2={ (a, c), (b, d) }

R^3={ (a, d) }

R^4=∅



Then

$R^* = R \cup R^2 \cup R^3 \cup R^4$

={(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)}

Method 2: Binary Matrices

Constructing $R^* = R \cup R^2 \cup \ldots \cup R^n$ using Binary Matrices

Given A and the relation R on A, construct the matrix M_R associated to R.

Then build the powers

$R \leftrightarrow M_R^{[2]} = M_R \odot M_R$

$R \leftrightarrow M_R^{[3]} = M_R \odot M_R \odot M_R$

.
.
.

$R \leftrightarrow M_R^{[n]} = M_R \odot \ldots \odot M_R$ (N Times)

The matrix associated to $R^* = R \cup R^2 \cup \ldots \cup R^n$ is

$M_{R^*} = M_R \vee M_R^{[2]} \vee \ldots \vee M_R^{[n]}$

Once we get M_{R^*} it is very easy to write down R^*

Example: Matrices

- A={a,b,c,d}
- R={(a,b),(b,c),(c,d)}

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_R^{[3]} = M_R \odot M_R^{[2]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note the order

Example: Matrices (2)

$$M_R^{[4]} = M_R \odot M_R^{[3]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we get:

$M_{R^*} = M_R \vee M_{R^{[2]}} \vee M_{R^{[3]}} \vee M_{R^{[4]}}$

which gives

R^* ={(a,b),(b,c),(c,d),(a,c),(b,d),(a,d)}

Homework for 8.4

- 1
- 3
- 5
- 9(on 6)
- 19 (a,b)
- 25(a,b)

Chapter 8: Section 8.5

Equivalence Relations

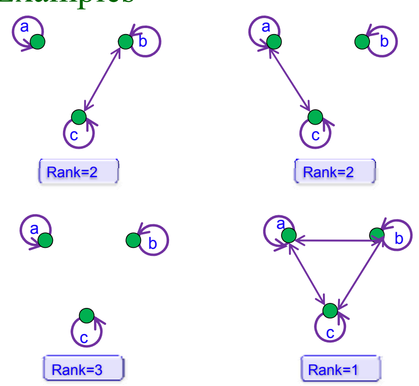
What is an Equivalence Relation?

Definition:
A relation on set A is called an ***equivalence relation*** if it is **reflexive, symmetric and transitive**.

It is easy to recognize equivalence relations using digraphs.

- The subset of all elements related to a particular element forms a universal relation (contains all possible arcs) on that subset. The (sub)digraph representing the subset is called a complete (sub)digraph. All arcs ***are*** present.
- The number of such subsets is called the *rank of the* equivalence relation

Examples



Congruence modulo m

Fix m, a positive integer.

Let a,b be integers.

We say that $a \underset{m}{\equiv} b$ ("a congruent to b modulo m")

if m|(b-a).

The condition "m divides b-a" means that you can find another integer k such that

(b-a)=mk

Examples

1 ≡ 7 modulo 6 because 7-1=6 is divisible by 6

42 ≡ 30 modulo 6 because 30-42=-6 is divisible by 6

1 is not congruent to 14 modulo 6 because 14-1 is not div by 6

Example

Let A= ℤ & R {(a,b) | a≡b modulo 6}

Notice that

- R is reflexive – because a≡a modulo 6 , for all a's
  - (this means that 6 divides a-a)
- R is symmetric because a≡b modulo 6 implies that b≡a modulo 6
  - (this means that if 6 divides (b-a), then 6 also divides (a-b))
- R is transitive, because if a≡b mod 6 and b≡c modulo 6 then a≡c modulo 6
  - Assumptions: b-a = 6k

    c-b=6t for some kt∈ℤ

    then we can write: c-a=(c-b)+(b-a)=6t+6k ➔ 6 divides (c-a)

Then R = "congruance modulo 6" is an equivalence relation

Equivalence Relations

Definition:
If R is an equivalence relation and (a,b)∈R, then we say that "a is equivalent to b"

- Notation: a~b (instead of aRb)

What is an Equivalence Class?

Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the ***equivalence class*** of a.

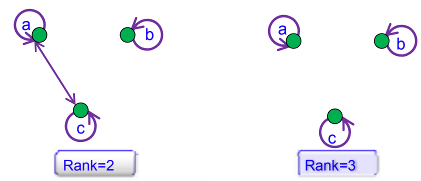
$[a]_R$ = {b∈A: b~a} = {b ∈ A: (a,b) ∈ R}

- Notation: $[a]_R$ or [a] for only 1 relation
- In other words

  Each of the subsets is called an ***equivalence class***
- A bracket around an **element** means the equivalence class in which the element lies.

  [x] = {y | (x, y) is in R}
- The element in the bracket is called a ***representative*** of the equivalence class. We could have chosen any one.

