CompSci 162  Unit 4 Diagnostic Exam 3  Spring 2023

DO NOT OPEN EXAM UNTIL INSTRUCTED TO DO SO

SILENCE AND STORE ALL ELECTRONICS

Format Style: Answer underneath the question

This is a diagnostic exam intended to help you evaluate your readiness for the real exam.

- On the real test, remember to look for your name ☺
- If you believe a question is ambiguous, write at least two reasonable interpretations and indicate clearly which one you will be using. Then answer your question with that assumption. Unless your interpretation makes the problem much more trivial than intended, we will grade your response as if one of us had made that clarification.
- We will only grade responses underneath each question, for that question.
- The following problems are \( \mathcal{NP} \)-complete:
  - SAT: Given a boolean formula \( \phi \), is it satisfiable?
    * Also 3-Sat, where the formula is a conjunction of disjunctions of size three.
  - Independent Set: Given a graph \( G \) and an integer \( k \), determine if there is some subset of the vertices \( V' \), \( |V'| \geq k \) such that no two vertices in \( V' \) share an edge.
  - Vertex Cover: Given a graph \( G \) and integer \( k \), determine if there is some subset of the vertices \( V' \), \( |V'| \leq k \) such that each edge is incident to some vertex in \( V' \).
  - Directed Hamiltonian Path: does this graph \( G \) contain a simple path that includes every vertex?
    * Also undirected Hamiltonian Path
    * Also Hamiltonian Cycle (directed and undirected)
  - 3-Color: Given an undirected graph \( G \), determine whether there is a mapping of vertices in \( G \) to 3 distinct colors such that no edge is monochromatic.
  - Subset Sum: Given a set \( S \) of numbers, as well as a number \( T \), determine if there is a subset of \( S \) that sums to exactly \( T \).
- It has been great having you this quarter. ☺
Scratch paper. Nothing you write on this side of the page will be graded.
1. (3 points) The **Minimum Cut** problem is as follows. We are given a graph $G$ and two distinct vertices $s,t$. Each edge has a *cost* on it. Our goal is to select a subset of the edges such that $s$ and $t$ no longer have a path between them in the graph. The goal is to select the subset of the edges with minimum total cost. There is an algorithm that successfully optimizes this with running time $O(n^3)$, where $n$ is the number of vertices in the graph.

   (a) Express that optimization problem as a decision problem.

   (b) Is the decision version of the problem in $\mathcal{P}$? Why or why not?

   (c) Is the optimization version of the problem in $\mathcal{NP}$? Why or why not?

2. (2 points) Consider the problem of **PRIMES**: $L_1 = \{\langle x \rangle : x$ is the binary representation of a prime number $\}$. Recall also that a prime number is a positive integer $p > 1$ such that the only positive integers that divide $p$ without leaving a remainder are 1 and $p$.

   The following describes a Turing Machine that decides PRIMES:

   (a) If the exact string $x$ is 10 (i.e., $x = 2$), accept.
   (b) If the last bit of $x$ is 0 (i.e., $x$ is even), reject.
   (c) Otherwise, check each odd $i$ from 3 to $\lceil \sqrt{x} \rceil$ in order.
       i. If $i$ divides $x$ without leaving a remainder, reject.
   (d) If we finish the previous step without a rejection, accept.

   If the operation in step 2(c)i takes time linear in the number of bits for the input, and computing $\lceil \sqrt{x} \rceil$ can be done in $O(1)$, the time complexity of this is $O(\sqrt{x} \log x)$, where $x$ is the *value* of the input, represented in binary on the input tape.

   Is this a polynomial time complexity, as per the definition from lecture and the reading?
Scratch paper. Nothing you write on this side of the page will be graded.
3. (3 points) Recall the Hamiltonian Path problem from lecture. There is an algorithm to solve this in linear time if the input graph is both directed and acyclic (has no cycles).

Does the existence of that algorithm prove $P = NP$? Why or why not? You DO NOT need to, nor should you, provide the algorithm described.

4. (2 points) In a graph, a Tonian Path is a path that includes exactly half the vertices without repeating one. Show that the problem of determining if a given graph $G$ has a Tonian Path is $NP$-complete.