First-Order Logic
Semantics

Reading: Chapter 8, 9.1-9.2, 9.5.1-9.5.5

FOL Syntax and Semantics read: 8.1-8.2
FOL Knowledge Engineering read: 8.3-8.5
FOL Inference read: Chapter 9.1-9.2, 9.5.1-9.5.5

(Please read lecture topic material before and after each lecture on that topic)
Outline

• Propositional Logic is **Useful** --- but has **Limited Expressive Power**

• First Order Predicate Calculus (FOPC), or First Order Logic (FOL).
  – FOPC has greatly expanded expressive power, though still limited.

• New Ontology
  – The world consists of **OBJECTS** (for propositional logic, the world was facts).
  – **OBJECTS** have **PROPERTIES** and engage in **RELATIONS** and **FUNCTIONS**.

• New Syntax
  – **Constants**, **Predicates**, **Functions**, **Properties**, **Quantifiers**.

• New Semantics
  – Meaning of new syntax.

• Knowledge engineering in FOL

• Inference in FOL
You will be expected to know

- FOPC syntax and semantics
  - Syntax: Sentences, predicate symbols, function symbols, constant symbols, variables, quantifiers
  - Semantics: Models, interpretations
- De Morgan’s rules for quantifiers
  - connections between $\forall$ and $\exists$
- Nested quantifiers
  - Difference between “$\forall x \exists y \ P(x, y)$” and “$\exists x \forall y \ P(x, y)$”
  - $\forall x \exists y \ Likes(x, y)$
  - $\exists x \forall y \ Likes(x, y)$
- Translate simple English sentences to FOPC and back
  - $\forall x \exists y \ Likes(x, y) \iff \text{“Everyone has someone that they like.”}$
  - $\exists x \forall y \ Likes(x, y) \iff \text{“There is someone who likes every person.”}$
Outline

- Review: \( KB \models S \) is equivalent to \( \models (KB \Rightarrow S) \)
  - So what does \( \models (KB \Rightarrow S) \) mean?

- Review: Follows, Entails, Derives
  - Follows: “Is it the case?”
  - Entails: “Is it true?”
  - Derives: “Is it provable?”

- Semantics of FOL (FOPC)
  - Model, Interpretation
FOL (or FOPC) Ontology:
What kind of things exist in the world?
What do we need to describe and reason about?
Objects --- with their relations, functions, predicates, properties, and general rules.
**Review: KB |= S means |= (KB ⇒ S)**

- KB |= S is read “KB entails S.”
  - Means “S is true in every world (model) in which KB is true.”
  - Means “In the world, S follows from KB.”

- KB |= S is equivalent to |= (KB ⇒ S)
  - Means “(KB ⇒ S) is true in every world (i.e., is valid).”

- And so: {} |= S is equivalent to |= ({} ⇒ S)

- So what does ({} ⇒ S) mean?
  - Means “True implies S.”
  - Means “S is valid.”
  - In Horn form, means “S is a fact.” p. 256 (3rd ed.; p. 281, 2nd ed.)

- Why does {} mean True here, but False in resolution proofs?
Review: \((\text{True} \Rightarrow S)\) means “\(S\) is a fact.”

- By convention,
  - The null conjunct is “syntactic sugar” for True.
  - The null disjunct is “syntactic sugar” for False.
  - Each is assigned the truth value of its identity element.
    - For conjuncts, True is the identity: \((A \land \text{True}) \equiv A\)
    - For disjuncts, False is the identity: \((A \lor \text{False}) \equiv A\)

- A KB is the conjunction of all of its sentences.
  - So in the expression: \(\{\} \models S\)
    - We see that \(\{\}\) is the null conjunct and means True.
    - The expression means “\(S\) is true in every world where True is true.”
      - I.e., “\(S\) is valid.”
      - Better way to think of it: \(\{\}\) does not exclude any worlds (models).

- In Conjunctive Normal Form each clause is a disjunct.
  - So in, say, \(\text{KB} = \{ (P \land Q) \lor (\neg Q \land R) \lor (\neg X \lor Y) \land (\neg Z) \}\)
    - We see that \((\lor)\) is the null disjunct and means False.
Side Trip: Functions AND, OR, and null values
(Note: These are “syntactic sugar” in logic.)

function AND(arglist) returns a truth-value
    return ANDOR(arglist, True)

function OR(arglist) returns a truth-value
    return ANDOR(arglist, False)

function ANDOR(arglist, nullvalue) returns a truth-value
    /* nullvalue is the identity element for the caller. */
    if (arglist = {}) then return nullvalue
    if (FIRST(arglist) = NOT(nullvalue) ) then return NOT(nullvalue)
    return ANDOR( REST(arglist) )
Side Trip: We only need one logical connective. (Note: AND, OR, NOT are “syntactic sugar” in logic.)

