Constraint Satisfaction Problems (CSPs)

This lecture topic (two lectures)
Chapter 6.1 – 6.4, except 6.3.3

Next lecture topic (two lectures)
Chapter 7.1 – 7.5

(Please read lecture topic material before and after each lecture on that topic)

Outline

- What is a CSP
- Backtracking for CSP
- Local search for CSPs
- (Removed) ~~Problem structure and decomposition~~

You Will Be Expected to Know

- Basic definitions (section 6.1)
- Node consistency, arc consistency, path consistency (6.2)
- Backtracking search (6.3)
- Variable and value ordering: minimum-remaining values, degree heuristic, least-constraining-value (6.3.1)
- Forward checking (6.3.2)
- Local search for CSPs: min-conflict heuristic (6.4)

Constraint Satisfaction Problems

- What is a CSP?
  - Finite set of variables $X_1, X_2, \ldots, X_n$
  - Nonempty domain of possible values for each variable $D_1, D_2, \ldots, D_n$
  - Finite set of constraints $C_1, C_2, \ldots, C_m$
    - Each constraint C_i limits the values that variables can take,
    - e.g., $X_1 \neq X_2$
  - Each constraint C_i is a pair <scope, relation>
    - Scope = Tuple of variables that participate in the constraint.
    - Relation = List of allowed combinations of variable values.
      May be an explicit list of allowed combinations.
      May be an abstract relation allowing membership testing and listing.

- CSP benefits
  - Standard representation pattern
  - Generic goal and successor functions
  - Generic heuristics (no domain specific expertise).

Sudoku as a Constraint Satisfaction Problem (CSP)

	1	2	3	4	5	6	7	8	9
A		6		1		4		5	
B			8	3		5	6		
C	2								1
D	8			4		7			6
E			6				3		
F	7			9		1			4
G	5								2
H			7	2		6	9		
I		4		5		8		7	

- Variables: 81 variables
  - A1, A2, A3, ..., I7, I8, I9
  - Letters index rows, top to bottom
  - Digits index columns, left to right
- Domains: The nine positive digits
  - A1 $\in$ {1, 2, 3, 4, 5, 6, 7, 8, 9}
  - Etc.
- Constraints: 27 *Alldiff* constraints
  - *Alldiff*(A1, A2, A3, A4, A5, A6, A7, A8, A9)
  - Etc.

CSPs --- what is a solution?

- A *state* is an *assignment* of values to some or all variables.
  - An assignment is *complete* when every variable has a value.
  - An assignment is *partial* when some variables have no values.

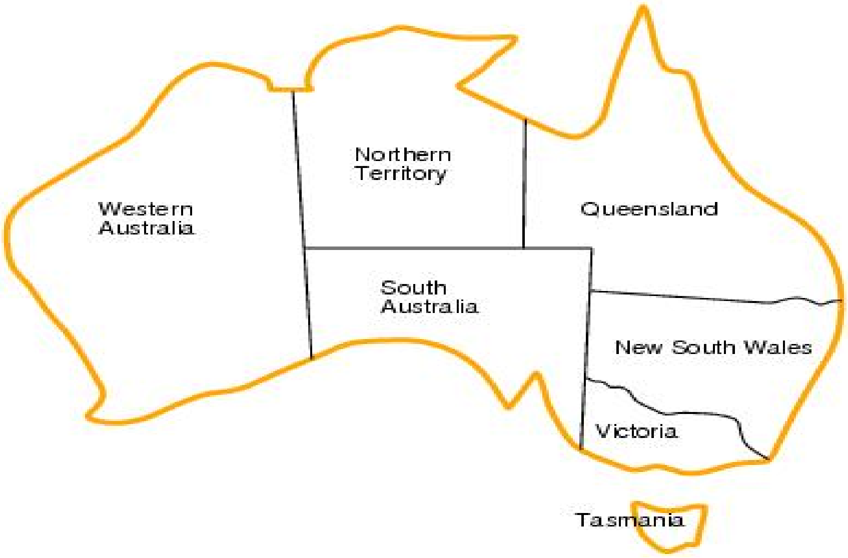
- ***Consistent assignment***
  - assignment does not violate the constraints

- A ***solution*** to a CSP is a complete and consistent assignment.

- Some CSPs require a solution that maximizes an *objective function*.

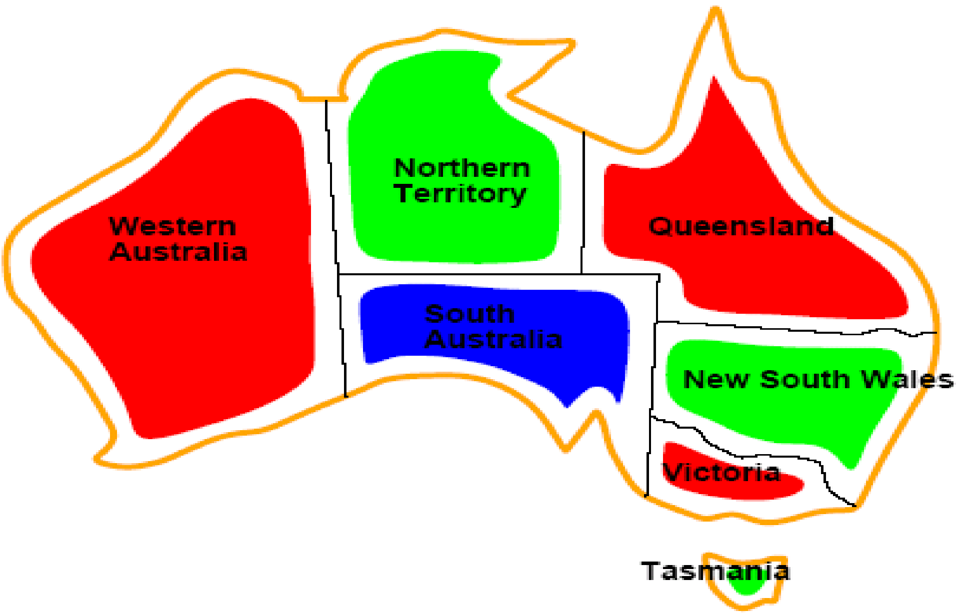
- Examples of Applications:
  - Scheduling the time of observations on the Hubble Space Telescope
  - Airline schedules
  - Cryptography
  - Computer vision -> image interpretation
  - Scheduling your MS or PhD thesis exam ☺

CSP example: map coloring



- Variables: *WA, NT, Q, NSW, V, SA, T*
- Domains: $D_i = \{red, green, blue\}$
- Constraints:adjacent regions must have different colors.
  - E.g. $WA \neq NT$

CSP example: map coloring



- Solutions are assignments satisfying all constraints, e.g.
  {WA=red,NT=green,Q=red,NSW=green,V=red,SA=blue,T=green}

Graph coloring

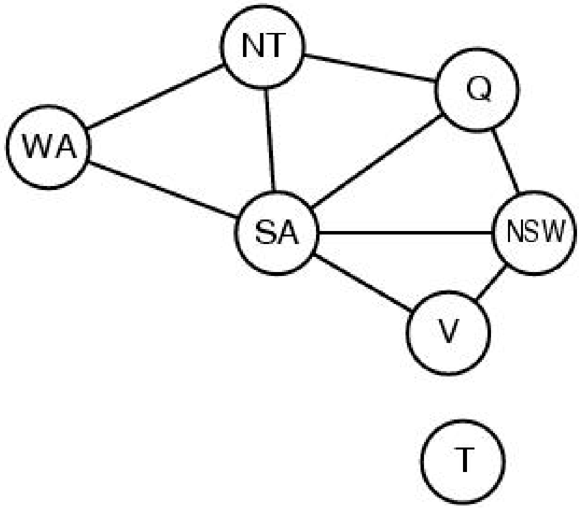
- More general problem than map coloring
- Planar graph = graph in the 2d-plane with no edge crossings
- Guthrie's conjecture (1852)

  Every planar graph can be colored with 4 colors or less

  - Proved (using a computer) in 1977 (Appel and Haken)

Constraint graphs

- Constraint graph:
  - nodes are variables
  - arcs are binary constraints



- Graph can be used to simplify search
  e.g. Tasmania is an independent subproblem

(will return to graph structure later)

Varieties of CSPs

- Discrete variables
  - Finite domains; size $d \Rightarrow O(d^n)$ complete assignments.
    - E.g. Boolean CSPs: Boolean satisfiability (NP-complete).
  - Infinite domains (integers, strings, etc.)
    - E.g. job scheduling, variables are start/end days for each job
    - Need a constraint language e.g $StartJob_1 + 5 \leq StartJob_3$.
    - Infinitely many solutions
    - Linear constraints: solvable
    - Nonlinear: no general algorithm
- Continuous variables
  - e.g. building an airline schedule or class schedule.
  - Linear constraints solvable in polynomial time by LP methods.

Varieties of constraints

- Unary constraints involve a single variable.
  - e.g. *SA* $\neq$ *green*

- Binary constraints involve pairs of variables.
  - e.g. *SA* $\neq$ *WA*

- Higher-order constraints involve 3 or more variables.
  - Professors A, B,and C cannot be on a committee together
  - Can always be represented by multiple binary constraints

- Preference (soft constraints)
  - e.g. *red* is better than *green* often can be represented by a cost for each variable assignment
  - combination of optimization with CSPs

CSPs Only Need Binary Constraints!!

- Unary constraints: Just delete values from variable's domain.
- Higher order (3 variables or more): reduce to binary constraints.
- Simple example:
  - Three example variables, X, Y, Z.
  - Domains Dx={1,2,3}, Dy={1,2,3}, Dz={1,2,3}.
  - Constraint C[X,Y,Z] = {***X+Y=Z***} = {(1,1,2), (1,2,3), (2,1,3)}.
  - Plus many other variables and constraints elsewhere in the CSP.

  - Create a new variable, W, taking values as triples (3-tuples).
  - Domain of W is Dw = {(1,1,2), (1,2,3), (2,1,3)}.
    - Dw is exactly the tuples that satisfy the higher order constraint.
  - Create three new constraints:
    - C[X,W] = { [1, (1,1,2)], [1, (1,2,3)], [2, (2,1,3)] }.
    - C[Y,W] = { [1, (1,1,2)], [2, (1,2,3)], [1, (2,1,3)] }.
    - C[Z,W] = { [2, (1,1,2)], [3, (1,2,3)], [3, (2,1,3)] }.
  - Other constraints elsewhere involving X, Y, or Z are unaffected.

CSP Example: Cryptharithmetic puzzle

```
    T W O
  + T W O
  -------
  F O U R
```

Variables: $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

Domains: $\{0,1,2,3,4,5,6,7,8,9\}$

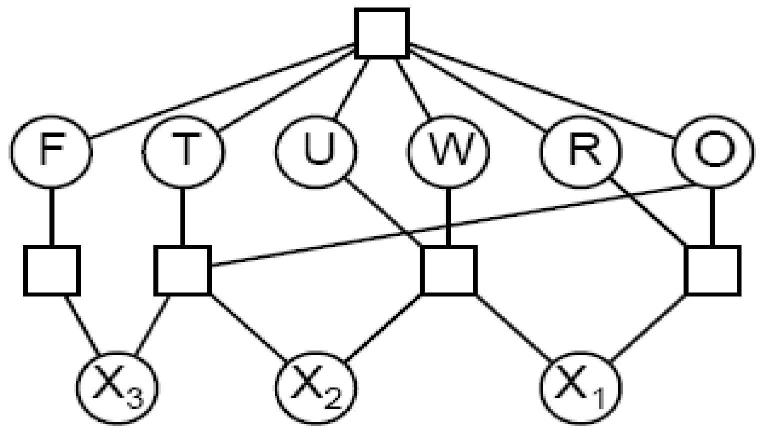
Constraints

alldiff(F,T,U,W,R,O)

$O + O = R + 10 \cdot X_1$, etc.

CSP Example: Cryptharithmetic puzzle

$$\begin{array}{rcccc} & T & W & O \\ + & T & W & O \\ \hline F & O & U & R \end{array}$$



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Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints

alldiff(F, T, U, W, R, O)

$O + O = R + 10 \cdot X_1$, etc.

CSP as a standard search problem

- A CSP can easily be expressed as a standard search problem.

- Incremental formulation

  - *Initial State*: the empty assignment {}

  - *Actions (3rd ed.), Successor function (2nd ed.)*: Assign a value to an unassigned variable provided that it does not violate a constraint

  - *Goal test*: the current assignment is complete
    (by construction it is consistent)

  - *Path cost*: constant cost for every step (not really relevant)

- Can also use complete-state formulation
  - Local search techniques (Chapter 4) tend to work well

CSP as a standard search problem

- Solution is found at depth *n* (if there are *n* variables).

- Consider using BFS
  - Branching factor *b* at the top level is *nd*
  - At next level is (n-1)d
  - ....

- end up with $n!d^n$ leaves even though there are only d^n complete assignments!

