The Selection Problem

- Given a list $S$ and numeric $k$
- Want: if we sorted $S$, what is $S_k$?
- Brute force:
  - Sort $S$ in $\Theta(n \log n)$
  - Return $S_k$
- Can we do better?
Randomized Selection

quickSelect(S, k)

If $n$ is small, brute force and return.
Pick a random $x \in S$ and put rest into:
- $L$, elements smaller than $x$
- $G$, elements greater than $x$

if $k \leq |L|$ then

return quickSelect($L$, k)

else if $k == |L| + 1$ then

return $x$

else

return quickSelect($G$, $k - (|L| + 1)$)

pivot

linear

array:

$L | x | G$
Randomized Selection

► What is the worst-case running time?
\[ \Theta(n^2) \]

► What would cause that bad time?
bad pivot (extreme)

► Estimate the expected running time?

*Hint: on average, the pivot is the median.*

\[ \Rightarrow T(n) = 1T(\frac{n}{2}) + \Theta(n) \]

is \( \Theta(n) \)
Deterministic Selection

Instead of picking $x$ at random:

- Divide $S$ into $g = \lceil n/5 \rceil$ groups
- Each group has 5 elements (except maybe $g^{th}$)
- Find median of each group of 5
- Find median of those medians
- Let $x$ be that median.

We call this the “medians of 5” method.
## Selecting Median of 5 Example

<table>
<thead>
<tr>
<th></th>
<th>870</th>
<th>647</th>
<th>845</th>
<th>742</th>
<th>372</th>
<th>882</th>
<th>691</th>
<th>341</th>
<th>461</th>
<th>596</th>
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<td>729</td>
<td>101</td>
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<td>397</td>
<td>825</td>
<td>587</td>
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<td>839</td>
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742  596  151  397
Deterministic Select

\text{DeterministicSelect}(S, k)

If \( n \) is small, brute force and return.
Pick \( x \in S \) via medians-of-5 and put rest into:
\( L \), elements smaller than \( x \)
\( G \), elements greater than \( x \)
\textbf{if} \( k \leq |L| \) \textbf{then}
\hspace{1em} \text{DeterministicSelect}(L, k)
\textbf{else if} \( k == |L| + 1 \) \textbf{then}
\hspace{1em} \text{return } x
\textbf{else}
\hspace{1em} \text{DeterministicSelect}(G, k - (|L| + 1))
Deterministic Selection

Let’s visualize: how does pivot compare to list?

\[ \frac{n}{10} \text{ : Smaller than } x \text{ AND in medians list} \]

\[ \frac{2n}{10} \text{ : Smaller than } x \text{ AND in one of those lists.} \]

also \[ \frac{3n}{10} \text{ larger than } x \text{ (same argument)} \]

and \[ \frac{4n}{10} \text{ I don’t know.} \]
Each column was a group of five.
Each column is sorted
Columns are ordered based on median-of-5
Which cells are in $L$? $G$? Either?

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Deterministic Selection

- How few elements must be smaller than pivot? \( \frac{3n}{10} \)
- How few must be non-smaller than pivot? \( \frac{3n}{10} \)
- How many could be in either group? \( \frac{4n}{10} \)

\[ T(n) \leq T\left(\frac{n}{5}\right) + T\left(\frac{3n}{10}\right) + n \text{ is } \mathcal{O}(n) \]
Integer Multiplication

- Given two $n$-bit integers $X$, $Y$, compute $X \cdot Y$
- Example: What is $13 \cdot 11$?

\[
\begin{array}{c}
13 \\
\times 11 \\
\hline
13 \\
13 \times 13 \\
\hline
143
\end{array}
\]
What is a Computer Anyway?

\[(12 - 1)(12 + 1) = 12^2 - 1^2 = 144 - 1 = 143\]

Example: What is $13 \cdot 11$?

\[
\begin{array}{c}
11 \\
5 \\
2 \\
1 \\
\hline
104 \\
\hline
143
\end{array}
\]

\[
\begin{array}{c}
13 \\
26 \\
52 \\
\hline
143
\end{array}
\]