Both NAND and NOR are logically complete.

- NAND is also called the “Sheffer stroke”
- NOR is also called “Pierce’s arrow”

\[(\neg A) = (\text{NAND } A \text{ TRUE}) = (\text{NOR } A \text{ FALSE})\]

\[(A \land B) = (\text{NAND } \text{ TRUE } (\text{NAND } A \text{ B}))\]
\[= (\text{NOR } (\text{NOR } A \text{ FALSE }) (\text{NOR } B \text{ FALSE }))\]

\[(A \lor B) = (\text{NAND } (\text{NAND } A \text{ TRUE }) (\text{NAND } B \text{ TRUE }))\]
\[= (\text{NOR } \text{ FALSE } (\text{NOR } A \text{ B }))\]
If KB is true in the real world, then any sentence $\alpha$ entailed by KB and any sentence $\alpha$ derived from KB by a sound inference procedure is also true in the real world.
Schematic Example: Follows, Entails, and Derives

Inference

“Mary is Sue’s sister and Amy is Sue’s daughter.”

“An aunt is a sister of a parent.”

Derives

Entails

Follows

Is it provable?

Is it true?

Is it the case?

World

Mary \(\xrightarrow{\text{Sister}}\) Sue

Daughter

Amy

Mary \(\xrightarrow{\text{Aunt}}\) Amy
Models are formal worlds in which truth can be evaluated.

We say $m$ is a model of a sentence $\alpha$ if $\alpha$ is true in $m$.

$M(\alpha)$ is the set of all models of $\alpha$.

Then $KB \models \alpha$ iff $M(KB) \subseteq M(\alpha)$.

- E.g. $KB = \text{"Mary is Sue’s sister and Amy is Sue’s daughter."}$
- $\alpha = \text{"Mary is Amy’s aunt."}$

Think of $KB$ and $\alpha$ as constraints, and of models $m$ as possible states.

$M(KB)$ are the solutions to $KB$ and $M(\alpha)$ the solutions to $\alpha$.

Then, $KB \models \alpha$, i.e., $\models (KB \Rightarrow \alpha)$, when all solutions to $KB$ are also solutions to $\alpha$. 
Semantics: Worlds

• The world consists of objects that have properties.
  – There are relations and functions between these objects
  – Objects in the world, individuals: people, houses, numbers, colors, baseball games, wars, centuries
    • Clock A, John, 7, the-house in the corner, Tel-Aviv, Ball43
  – Functions on individuals:
    • father-of, best friend, third inning of, one more than
  – Relations:
    • brother-of, bigger than, inside, part-of, has color, occurred after
  – Properties (a relation of arity 1):
    • red, round, bogus, prime, multistoried, beautiful
An *interpretation* of a sentence (wff) is an assignment that maps
- Object constant symbols to objects in the world,
- n-ary function symbols to n-ary functions in the world,
- n-ary relation symbols to n-ary relations in the world

Given an interpretation, an atomic sentence has the value “true” if it denotes a relation that holds for those individuals denoted in the terms. Otherwise it has the value “false.”
- Example: Kinship world:
  - Symbols = Ann, Bill, Sue, Married, Parent, Child, Sibling, ...
  - World consists of individuals in relations:
    - Married(Ann,Bill) is false, Parent(Bill,Sue) is true, ...
Truth in first-order logic

- Sentences are true with respect to a model and an interpretation.
- Model contains objects (domain elements) and relations among them.
- Interpretation specifies referents for:
  - constant symbols → objects
  - predicate symbols → relations
  - function symbols → functional relations
- An atomic sentence \( \text{predicate}(\text{term}_1, \ldots, \text{term}_n) \) is true iff the objects referred to by \( \text{term}_1, \ldots, \text{term}_n \) are in the relation referred to by \( \text{predicate} \).
Semantics: Models

- An interpretation satisfies a wff (sentence) if the wff has the value “true” under the interpretation.
- Model: A domain and an interpretation that satisfies a wff is a model of that wff
- Validity: Any wff that has the value “true” under all interpretations is valid
- Any wff that does not have a model is inconsistent or unsatisfiable
- If a wff w has a value true under all the models of a set of sentences KB then KB logically entails w
Models for FOL: Example

crown

person

brother

on head

person

king

left leg

left leg
FOL (or FOPC) Ontology:
What kind of things exist in the world?
What do we need to describe and reason about?
Objects --- with their relations, functions, predicates, properties, and general rules.
Summary

• First-order logic:
  – Much more expressive than propositional logic
  – Allows objects and relations as semantic primitives
  – Universal and existential quantifiers

• Syntax: constants, functions, predicates, equality, quantifiers

• Nested quantifiers

• Translate simple English sentences to FOPC and back