Commutativity

- CSPs are commutative.
  - The order of any given set of actions has no effect on the outcome.
  - Example: choose colors for Australian territories one at a time
    - [WA=red then NT=green] same as [NT=green then WA=red]

- All CSP search algorithms can generate successors by considering assignments for only a single variable at each node in the search tree

  $\Rightarrow$ there are d^n leaves

(will need to figure out later which variable to assign a value to at each node)

Backtracking search

- Similar to Depth-first search, generating children one at a time.
- Chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.
- Uninformed algorithm
  - No good general performance

Backtracking search

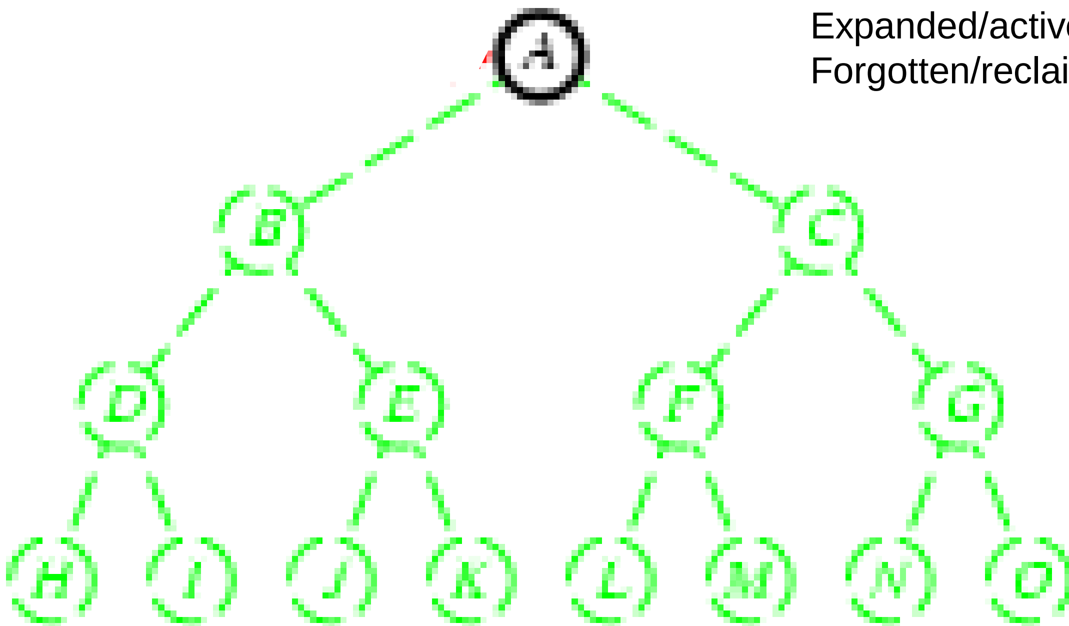
```
function BACKTRACKING-SEARCH(csp) return a solution or failure
    return RECURSIVE-BACKTRACKING({} , csp)

function RECURSIVE-BACKTRACKING(assignment, csp) return a solution or failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp],assignment,csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment according to CONSTRAINTS[csp]
                then
                add {var=value} to assignment
                result ← RECURSIVE-BACTRACKING(assignment, csp)
                if result ≠ failure  then return result
                remove {var=value} from assignment
    return failure
```

Backtracking search

- Expand *deepest* unexpanded node
- Generate ***only one*** child at a time.
- *Goal-Test* when inserted.
  - For CSP, Goal-test at bottom

Future= green dotted circles
Frontier=white nodes
Expanded/active=gray nodes
Forgotten/reclaimed= black nodes



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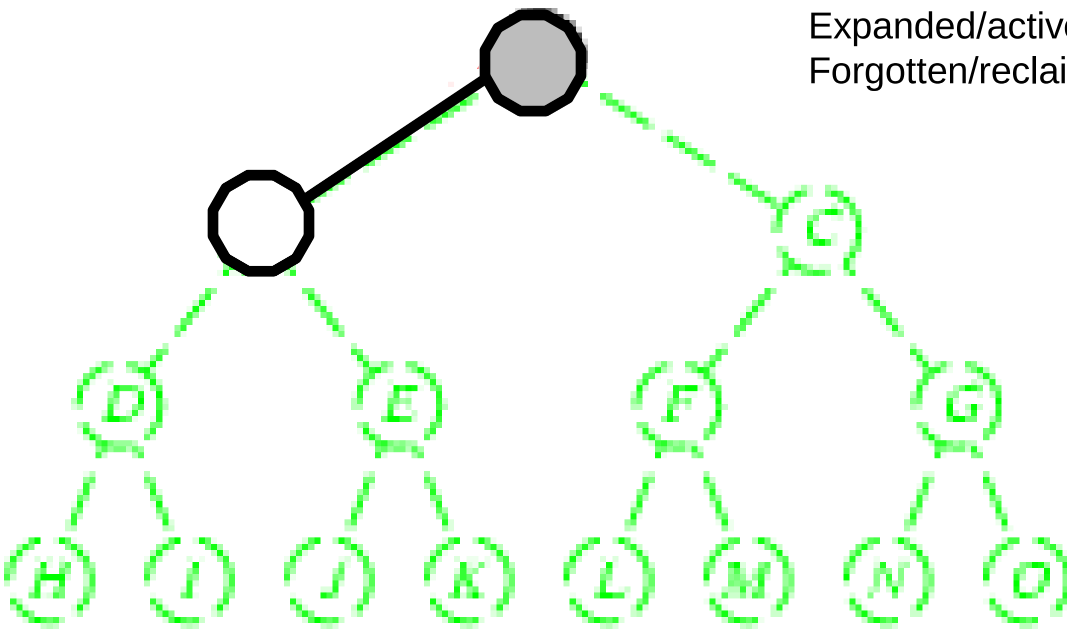

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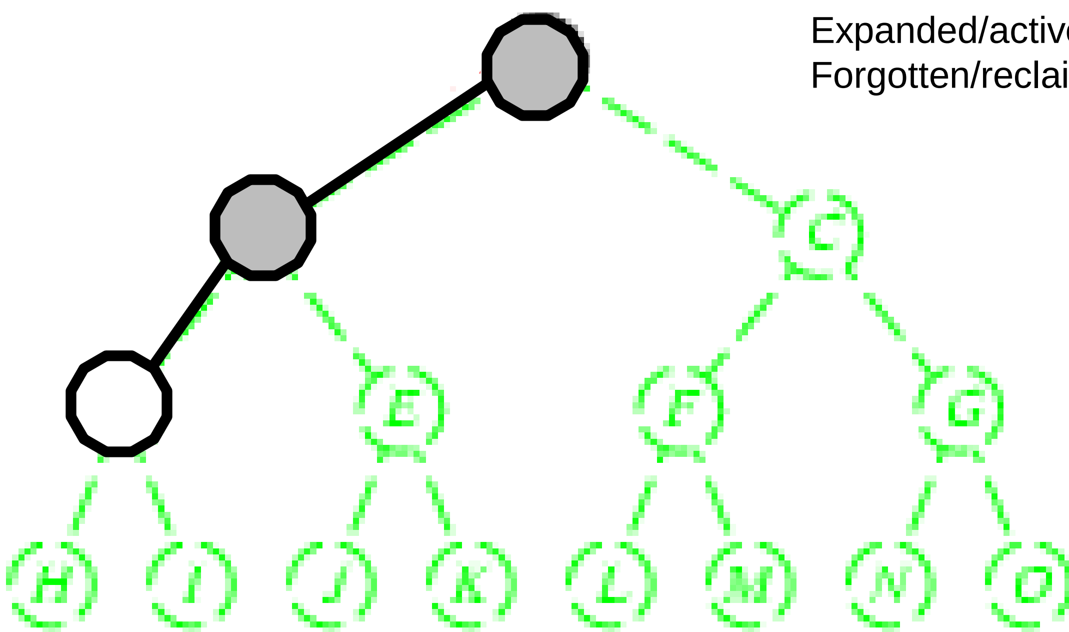

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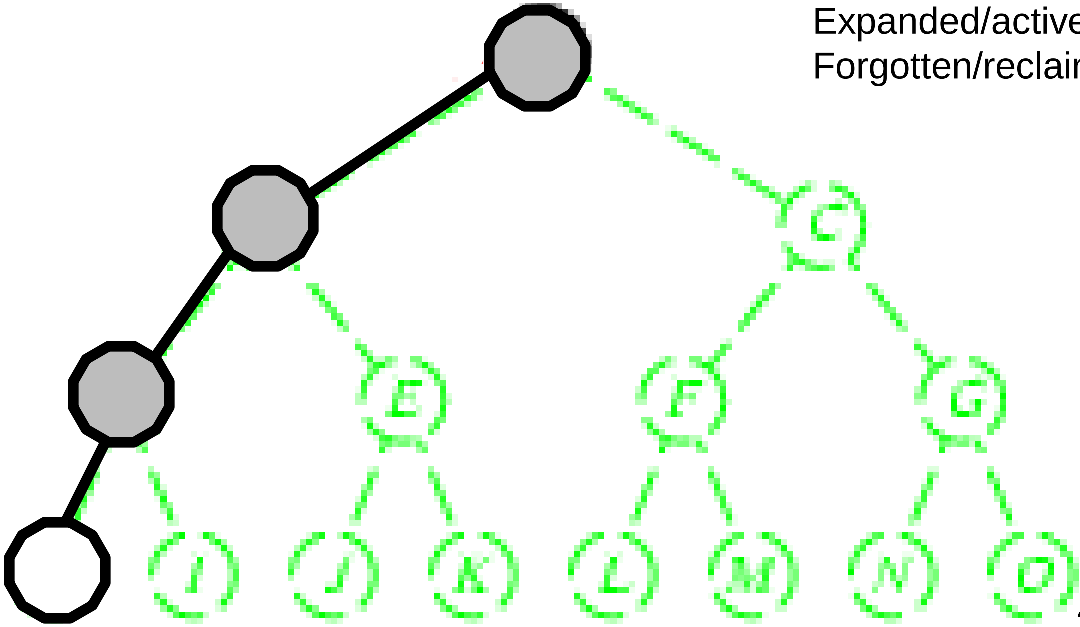

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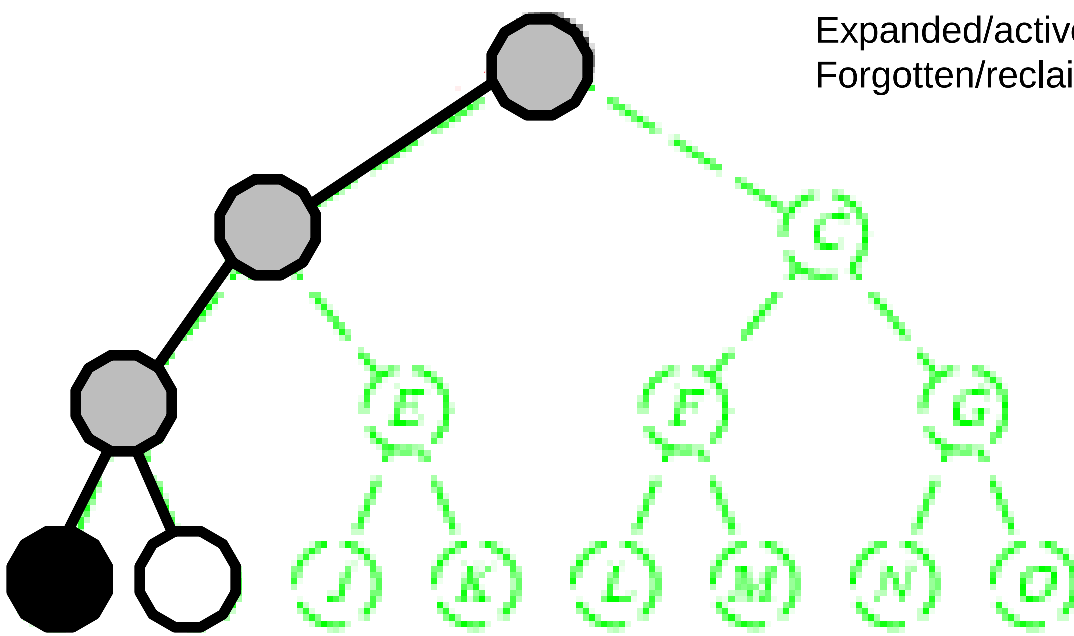

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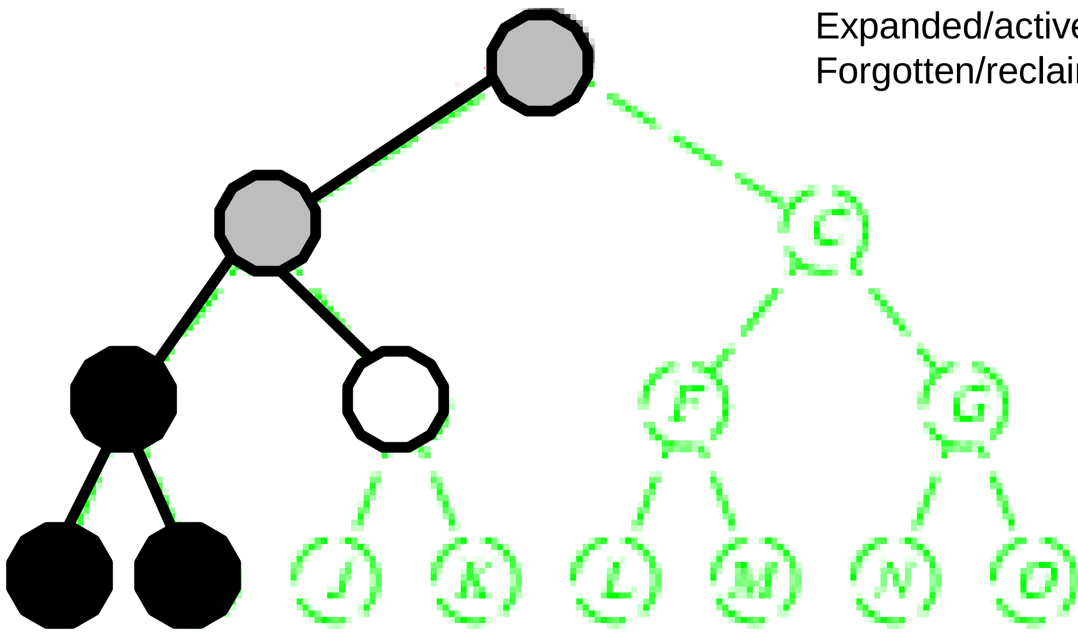

