ICS 6B
Boolean Algebra & Logic

Lecture Notes for Summer Quarter, 2008

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Set 8 – Ch. 8.6, 11.6

Announcements

- Regrades for Quiz #3 and Homeworks #4 & 5 are due Today

Today's Lecture

- Chapter 8 (8.6), Chapter 11 (11.1)
  - Partial Orderings (8.6)
  - Boolean Functions (11.1)

Chapter 8: Section 8.6

Partial Orderings (Continued)

What is a Partial Order?

Let R be a relation on A.
The R is a ***partial order*** iff R is:
reflexive, antisymmetric, & transitive

- (A,R) is called a partially ordered set or "poset"
- Notation:
  - If (A,R) is a poset and a and b are 2 elements of A such that $(a,b) \in R$, we write $a \preccurlyeq b$ (instead of aRb)

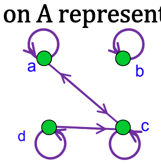
NOTE: it is not ***required*** that two things be related under a partial order.
  - That's the "partial" of it.

Some more defintions

- If (A,R) is a poset and a,b are $\in$ A, we say that:
  - "a and b are comparable" if $a \preccurlyeq b$ or $b \preccurlyeq a$
    - (i.e. if $(a,b) \in R$ and $(b,a) \in R$)
  - "a and b are incomparable" if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$
    - (i.e if $(a,b) \notin R$ and $(b,a) \notin R$
- If two objects are ***always*** related in a poset it is called a ***total order***, ***linear order*** or ***simple order***.
  - In this case (A,R) is called a *chain*.
  - (i.e if any two elements of A are comparable)
  - So for all a,b $\in$ A, it is true that $(a,b) \in R$ or $(b,a) \in R$

Now onto more examples…

Let A={a,b,c,d} and let R be the relation on A represented by the digraph



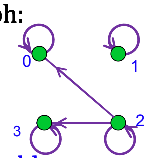
The R is reflexive, but not antisymmetric (a,c) &(c,a) and not transitive (d,c)(c,a), but not (d,a)

More Examples

Let A={0,1,2,3} and

Let R={(0,0)(1,1), (2,0),(2,2),(2,3)(3,3)}

We draw the associated digraph:



It is easy to check that R is Refl. , antisym. & trans.

So (A,R) is a poset.

The elements 1,3 are incomparable

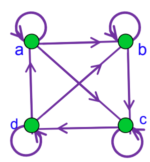
(because $(1, 3)\notin R$ and $(3, 1)\notin R$) so (A,R) is not a total ordered set.

More Examples (3)

Let A={a,b,c,d} and let R be the related Matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

To check properties more easily lets convert it to a diagraph



We see that R is reflexive (loops at every vertex) and antisymmetric (no double arrows)

It is not transitive (eg. (c,d) & (d,a), but no (c,a))

Another example

A={all people}

R=the relation “a not taller than b”

Then

R is reflexive (a is not taller than a)

R is not antisymmetric

- If “a is not taller than b” and “b is not taller than a”, then a and b have the same height, but a is not necessarily equal to b – it could be 2 people of the same height.

Not a poset!

Lexicographic Order

- Lexicographic order is how we order words in the dictionary
  - It is AKA dictionary order or alphabetic order
  - First, we compare the 1st letters, if they are equal then we check the 2nd pair and etc.
    - E.g Let S={1,2,3}
    - The lexicographical order of P(S) ={},{1},{1,2},{1,2,3},{1,3},{2},{2,3},{3}
- We can apply this to any poset
  - On a poset it is a natural order structure on the Cartesian product.

Lexicographic Order

Suppose that $(A_1,\preccurlyeq_1)$ and $(A_2,\preccurlyeq_2)$ are two posets.

We construct a partial ordering on the Cartesian product $A_1 x A_2$.

Given 2 elements (a_1, a_2) and (b_1, b_2) in $A_1 x A_2$

We say that $(a_1, a_2) \preccurlyeq (b_1, b_2)$

Iff $a_1 \preccurlyeq b_1$ (but $a_1 \neq b_1$)

or $a_1 = b_1$ and $a_2 \preccurlyeq b_2$.

