CompSci 260P
Spring 2023 Lecture 4:
Introducing Complexity
About $NP$-complete Problems

- In general, $NP$-complete means:
  - Can certify a “yes” in polynomial space
  - Can verify that certificate in polynomial time
  - Can use it to solve another problem known to be $NP$-complete

- Informally, $NP$-complete means:
  - We do not know an efficient solution
  - We do not know that one does not exist

- 1972, Cook-Levin: (effectively) 3-SAT is $NP$-complete
3-Sat

\[ \phi = (x_2 \lor x_3 \lor x_4)(\overline{x_2} \lor x_3 \lor \overline{x_4})(\overline{x_1} \lor x_3 \lor x_5) \\
(\overline{x_1} \lor x_2 \lor x_5)(\overline{x_3} \lor x_4 \lor x_5)(\overline{x_2} \lor x_4 \lor \overline{x_5}) \\
(x_1 \lor \overline{x_2} \lor x_5)(x_3 \lor \overline{x_4} \lor x_5) \]

<table>
<thead>
<tr>
<th>Variable</th>
<th>Truth Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x_1)</td>
<td>True or False</td>
</tr>
<tr>
<td>(x_2)</td>
<td>True or False</td>
</tr>
<tr>
<td>(x_3)</td>
<td>True or False</td>
</tr>
<tr>
<td>(x_4)</td>
<td>True or False</td>
</tr>
<tr>
<td>(x_5)</td>
<td>True or False</td>
</tr>
</tbody>
</table>

\(n\) Variables
\(k\) clauses
for Ind Set
and 3-color:

\(|V| = ?\)
\(|E| = ?\)
Independent Set

Find an independent set of size 4 in this graph:
Verifier for Independent Set

Certificate: $V'$, a set of vertices.

Verifier:

if $V' \not\subseteq V$ then
  return false

if $|V'| \neq k$ then
  return false

for all edges $e = (u, v) \in E$ do
  if $u \in V'$ and $v \in V'$ then
    return false

return true
Can Independent Set solve 3-Sat?

Given a solution to **Independent Set**, use it to write an algorithm that solves 3-Sat.

**The Plan:** Build a graph such that:

- The **Independent Set** solver will select one variable from each clause, when given the graph and an appropriate value of $k$ as input.
- Before you use the solver, modify the graph so that $x_i$ and $\overline{x_i}$ won’t both be selected, for any $i$.
- It is okay to select $x_i$ two or more times, or $\overline{x_i}$ two or more times, as long as you don’t select $x_i$ and $\overline{x_i}$.
Reduction

\[ \phi = (x_2 \lor x_3 \lor x_4)(\overline{x_2} \lor x_3 \lor \overline{x_4})(\overline{x_1} \lor x_3 \lor x_5)(\overline{x_1} \lor x_2 \lor x_5)(\overline{x_3} \lor x_4 \lor x_5)(\overline{x_2} \lor \overline{x_4} \lor \overline{x_5})(x_1 \lor \overline{x_2} \lor x_5)(x_3 \lor \overline{x_4} \lor x_5) \]
3-SAT($n$ variables, $k$ clauses)

for each clause $A \lor B \lor C$ do
    Create 3 vertices // one each $A$, $B$, $C$
    Add edges $(A, B)$, $(B, C)$, $(A, C)$
for each variable $x_i$ do
    Connect any “$x_i$” node to all $\overline{x_i}$ nodes
return INDEPENDENT SET($G, k$)
Could we get false positives?

Show that, if that graph $G$ has an independent set of size $k$, then the $3$-$\text{SAT}$ instance truly has a satisfying assignment.
Could we get false negatives?

Show that, if the 3-$SAT$ instance we have as input has a satisfying assignment, the graph we build will have an independent set of size $k$. 
Use 3-Color to get a truth value assignment on $n$ variables

- Remember, all $2^n$ TVAs should be possible.
- Running time polynomial plus call to 3-Color
Amend the 3-Color usage to get a satisfying truth value assignment on \( n \) variables. For each clause \( A \lor B \lor C \), add as follows:

- **Example:** if \( (\overline{x_2} \lor x_3 \lor \overline{x_4}) \), then \( A = \overline{x_2} \) etc.

1. If \( B, C \) both false, this is false.
2. If \( A = \text{false} \), this is neutral.
3. Add this and if \( A \) \( BC \) all false, no coloring.
4. This is false.