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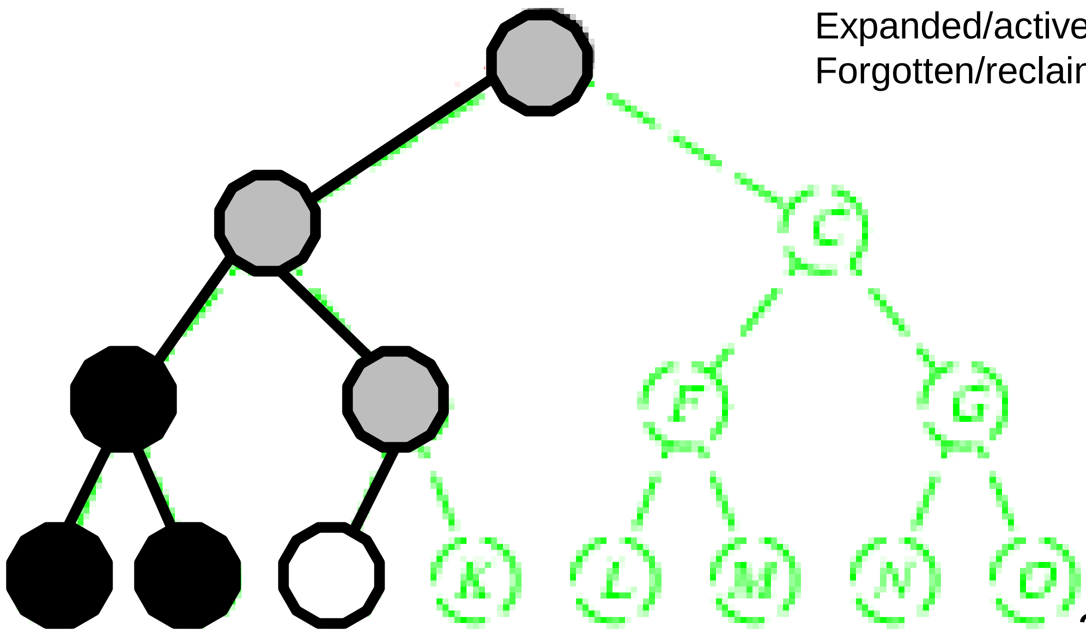

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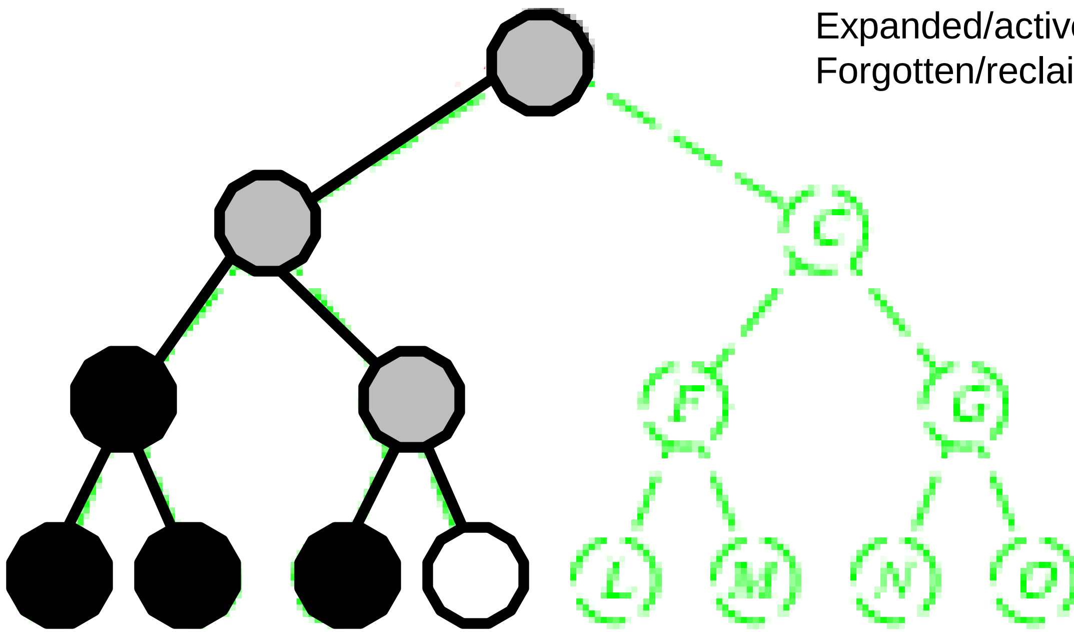

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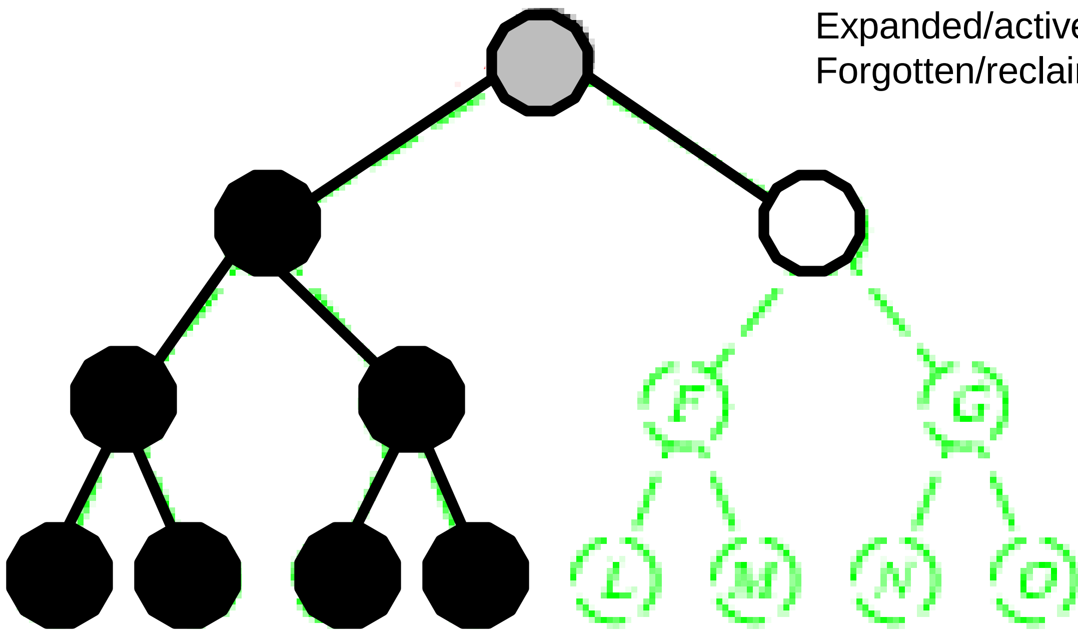

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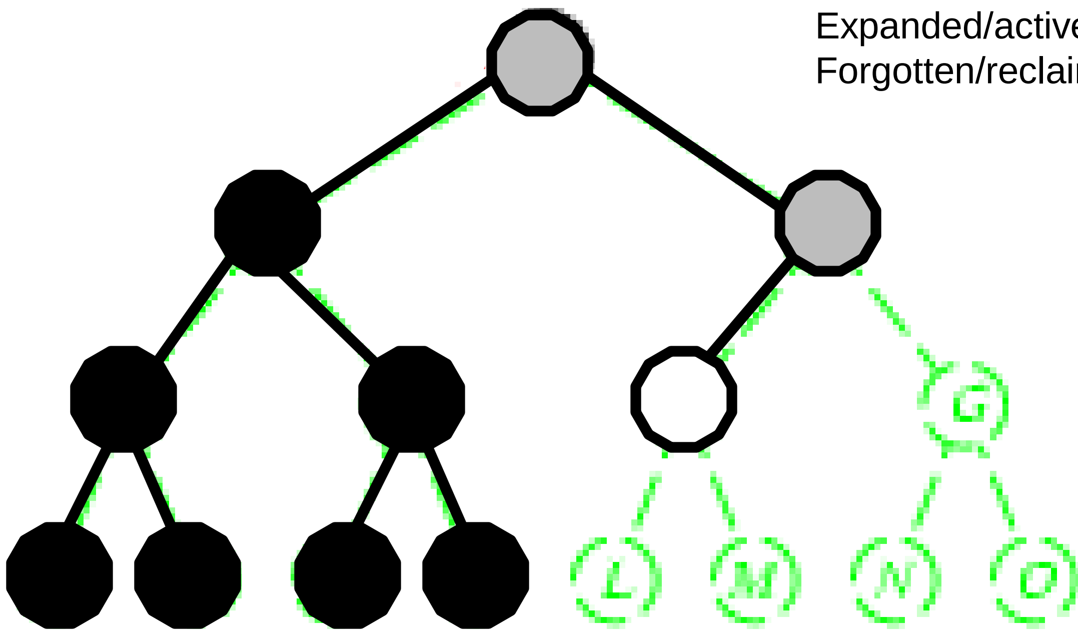

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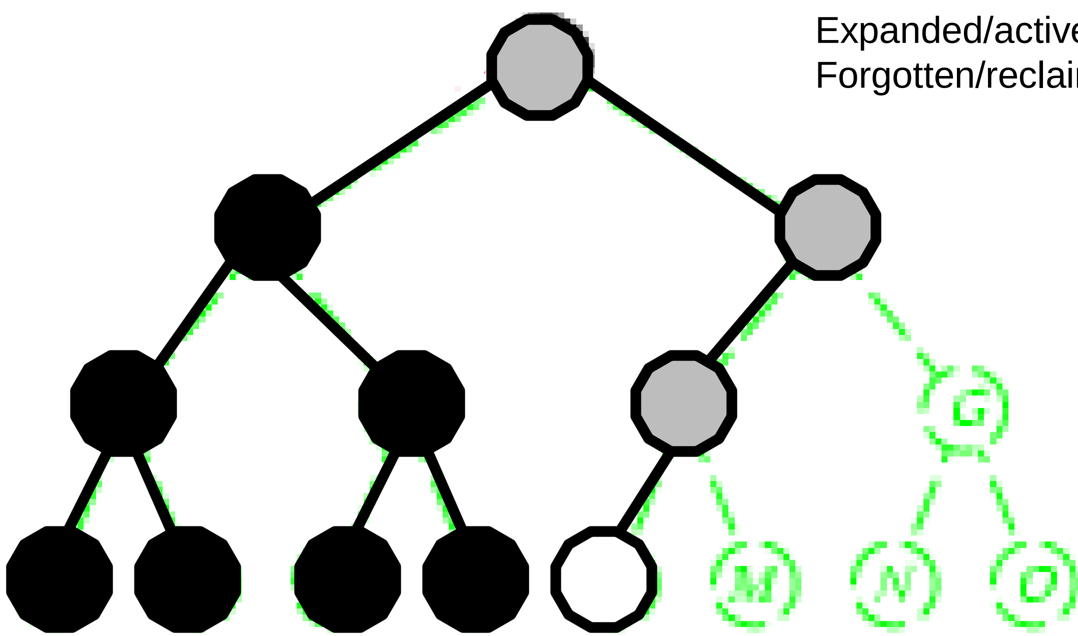

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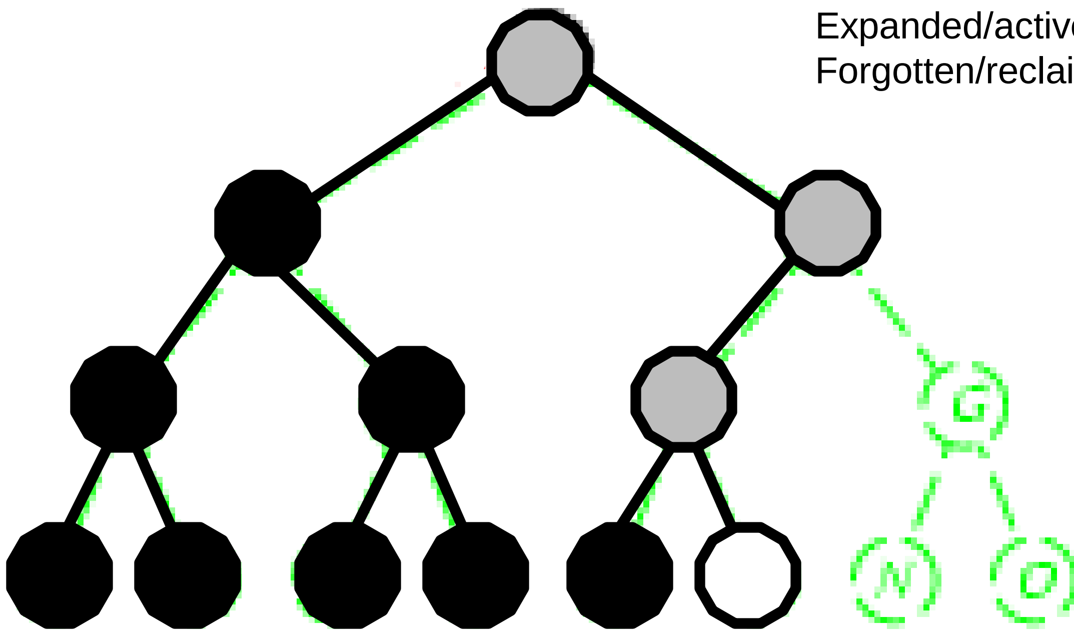


Backtracking search (Figure 6.5)

function BACKTRACKING-SEARCH(*csp*) **return** a solution or failure
    return RECURSIVE-BACKTRACKING(*{}* , *csp*)

function RECURSIVE-BACKTRACKING(*assignment, csp*) **return** a solution or failure
    if *assignment* is complete **then return** *assignment*
    var ← **SELECT-UNASSIGNED-VARIABLE(VARIABLES**[*csp*],*assignment*,*csp*)
    for each *value* **in** **ORDER-DOMAIN-VALUES**(*var, assignment, csp*) **do**
        if *value* is consistent with *assignment* according to CONSTRAINTS[*csp*]
            then
            add *{var=value}* to assignment
            result ← RECURSIVE-BACTRACKING(*assignment, csp*)
            if *result* ≠ *failure* **then return** *result*
            remove *{var=value}* from *assignment*
    return *failure*

Comparison of CSP algorithms on different problems

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA	(> 1,000K)	(> 1,000K)	2K	60	64
n-Queens	(> 40,000K)	13,500K	(> 40,000K)	817K	4K
Zebra	3,859K	1K	35K	0.5K	2K
Random 1	415K	3K	26K	2K	
Random 2	942K	27K	77K	15K	

Median number of consistency checks over 5 runs to solve problem

Parentheses -> no solution found

USA: 4 coloring
n-queens: n = 2 to 50
Zebra: see exercise 6.7 (3rd ed.); exercise 5.13 (2nd ed.)

Random Binary CSP (adapted from http://www.unitime.org/csp.php)

- A random binary CSP is defined by a four-tuple (n, d, p1, p2)
  - n = the number of variables.
  - d = the domain size of each variable.
  - p1 = probability a constraint exists between two variables.
  - p2 = probability a pair of values in the domains of two variables connected by a constraint is incompatible.
    - Note that R&N lists compatible pairs of values instead.
    - Equivalent formulations; just take the set complement.
- (n, d, p1, p2) are used to generate randomly the binary constraints among the variables.
- The so called model B of Random CSP (n, d, n1, n2)
  - n1 = p1n(n-1)/2 pairs of variables are randomly and uniformly selected and binary constraints are posted between them.
  - For each constraint, n2 = p2d^2 randomly and uniformly selected pairs of values are picked as incompatible.
- The random CSP as an optimization problem (minCSP).
  - Goal is to minimize the total sum of values for all variables.