In other words,

$(a_1, a_2) \preccurlyeq (b_1, b_2)$ iff

1. the first entries of (a_1, a_2) < the first entries of (b_1, b_2)

or

2. the first entries of (a_1, a_2) and (b_1, b_2) , and the 2nd entries of (a_1, a_2) < the 2nd entries of (b_1, b_2) entry

Examples

EX. $A_1 = A_2 = \mathbb{Z}$; $R_1 = R_2 = \leq$

Then $(1,3) \preccurlyeq (1,4)$; $(1,3) \preccurlyeq (2,0)$;

$(1,3) \preccurlyeq (1,3)$

In general, given (a_1, a_2) and (b_1, b_2),

compare a_1 and b_1

If $a_1 < b_1$ then

$(a_1, a_2) \preccurlyeq (b_1, b_2)$

ElseIf $a_1 = b_1$, then

If $b_1 < b_2$ then

$(a_1, a_2) \preccurlyeq (b_1, b_2)$

Example (2)

Which of the following are true?

$(3,5) \prec (4,8)$ True

$(4,4) \prec (2,8)$ False

$(3,8) \prec (4,5)$ True

$(1,2,4,10) \prec (1,2,5,8)$ True

$(2,4,5) \prec (2,3,6)$ False

$(1,5,8) \prec (2,3,4)$ True

Strings

- We apply this ordering to strings of symbols where there is an underlying 'alphabetical' or partial order (which is a total order in this case).

Example:

- Let $A = \{ a, b, c\}$ and suppose R is the natural alphabetical order on A:

  a R b and b R c.

  Then
  - Any shorter string is related to any longer string (comes before it in the ordering).
  - If two strings have the same length then use the induced partial order from the alphabetical order:

    aabc *R abac*

Well Ordered Set

- Let (A,R) be a poset

We say that A is a **well-ordered set** if any non-empty subset $B \subseteq A$ has a least element.

In other words,

$a \in B$ is a least element if $a \preccurlyeq b$ for all $b \in B$.

(recall that R is denoted by $\preccurlyeq$)

Example

$(\mathbb{Z}^+, \leq)$ is well ordered

- Any subset $B \subseteq \mathbb{Z}$ (which is not empty) has a least element
- Eg if B={2,3,4,5,27,248,1253} then the least element is a=2, because $a \leq b$ for all $b \in B$.

$(\mathbb{Z}, \leq)$ is not well ordered

- Choose $B = \mathbb{Z}$, there is no least element

Hasse Diagram

To every poset (A,R) we associate a Hasse diagram

- (A graph that carries less info than the diagraph)

To construct a Hasse diagram:

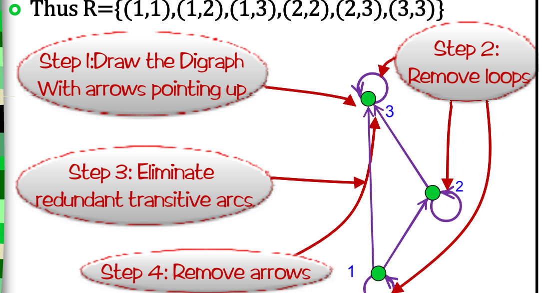
1) Construct a digraph representation of the poset (A, R) so that all arcs point up (except the loops).
2) Eliminate all loops
   - R is reflexive – SO we know they are there
3) Eliminate all arcs that are redundant because of transitivity
   - Keep (a,b) and (b,c) → remove (a,c)
4) Eliminate the arrows at the ends of arcs since everything points up. (and it is antisymmetric so arrows go only 1 way)

Example

Construct the Hasse diagram of

$A = \{1, 2, 3\}$, $R = \leq$

- Thus R={(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)}



Example (2)

Construct the Hasse diagram of (P({a, b, c}), ⊆)

- The elements of *P({a, b, c}) are*

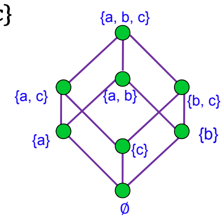
  ∅

  {a}, {b}, {c}

  {a, b}, {a, c}, {b, c}

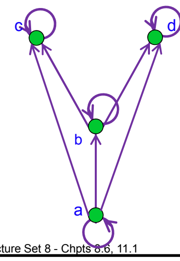
  {a, b, c}

- Basically, it shows a hierarchy



Example (3)

- Given the Hasse diagram write down all the ordered pairs in R



- Okay, so what do we know about this diagram
- We know it is antisymmetric
  - and that the arrows point up
- We know it is reflexive
- We also know it is transitive
- Now that we have the diagraph R is easy to find

R={(a,a),(a,b),(a,c),(a,d),(b,b),(b,c), (b,d),(c,c),(d,d)}

Maximal and Minimal Elements

Let (S, ≼) be a poset

An element a ∈ S is called maximal if there is no b such that a ≺ b.