Example – Equivalence Class



[a]={a,c}, [c]={a.c}. [b]={b}

[a]={a}, [b]={b}. [c]={c}

Example

If a is a person, and

R is the relation "having the same age", then

[a] ={all people that are the same age as a}

The distinct equivalence classes are:

- {all people who are age 0}
- {all people who are age 1}
- {all people who are age 2}
- ... and so on

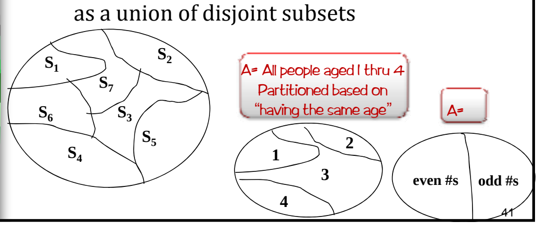
We notice that:

- Every person belongs to exactly 1 of these distinct equivalence classes
- Distinct equivalence classes are disjoint
  - They don't intersect
- The union of all distinct equivalence classes is the full set of people.

Equivalence Partitions

Let $S_1, S_2, \ldots, S_n$ be a collection of subsets of A. Then the collection forms a ***partition*** of A if the subsets are nonempty, disjoint and exhaust A.

- **In other words:**
  - The distinct equivalence classes partition A as a union of disjoint subsets



Formal Def of Partition

Theorem:
The equivalence classes of an equivalence relation R ***partition*** the set A into disjoint, nonempty subsets whose union is the entire set.

- **Notation A| R**
- **In other words,**
  - The distinct equivalence classes for a partition of A
- **Called:**
  - the quotient set, or
  - the partition of A induced by R, or
  - A modulo R

Examples

Let S be a set = {1,2,3,4,5}

Which are partitions of S?

- T_1={1,2}, T_2={3}, T_3={4,5} Yes
- T_1={1,2,3}, T_2={2,4}, T_3={5} No
- T_1={1}, T_2={2,3}, T_4={4} No

Homework for 8.5

- 1d
- 2d
- 3(a,b),
- 21
- 23
- 26 (on 1d),
- 35(a,c)
- 41(a-d)

Chapter 8: Section 8.6

Partial Orderings

What is a Partial Order?

Let R be a relation on A.
The R is a ***partial order*** iff R is:
reflexive, antisymmetric, & transitive

- (A,R) is called a partially ordered set or "poset"
- Notation:
  - If (A,R) is a poset and a and b are 2 elements of A such that (a,b)∈R, we write a ≼ b (instead of aRb)

NOTE: it is not ***required*** that two things be related under a partial order.

- That's the "partial" of it.

Some more defintions

- If (A,R) is a poset and a,b are ∈ A, we say that:
  - "a and b are comparable" if a≼b or b≼a
    - (i.e. if (a,b)∈R and (b,a)∈R)
  - "a and b are incomparable" if neither a≼b nor b≼a
    - (i.e if (a,b)∉R and (b,a)∉R
- If two objects are ***always*** related in a poset it is called a ***total order***, ***linear order*** or ***simple order***.
  - In this case (A,R) is called a *chain*.
  - (i.e if any two elements of A are comparable)
  - So for all a,b ∈ A, it is true that (a,b)∈R or (b,a)∈R

Example

- A=ℤ; R = the relation ≤
  - Then R is reflexive (a ≤a, ∀a∈ℤ)
  - R is antisymmetric (if a ≤b & b ≤a, then a=b)
  - R is transitive (if a ≤b and b ≤c, than a ≤c)
- So, (ℤ, ≤) is a poset.
  - In this case either a ≤b or b ≤a so two things are always related (maybe both if a=b)
  - So, any two a,b∈R are comparable
  - Hence, ≤ is a total order and , (ℤ,≤) is a chain

Note: (ℤ, ≥) is also a poset.. But > and < are not.. Why not?

Example 2

- If S is a set then $(P(S), \subseteq)$ is a poset.
  - It may not be the case that $A \subseteq B$ or $B \subseteq A$.
    - Eg. S={0, 1}, P(S)={∅, {0}, {1}, {0, 1} }
      - {0} and {1} are incomparable since $\{0\} \nsubseteq \{1\}$ and $\{1\} \nsubseteq \{0\}$
      - {0,1} and {1} are comparable since $\{1\} \subseteq \{0, 1\}$ so it is a poset, but
    - $\subseteq$ is not a total order.

Example 3

- A=ℤ

  R= divisibility relation: aRb iff a|b (a divides b)

  Then
  - R is not reflexive because 0 does not divide 0

  So, (ℤ, |) is not a poset
- **$A=\mathbb{Z}^+=\{a \in \mathbb{Z} : a > 0\}$, R=aRb iff a|b**

  Then
  - R is reflexive
  - R is antisymmetric because if a|b and b|a then a=b
  - R is transitive

  So $(\mathbb{Z}^+, |)$ is poset!