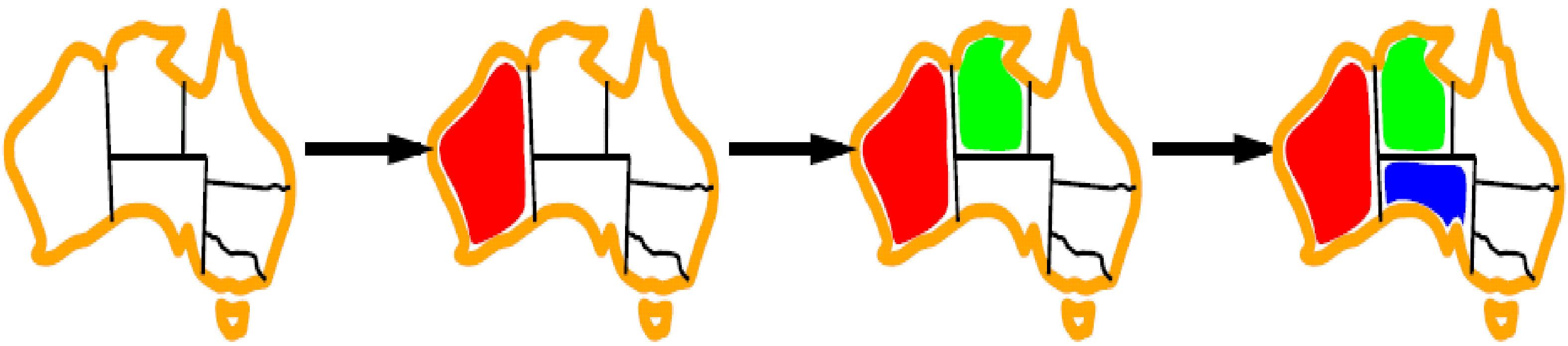
Improving CSP efficiency

- Previous improvements on uninformed search
  $\rightarrow$ introduce heuristics

- For CSPS, general-purpose methods can give large gains in speed, e.g.,
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

Note: CSPs are somewhat generic in their formulation, and so the heuristics are more general compared to methods in Chapter 4

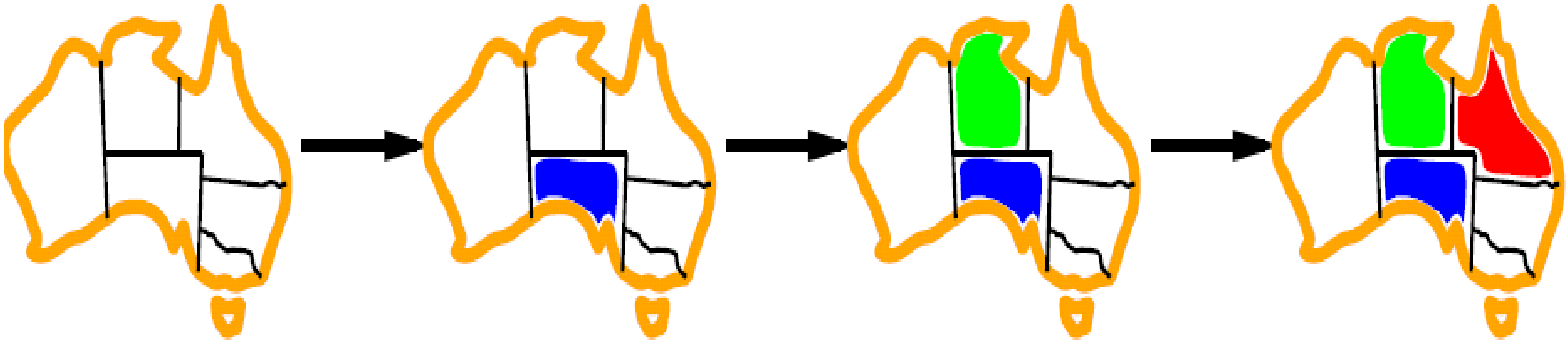
Minimum remaining values (MRV)



var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[*csp*],*assignment*,*csp*)

- A.k.a. most constrained variable heuristic
- *Heuristic Rule*: choose variable with the fewest legal moves
  - e.g., will immediately detect failure if X has no legal values

Degree heuristic for the initial variable

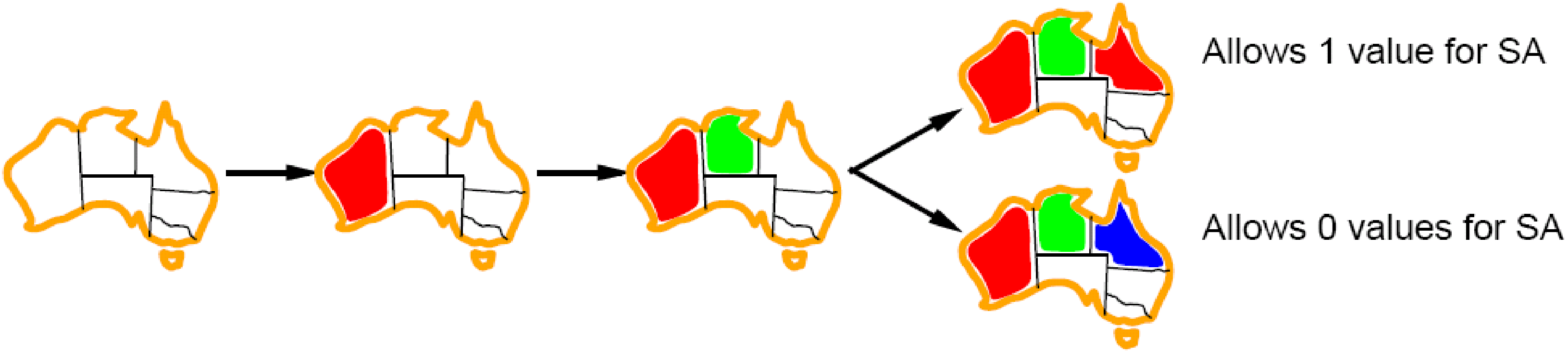


- *Heuristic Rule*: select variable that is involved in the largest number of constraints on other unassigned variables.

- Degree heuristic can be useful as a tie breaker.

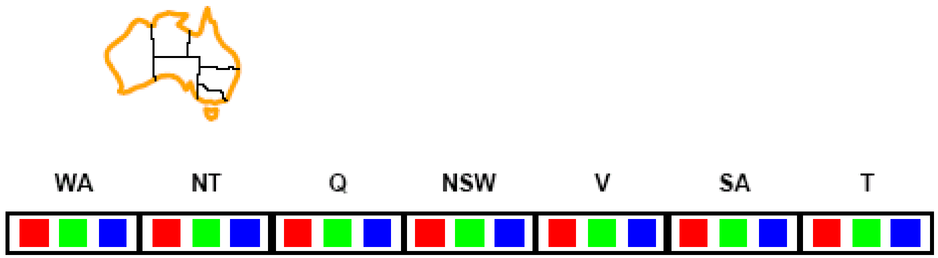
- *In what order should a variable's values be tried?*

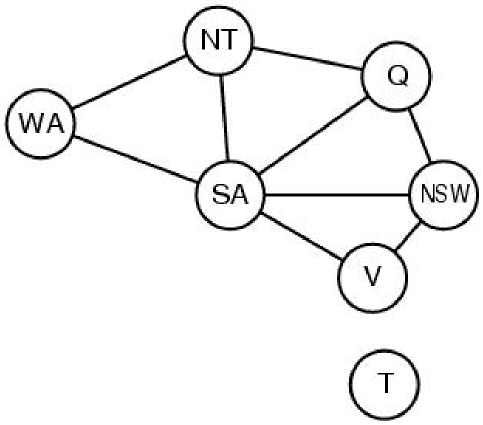
Least constraining value for value-ordering



- Least constraining value heuristic
- Heuristic Rule: given a variable choose the least constraining value
  - leaves the maximum flexibility for subsequent variable assignments

Forward checking



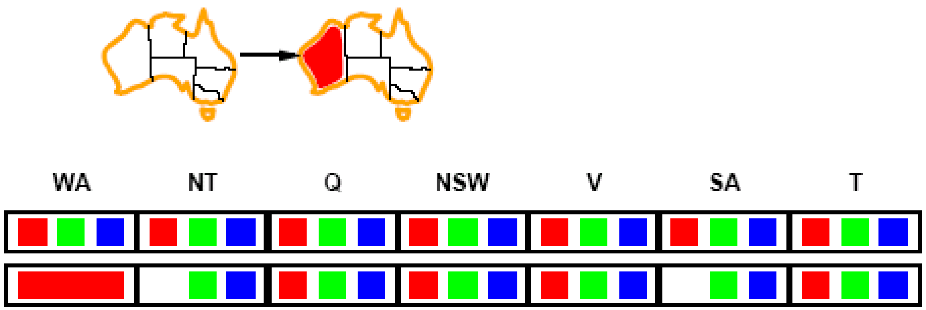


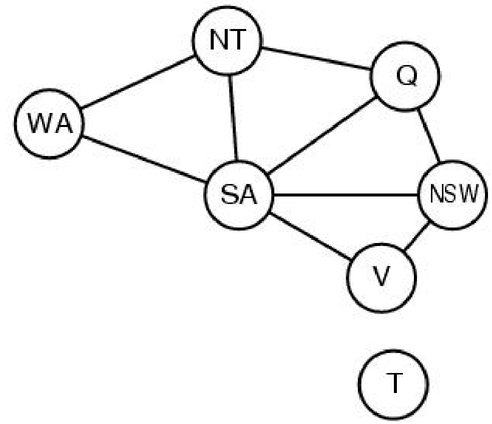
- Can we detect inevitable failure early?
  - *And avoid it later?*

- *Forward checking idea:* keep track of remaining legal values for unassigned variables.

- Terminate search when any variable has no legal values.

Forward checking

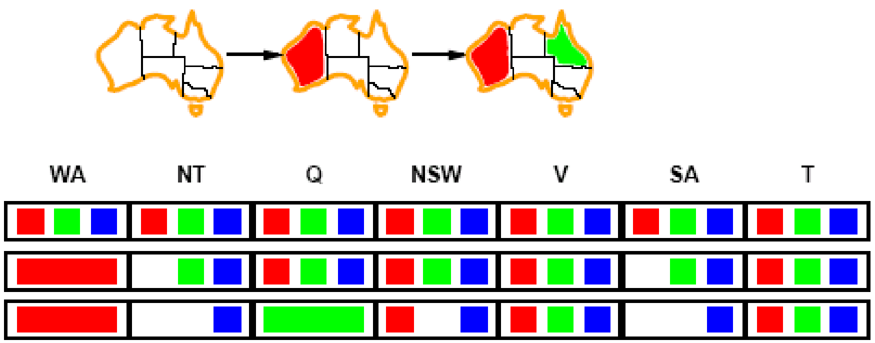


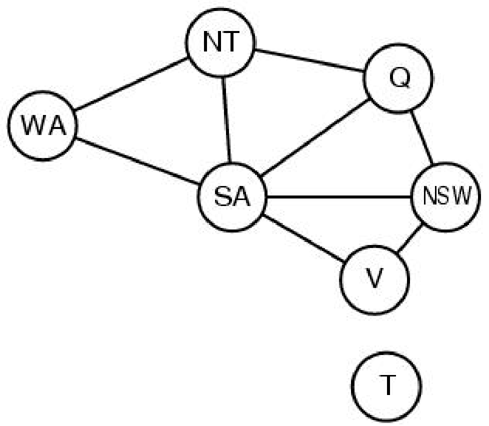


- Assign *{WA=red}*

- Effects on other variables connected by constraints to WA
  - *NT can no longer be red*
  - *SA can no longer be red*

Forward checking



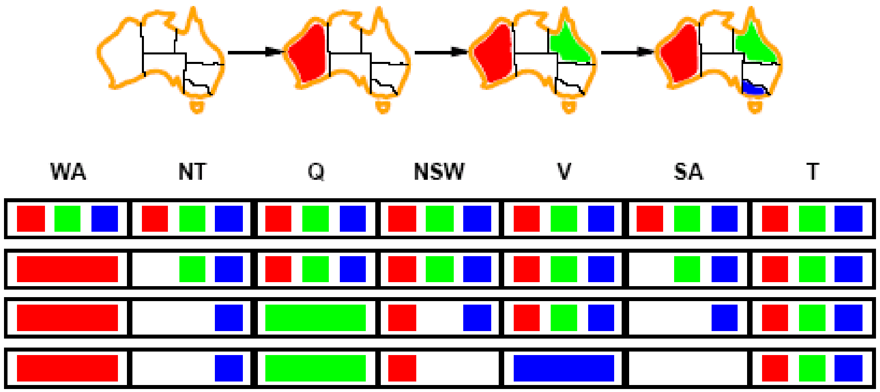


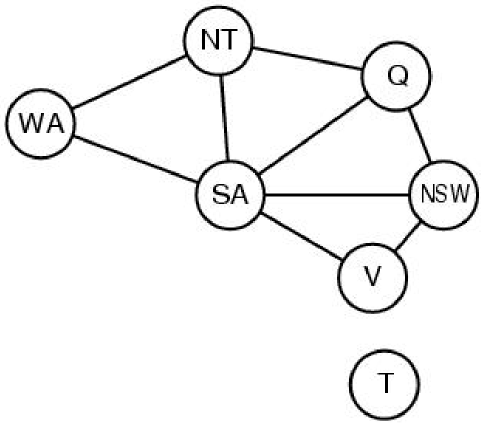
- Assign *{Q=green}*

- Effects on other variables connected by constraints with WA
  - *NT can no longer be green*
  - *NSW can no longer be green*
  - *SA can no longer be green*