In other words

a is not less than any element in the poset.

Similarly,

An element B ∈ S is called minimal if there is no b such that b ≺ a.

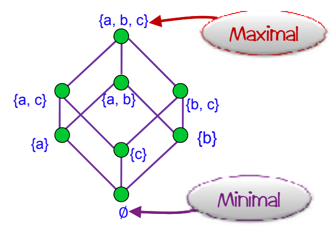
These are easy to find in a Hasse diagram
Because they are the "top" and "bottom" elements

Example

- In this diagram

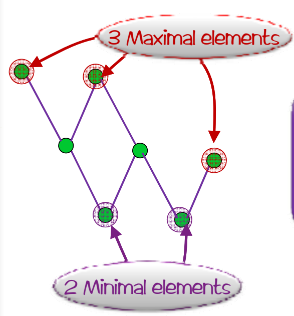
  What is the maximal element?

  What is the minimal element?



Note: there can be more than 1 minimal and maximal element in a poset

Example (2)



Note: Every poset has 1 or more maximal elements and 1 or more Minimal elements

Greatest and Least Elements

a∈S is called the greatest element of poset (S, ≼), if b ≼ a , ∀b ∈ S

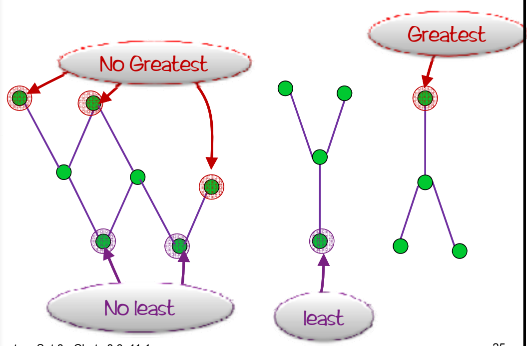
- If there is only 1 maximal element then it is the greatest.

a∈S is called the least element of poset (S, ≼), if a ≼ b , ∀b ∈ S

- If there is only 1 minimal element then it is the least

Note: The greatest and least may not exist

Examples



Upper and Lower Bounds

Let S be a subset of A in the poset (A, R).

Let A ⊆ S – be any subset

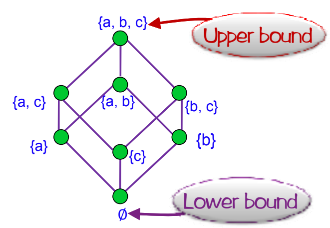
- An element b∈S is an upper bound for A
  if a ≼ b , ∀a ∈ S
  Note: B is not necessarily in A
- b is a lower bound for A
  if b ≼ a , ∀a ∈ S

Note: There can be more than 1 upper or lower bound

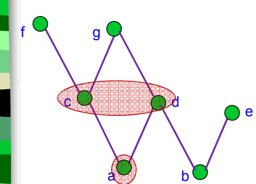
Example

- In this diagram
  What is the upper bound for all subsets?
  What is the lower bound for all subsets?

Note: Because we are dealing with subsets the bounds don't necessarily have to be in the subset



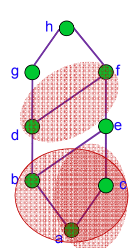
Example (2)



A= {c,d}
Upper bound:
g
Lower bound:
a
A= {a}
Upper bounds:
a,c,d,f,g
Lower bound:
a

Note: It must be reachable From all elements in the subset

Example (2)



A= {a,b,c}
Upper bound:
e, f, h
Lower bound:
a

A= {a, c, d, f}
Upper bounds:
h,f
Lower bound:
a

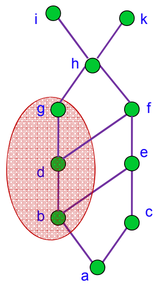
Least Upper Bound & Greatest Lower Bound

x∈S is called the least upper bound of A iff
1 – x is an upper bound for A
2 – x is less than any other upper bound of A

x∈S is called the greatest lower bound of A iff
1 – x is an lower bound for A
2 – x is greater than any other upper bound of A