- *MRV heuristic* would automatically select NT or SA next

Forward checking



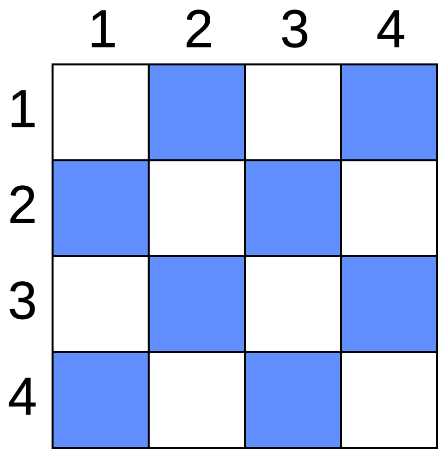


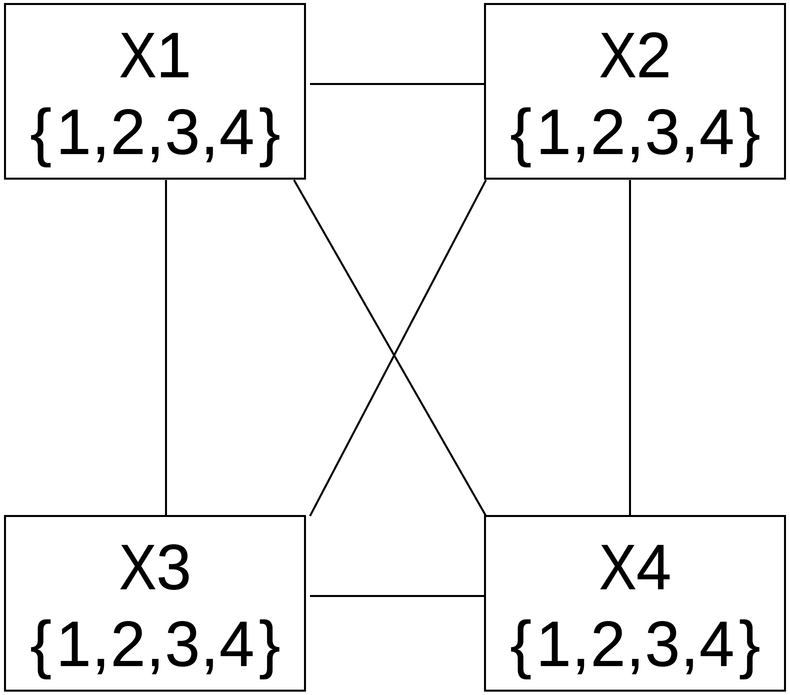
- If *V* is assigned *blue*

- Effects on other variables connected by constraints with WA
  - *NSW can no longer be blue*
  - *SA is empty*

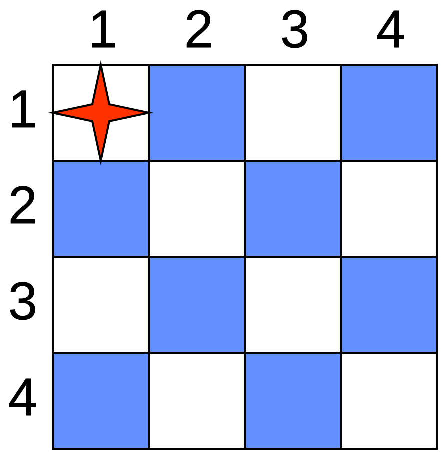
- FC has detected that partial assignment is *inconsistent* with the constraints and backtracking can occur.

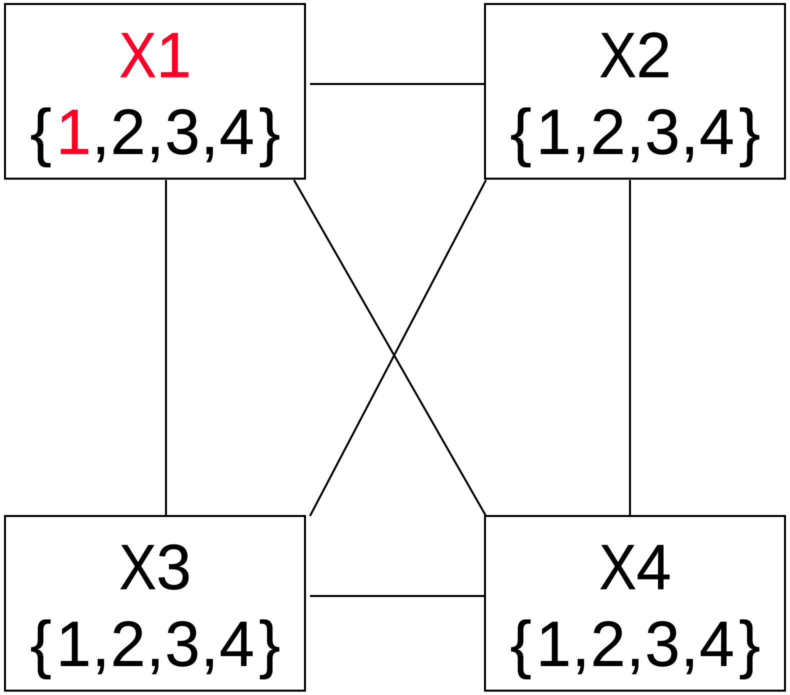
Example: 4-Queens Problem





Example: 4-Queens Problem





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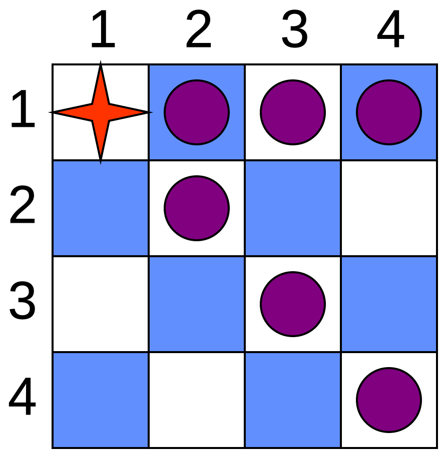

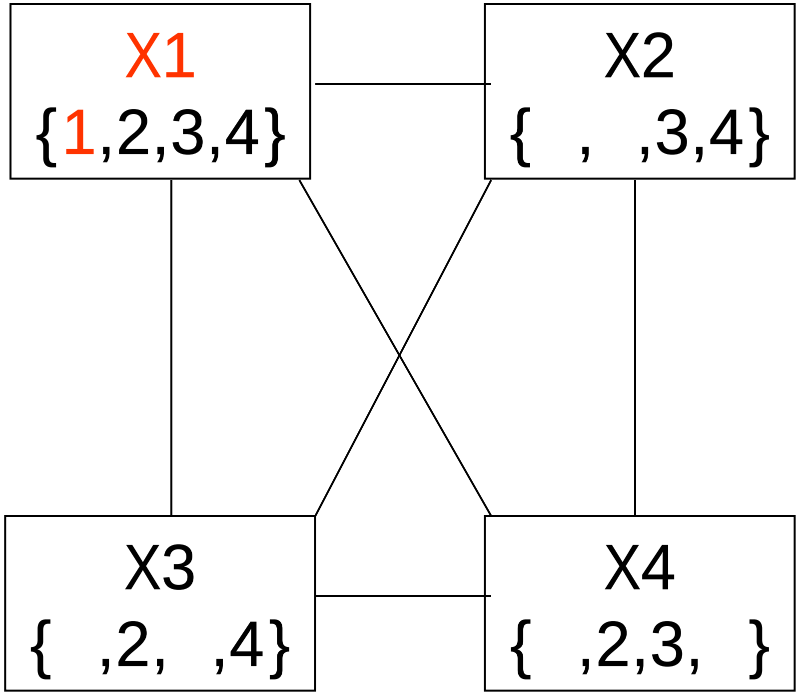

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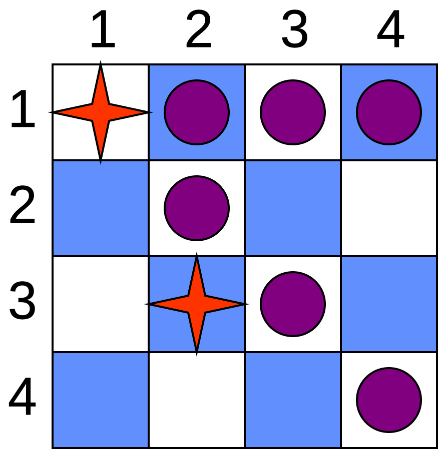

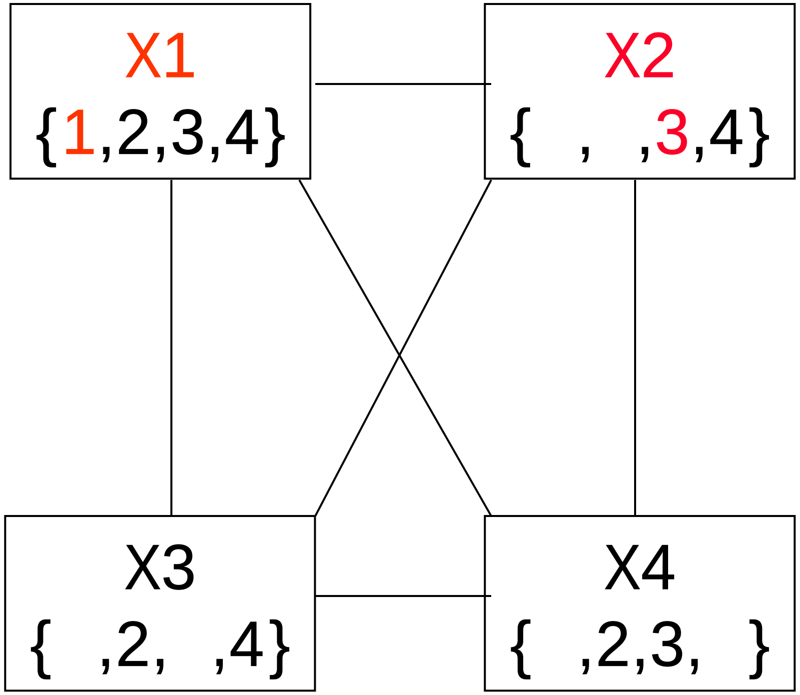

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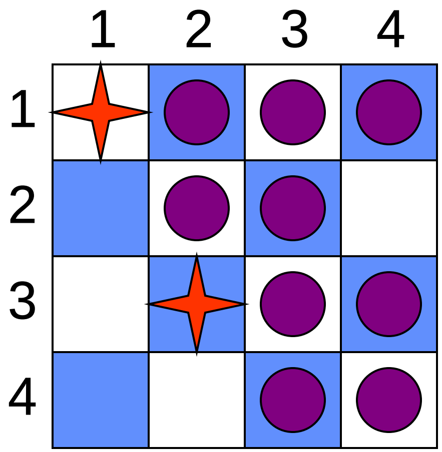

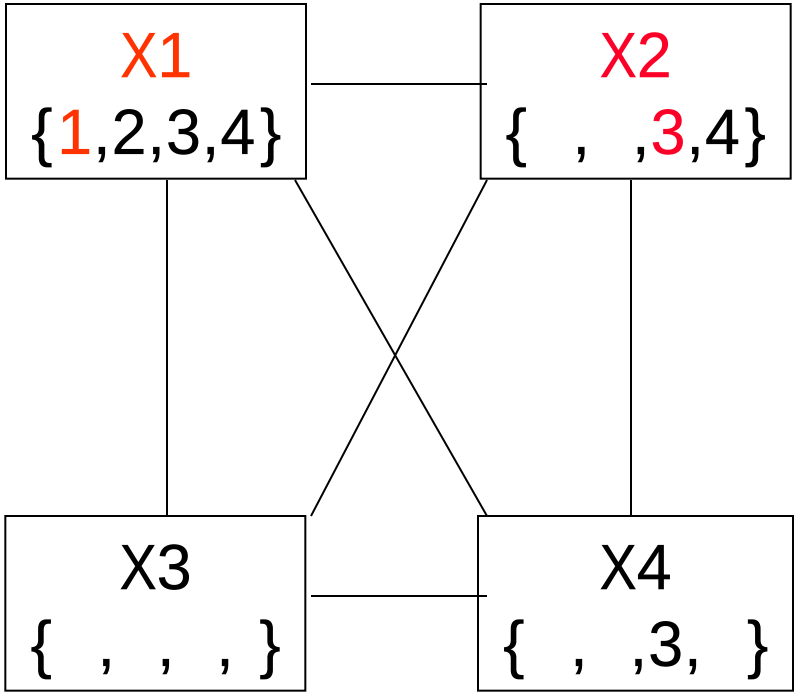

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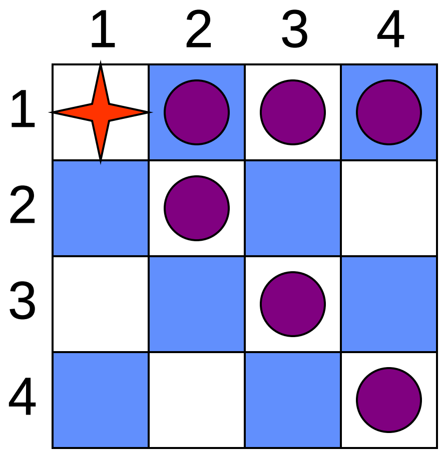

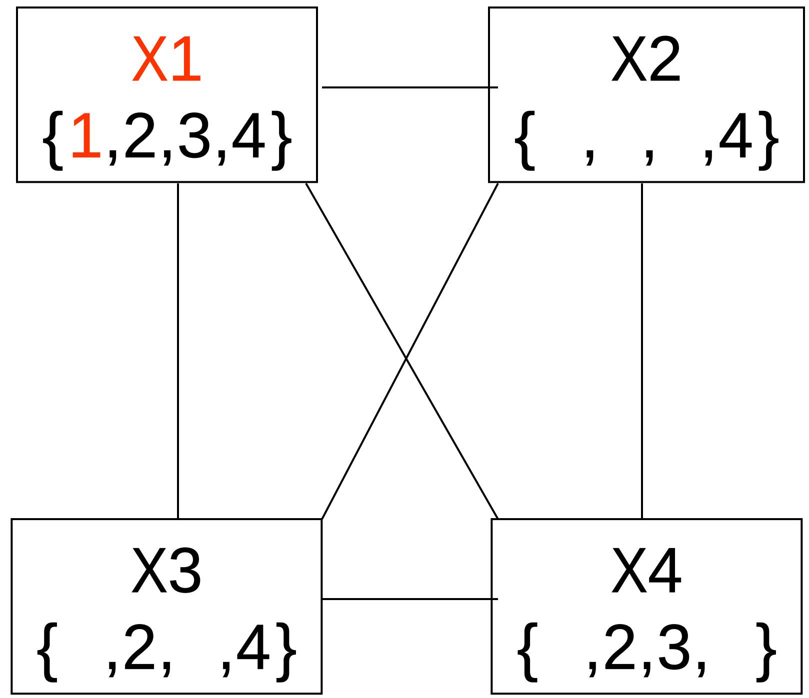

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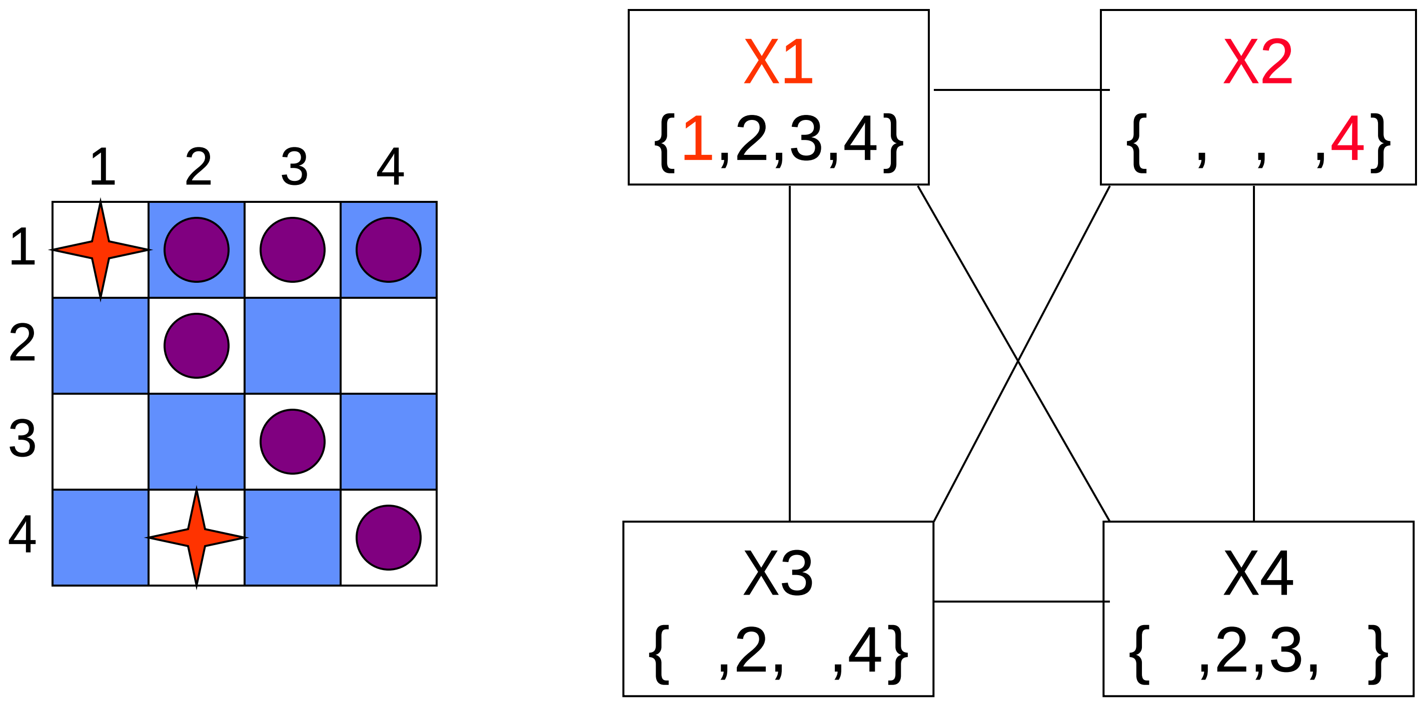
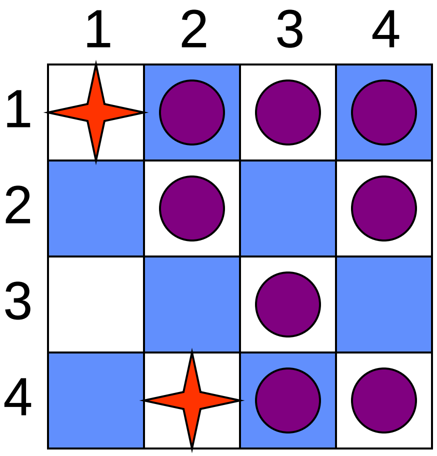
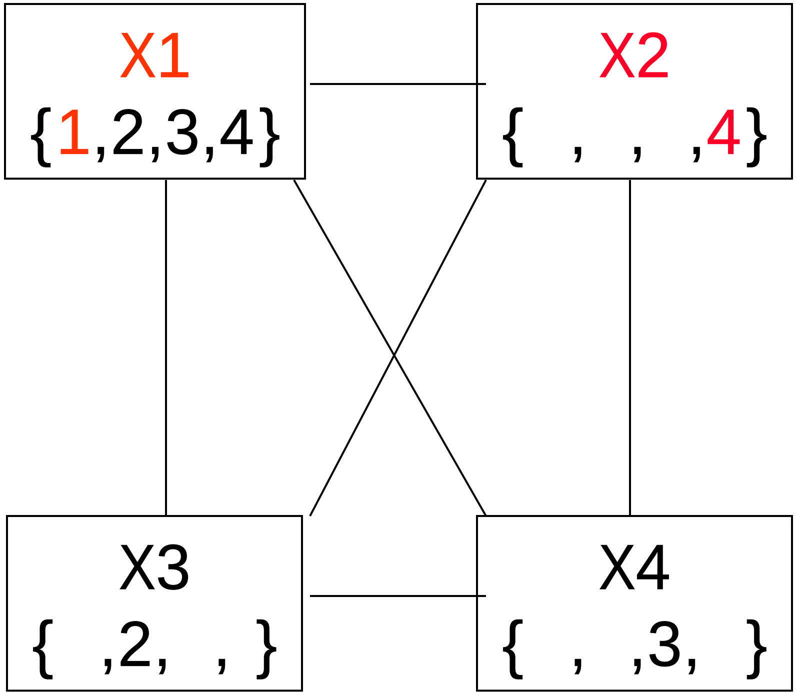


Example: 4-Queens Problem





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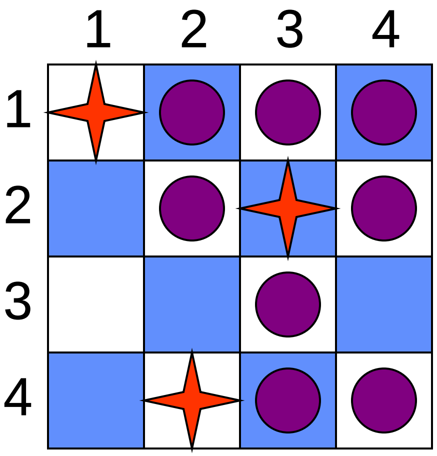

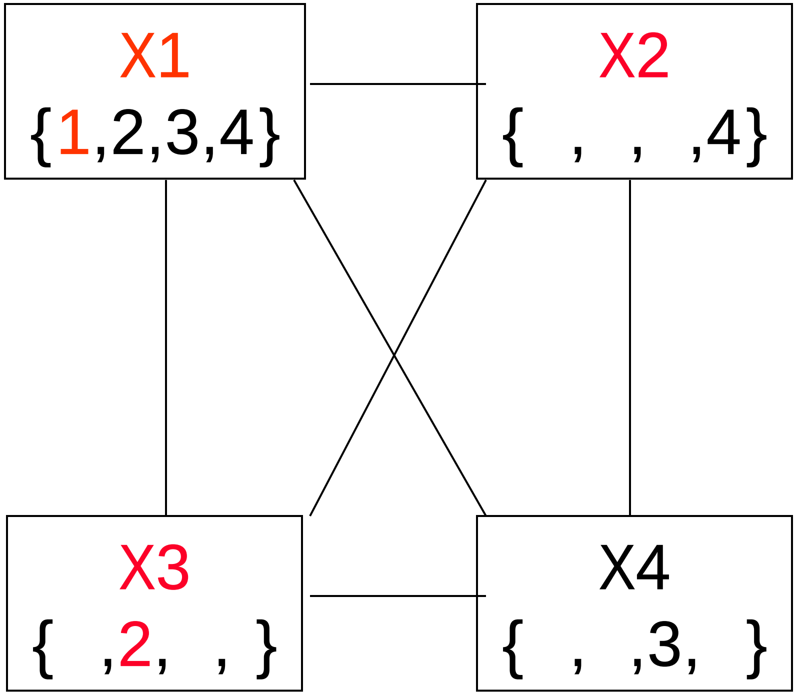

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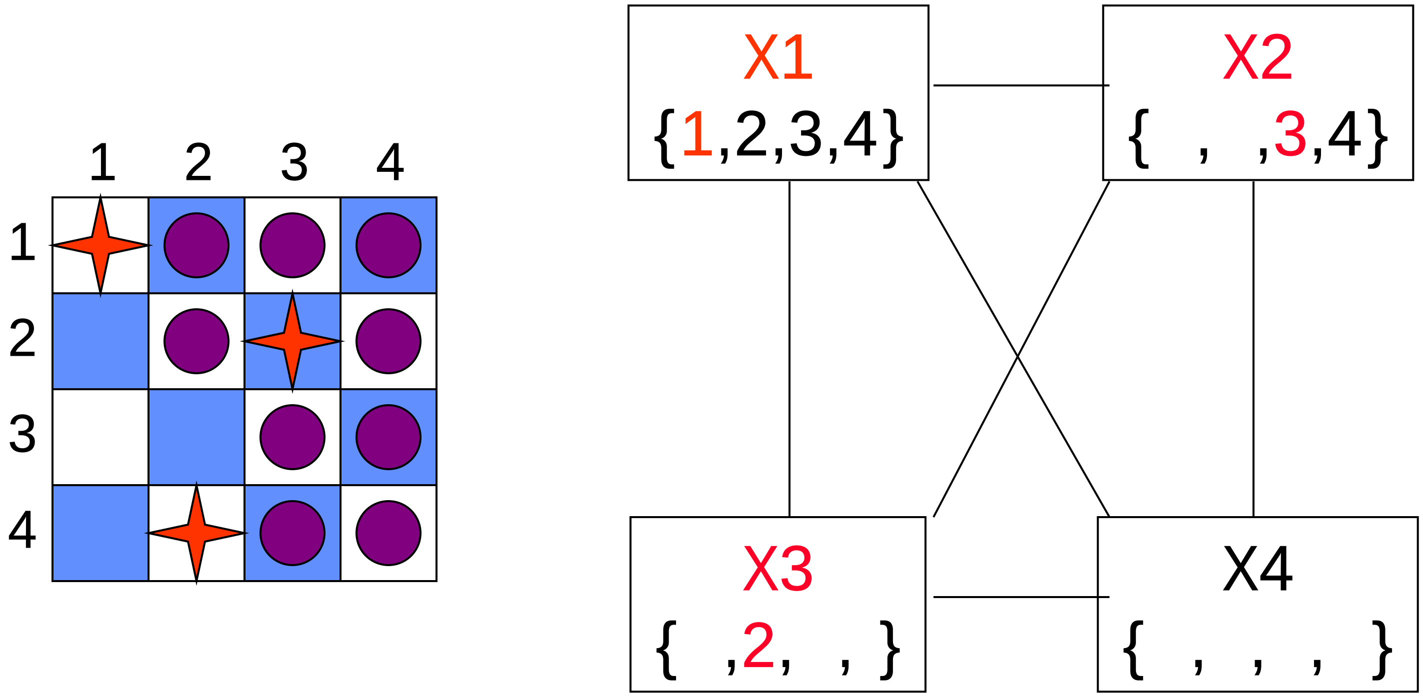


Comparison of CSP algorithms on different problems

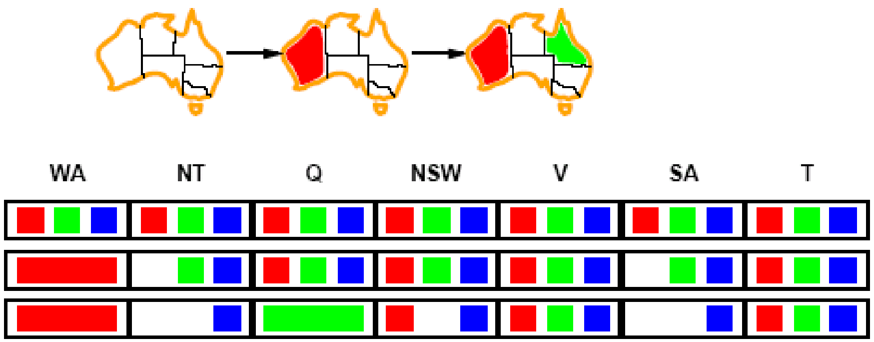
Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA	(> 1,000K)	(> 1,000K)	2K	60	64
n-Queens	(> 40,000K)	13,500K	(> 40,000K)	817K	4K
Zebra	3,859K	1K	35K	0.5K	2K
Random 1	415K	3K	26K	2K	
Random 2	942K	27K	77K	15K	

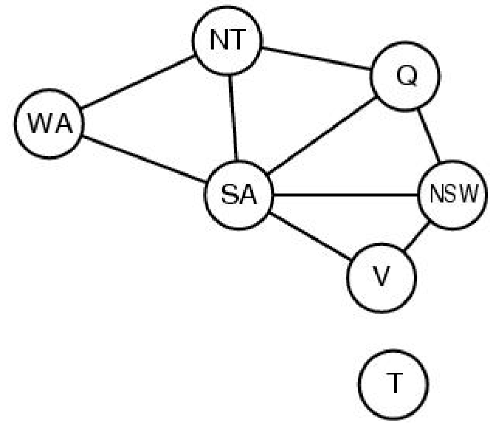
Median number of consistency checks over 5 runs to solve problem