Note: These bounds may not exist if they do they are unique

Example (2)



A= {b, d, g

Upper bound:

g, h, I, k

Lower bound:

b, a

Least upper bound:

g

Greatest lower bound:

b

Example (2)

$(S, \preccurlyeq) = (\mathbb{Z}^+, |)$ --- assume $\mathbb{Z}^+ > 0$

A={3,9,12}

Upper bounds:

common multiples of 3,9,12 ⇔

any positive integer which is divisible by 3,9,12 ⇔

any positive integer which is divisible by 36

Lower bounds: 3,1 (← common divisors of 3,9,12)

Least upper bound:

36 (= least common multiple of 3,9,12)

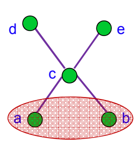
Greatest lower bound:

3 (= greatest common divisor of 3,9,12)

Lattices

A poset $(S, \preccurlyeq)$ is called a lattice if every pair of elements a, b ∈ S has both a greatest lower bound and a least upper bound

Example

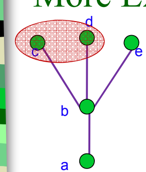


Is this a lattice?

what is the glb of {a,b}?

None – not a lattice

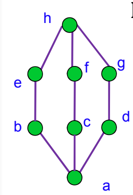
More Examples



Is this a lattice?

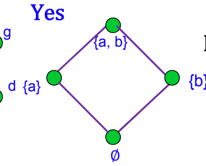
what is the lub of {b,c}?

None – not a lattice



Is this a lattice?

Yes



Is this a lattice?

Yes

Homework for 8.6

- 1(c-d)3(a-d),5,7,9,11,19,21,23(b-c), 25,33 (a-h),35(a-h),43(a-c)

Chapter 11: Section 11.1

Boolean Functions

Bit Operations

Boolean Sum - (denoted by + , OR, or ∨)

$1+0=1, \quad 0+1 = 1, \quad 1+1=1$

$0+0=0$ ← Only time it =s 0

Boolean Product- (denoted by · , AND, or ∧)

$1 \cdot 1=1$ ← Only time it =s 1

$1 \cdot 0=0, \quad 0 \cdot 1 = 0, \quad 0 \cdot 0=0$

Complement (notated by ‾, –NOT, or ¬)

$\overline{0} = 1$

$\overline{1} = 0$

Boolean Functions

Let B={0,1}. Then $B^n = \{x_1, x_2, \ldots, x_n\}$ |∈B for 1≤i≤n} is the set of all possible n-tuples of 0s and 1s. The variable x is called a ***Boolean variable*** if it assumes values only from B (
A function form B^n to B is called a ***Boolean Function*** of ***degree n***.

A Boolean variable x is a bit (binary digit)

– accepts values 0 and 1. So x ∈ {0,1}.

A Boolean function of ***degree 1*** is a function of 1 Boolean variable with the values in {0,1}

F: {0,1}⟶{0,1}, x⟼ F(x)

Examples

For Example:

$F_1(x) = \neg x$

x	$F_1(x) = \neg x$
1	0
0	1

$F_2(x) = \neg(\neg x)$

x	¬x	$F_2(x) = \neg(\neg x)$
1	0	1
0	1	0

$G(x) = x$

x	G(x)=x
1	1
0	0

Notice these are =

If G(x) takes exactly the same value as F_2. We say F_2 and G are the same function

Even if they are defined by different expressions!

Examples (2)

A Boolean function of degree 2 is a function of two variables with values in {0,1}

Example:

$F_1(x,y) = x+y$

x	y	$F_1(x,y)= x+y$
0	0	0
0	1	1
1	0	1
1	1	1

$F_1(x,y) = xy$

x	y	$F_1(x,y)= xy$
0	0	0
0	1	0
1	0	0
1	1	1

Order of Operations

- In order to evaluate these we need to understand the order of preference
  - Things in Parenthesis come first
  - Then
    - 1 – Complement
    - 2 – Product
    - 3 – Sum