Parentheses -> no solution found

USA: 4 coloring
n-queens: n = 2 to 50
Zebra: see exercise 5.13

Constraint propagation





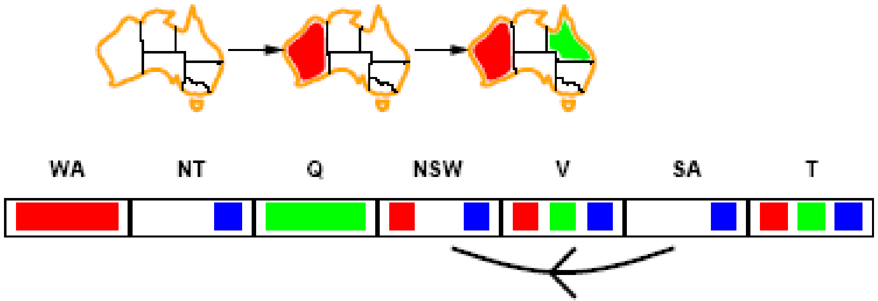
- Solving CSPs with combination of heuristics plus forward checking is more efficient than either approach alone

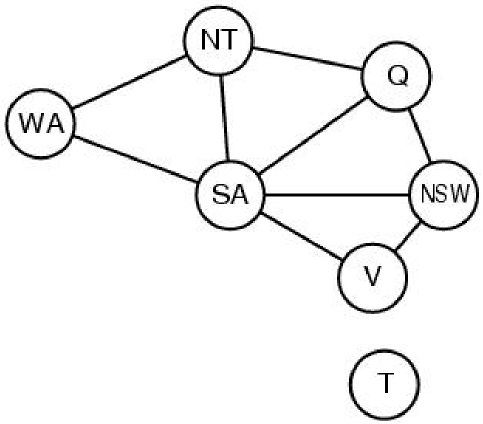
- FC checking does not detect all failures.
  - E.g., NT and SA cannot be blue

Constraint propagation

- Techniques like CP and FC are in effect eliminating parts of the search space
  - Somewhat complementary to search
- Constraint propagation goes further than FC by repeatedly enforcing constraints locally
  - Needs to be faster than actually searching to be effective
- Arc-consistency (AC) is a systematic procedure for constraint propagation

Arc consistency





- *An Arc* $X \rightarrow Y$ is consistent if

  for *every* value *x* of *X* there is some value *y* consistent with *x*

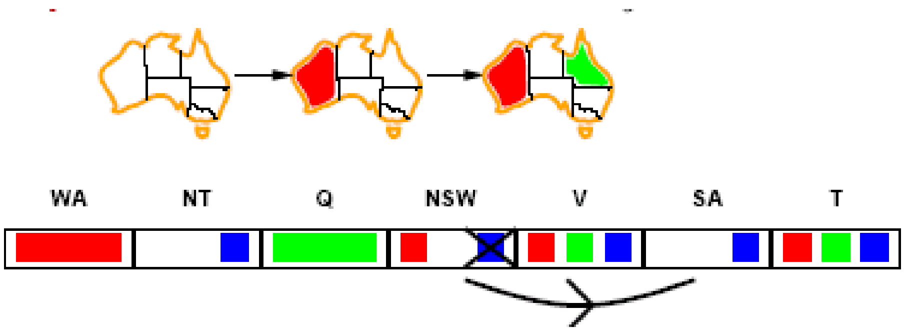
  (note that this is a directed property)

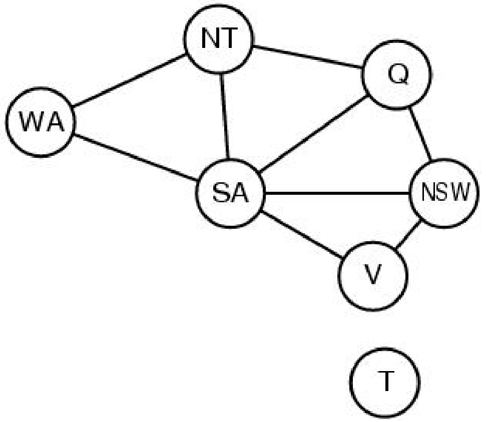
- Consider state of search after WA and Q are assigned:

SA $\rightarrow$ *NSW* is consistent if

SA=blue and *NSW=red*

Arc consistency





- $X \rightarrow Y$ is consistent if

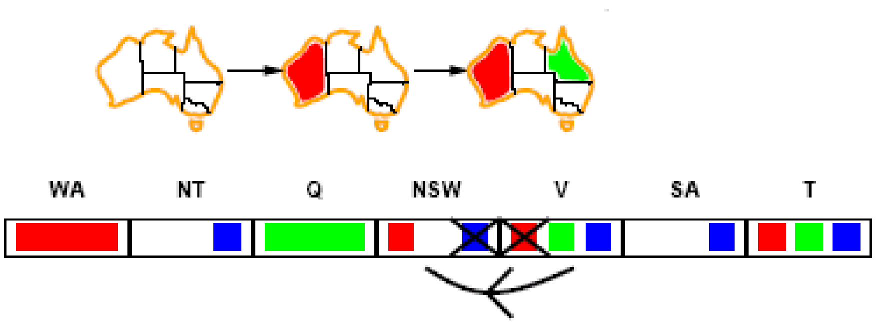
  for *every* value *x* of *X* there is some value *y* consistent with *x*

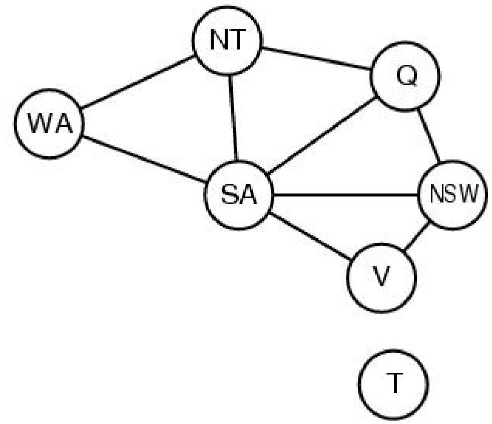
- *NSW* $\rightarrow$ *SA* is consistent if

  NSW=red and *SA=blue*

  NSW=blue and SA=???

Arc consistency

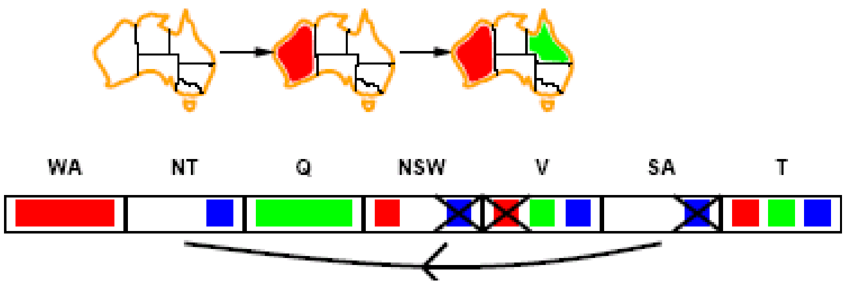


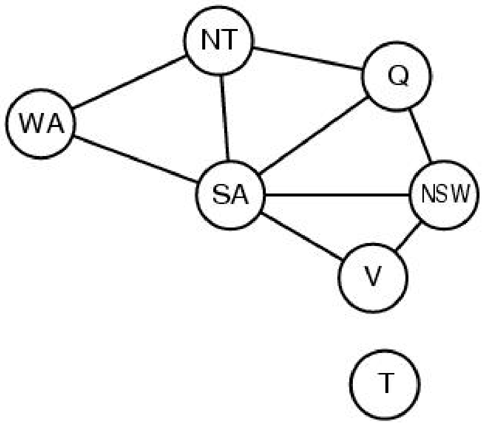


- Can enforce arc-consistency:
  Arc can be made consistent by removing *blue* from *NSW*

- Continue to propagate constraints….
  - Check $V \rightarrow NSW$
  - Not consistent for V = red
  - Remove red from *V*

Arc consistency





- Continue to propagate constraints....
- *SA* $\rightarrow$ *NT* is not consistent
  - and cannot be made consistent
- Arc consistency detects failure earlier than FC

Arc consistency checking

- Can be run as a preprocessor or after each assignment
  - Or as preprocessing before search starts
- AC must be run repeatedly until no inconsistency remains
- Trade-off
  - Requires some overhead to do, but generally more effective than direct search
  - In effect it can eliminate large (inconsistent) parts of the state space more effectively than search can
- Need a systematic method for arc-checking
  - If *X* loses a value, neighbors of *X* need to be rechecked:

    i.e. incoming arcs can become inconsistent again

    (outgoing arcs will stay consistent).

Arc consistency algorithm (AC-3)

function AC-3(*csp*) **returns** false if inconsistency found, else true, may reduce *csp* domains
    inputs: *csp*, a binary CSP with variables $\{X_1, X_2, ..., X_n\}$
    local variables: *queue,* a queue of arcs, initially all the arcs in *csp*
        /* *initial queue must contain both* (X_i, X_j) *and* (X_j, X_i) */
    while queue is not empty **do**
        $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
        if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
            if size of $D_i = 0$ **then return** false
            for each X_k **in** NEIGHBORS[X_i] − $\{X_j\}$ **do**
                add (X_k, X_i) to queue if not already there
    return true

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** *true* iff we delete a
        value from the domain of X_i
    removed ← *false*
    for each *x* **in** DOMAIN[X_i] **do**
        if no value *y* in DOMAIN[X_j] allows (*x*,*y*) to satisfy the constraints
            between X_i and X_j
        then delete *x* from DOMAIN[X_i]; *removed* ← *true*
    return *removed*

(from Mackworth, 1977)

Complexity of AC-3

- A binary CSP has at most n^2 arcs
- Each arc can be inserted in the queue d times (worst case)
  - (X, Y): only d values of X to delete
- Consistency of an arc can be checked in $O(d^2)$ time
- Complexity is $O(n^2 d^3)$
- Although substantially more expensive than Forward Checking, Arc Consistency is usually worthwhile.

K-consistency

- Arc consistency does not detect all inconsistencies:
  - Partial assignment *{WA=red, NSW=red}* is inconsistent.
- Stronger forms of propagation can be defined using the notion of k-consistency.
- A CSP is k-consistent if for any set of k-1 variables and for any consistent assignment to those variables, a consistent value can always be assigned to any kth variable.
  - E.g. 1-consistency = node-consistency
  - E.g. 2-consistency = arc-consistency
  - E.g. 3-consistency = path-consistency
- Strongly k-consistent:
  - k-consistent for all values {k, k-1, ...2, 1}

Trade-offs

- Running stronger consistency checks…
  - Takes more time
  - But will reduce branching factor and detect more inconsistent partial assignments
  - No “free lunch”
    - In worst case n-consistency takes exponential time
- Generally helpful to enforce 2-Consistency (Arc Consistency)
- Sometimes helpful to enforce 3-Consistency
- Higher levels may take more time to enforce than they save.

Further improvements

- Checking special constraints
  - Checking Alldif(…) constraint
    - *E.g. {WA=red, NSW=red}*
  - Checking Atmost(…) constraint
    - *Bounds propagation for larger value domains*

- Intelligent backtracking
  - Standard form is chronological backtracking i.e. try different value for preceding variable.
  - More intelligent, backtrack to conflict set.
    - Set of variables that caused the failure or set of previously assigned variables that are connected to X by constraints.
    - Backjumping moves back to most recent element of the conflict set.
    - Forward checking can be used to determine conflict set.

Local search for CSPs

- Use complete-state representation
  - Initial state = all variables assigned values
  - Successor states = change 1 (or more) values

- For CSPs
  - allow states with unsatisfied constraints (unlike backtracking)
  - operators **reassign** variable values
  - hill-climbing with n-queens is an example

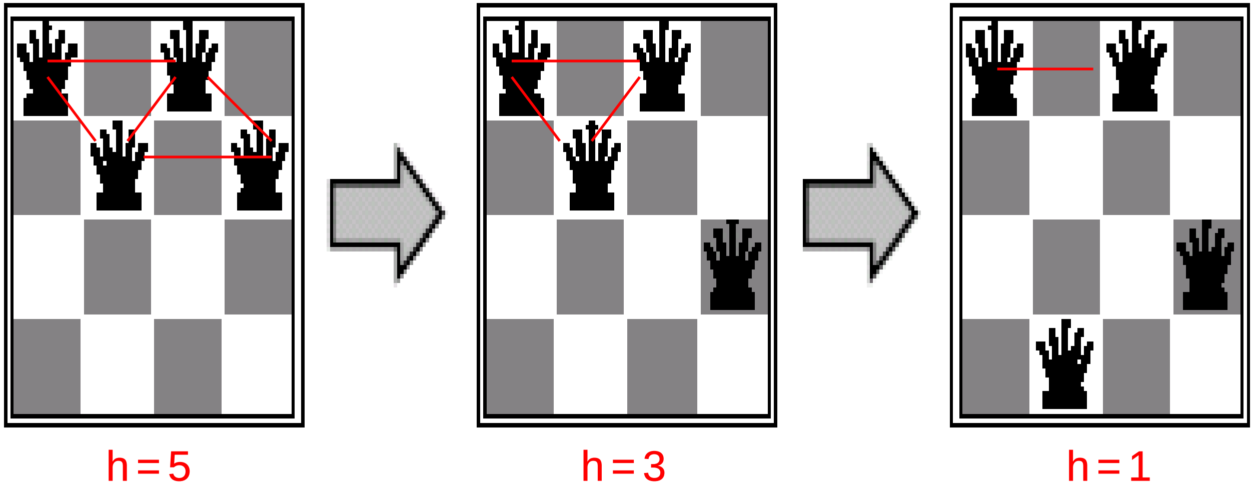
- Variable selection: randomly select any conflicted variable

- Value selection: *min-conflicts heuristic*
  - Select new value that results in a minimum number of conflicts with the other variables

Local search for CSP

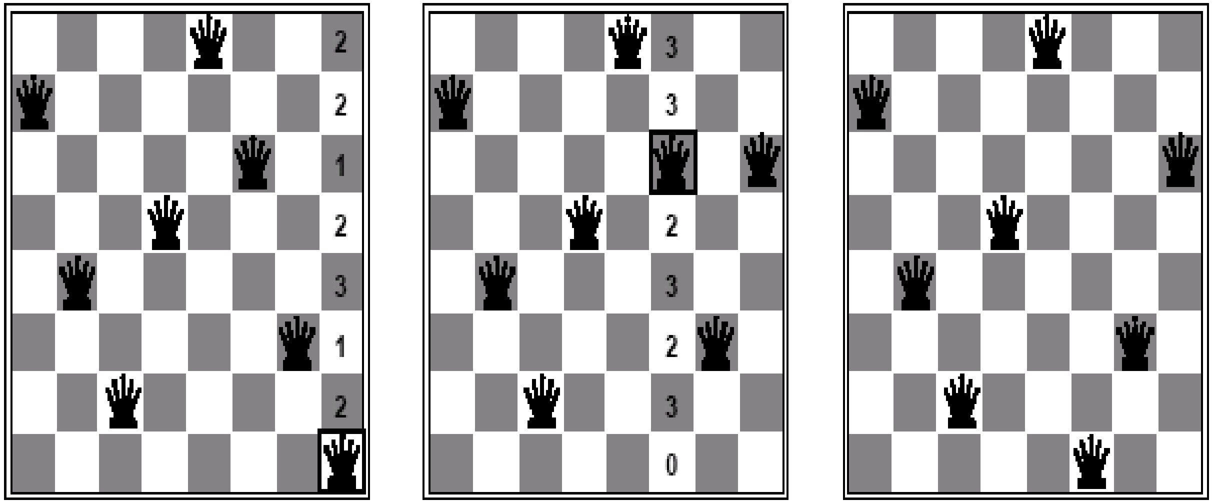
```
function MIN-CONFLICTS(csp, max_steps) return solution or failure
    inputs: csp, a constraint satisfaction problem
            max_steps, the number of steps allowed before giving up

    current ←  an initial complete assignment for csp
    for i = 1 to max_steps do
            if current is a solution for csp then return current
            var ←  a randomly chosen, conflicted variable from VARIABLES[csp]
            value  ←  the value v for var that minimize CONFLICTS(var,v,current,csp)
            set var = value in current
    return failure
```

Min-conflicts example 1



Use of min-conflicts heuristic in hill-climbing.

Min-conflicts example 2



- A two-step solution for an 8-queens problem using min-conflicts heuristic
- At each stage a queen is chosen for reassignment in its column
- The algorithm moves the queen to the min-conflict square breaking ties randomly.

Comparison of CSP algorithms on different problems

Problem	Backtracking	BT+MRV	Forward Checking	FC+MRV	Min-Conflicts
USA	(> 1,000K)	(> 1,000K)	2K	60	64
n-Queens	(> 40,000K)	13,500K	(> 40,000K)	817K	4K
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Median number of consistency checks over 5 runs to solve problem

Parentheses -> no solution found

USA: 4 coloring
n-queens: n = 2 to 50
Zebra: see exercise 6.7 (3rd ed.); exercise 5.13 (2nd ed.)

Advantages of local search

- Local search can be particularly useful in an online setting
  - Airline schedule example
    - E.g., mechanical problems require than 1 plane is taken out of service
    - Can locally search for another “close” solution in state-space
    - Much better (and faster) in practice than finding an entirely new schedule

- The runtime of min-conflicts is roughly independent of problem size.
  - Can solve the millions-queen problem in roughly 50 steps.

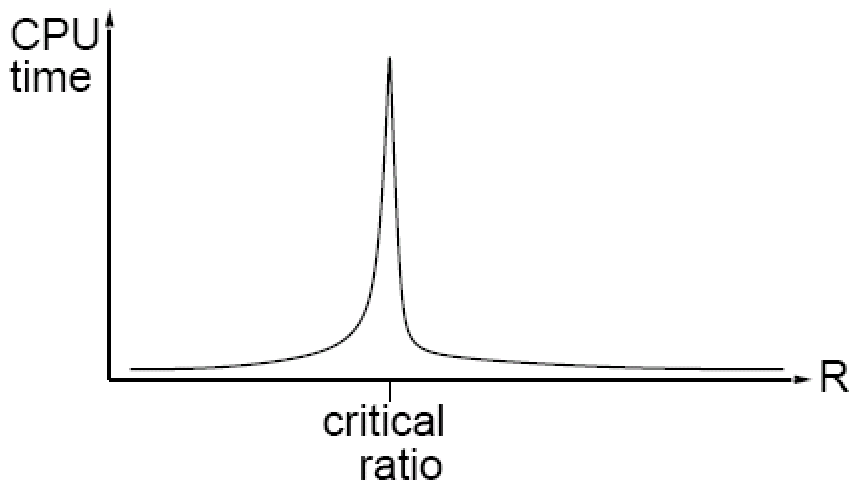
  - Why?
    - n-queens is easy for local search because of the relatively high density of solutions in state-space

Performance of min-conflicts

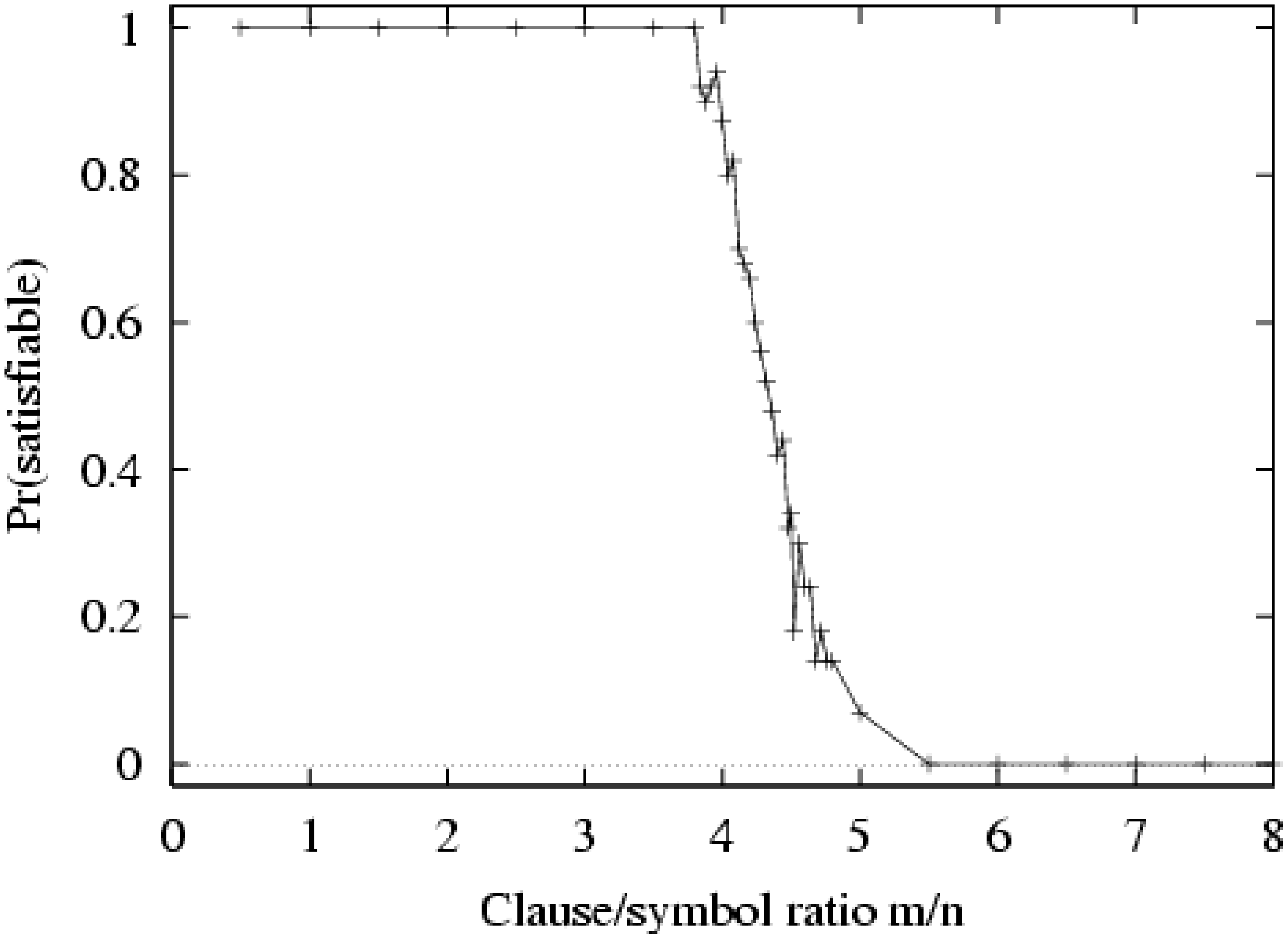
Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., $n = 10{,}000{,}000$)

The same appears to be true for any randomly-generated CSP
except in a narrow range of the ratio

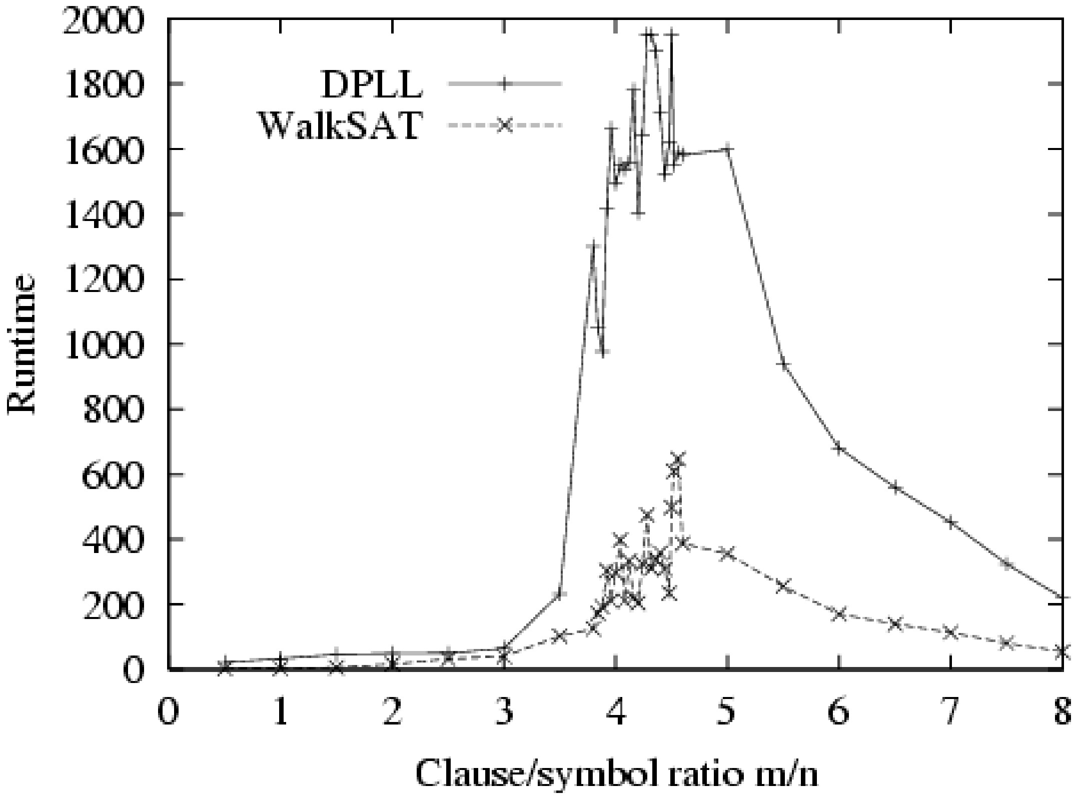
$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Hard satisfiability problems

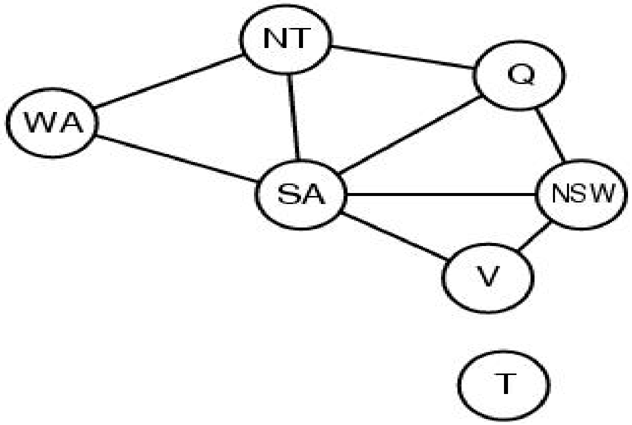


Hard satisfiability problems



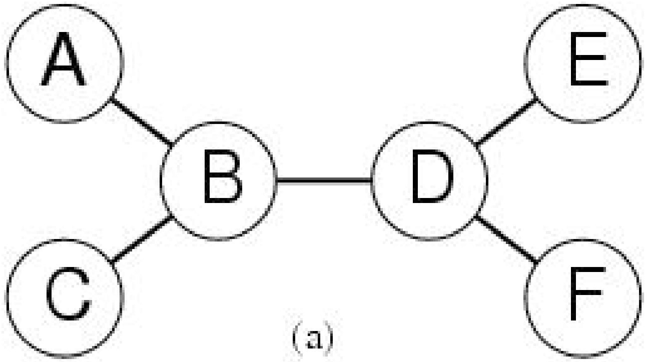
- Median runtime for 100 satisfiable random 3-CNF sentences, $n = 50$

Graph structure and problem complexity



- Solving disconnected subproblems
  - Suppose each subproblem has *c* variables out of a total of *n*.
  - Worst case solution cost is $O(n/c\ d^c)$, i.e. linear in *n*
    - Instead of $O(d^n)$, exponential in *n*
- E.g. $n= 80, c= 20, d=2$
  - 2^{80} = 4 billion years at 1 million nodes/sec.
  - $4 * 2^{20}$= .4 second at 1 million nodes/sec

Tree-structured CSPs



(a)

- Theorem:
  - if a constraint graph has no loops then the CSP can be solved in $O(nd^2)$ time
  - linear in the number of variables!

- Compare difference with general CSP, where worst case is $O(d^n)$

Summary

- CSPs
  - special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values
- Backtracking=depth-first search with one variable assigned per node
- Heuristics
  - Variable ordering and value selection heuristics help significantly
- Constraint propagation does additional work to constrain values and detect inconsistencies
  - Works effectively when combined with heuristics
- Iterative min-conflicts is often effective in practice.
- Graph structure of CSPs determines problem complexity
  - e.g., tree structured CSPs can be solved in linear time.