Propositional Logic: Methods of Proof (Part II)

This lecture topic:

Propositional Logic (two lectures)

Chapter 7.1-7.4 (previous lecture, Part I)

Chapter 7.5 (this lecture, Part II)

(optional: 7.6-7.8)

Next lecture topic:

First-order logic (two lectures)

Chapter 8

(Please read lecture topic material before and after each lecture on that topic)

Outline

- Basic definitions
  - Inference, derive, sound, complete
- Application of inference rules
  - Resolution
  - Horn clauses
  - ~~Forward & Backward chaining~~
- Model Checking
  - ~~Complete backtracking search algorithms~~
    - E.g., DPLL algorithm
  - ~~Incomplete local search algorithms~~
    - E.g., WalkSAT algorithm

You will be expected to know

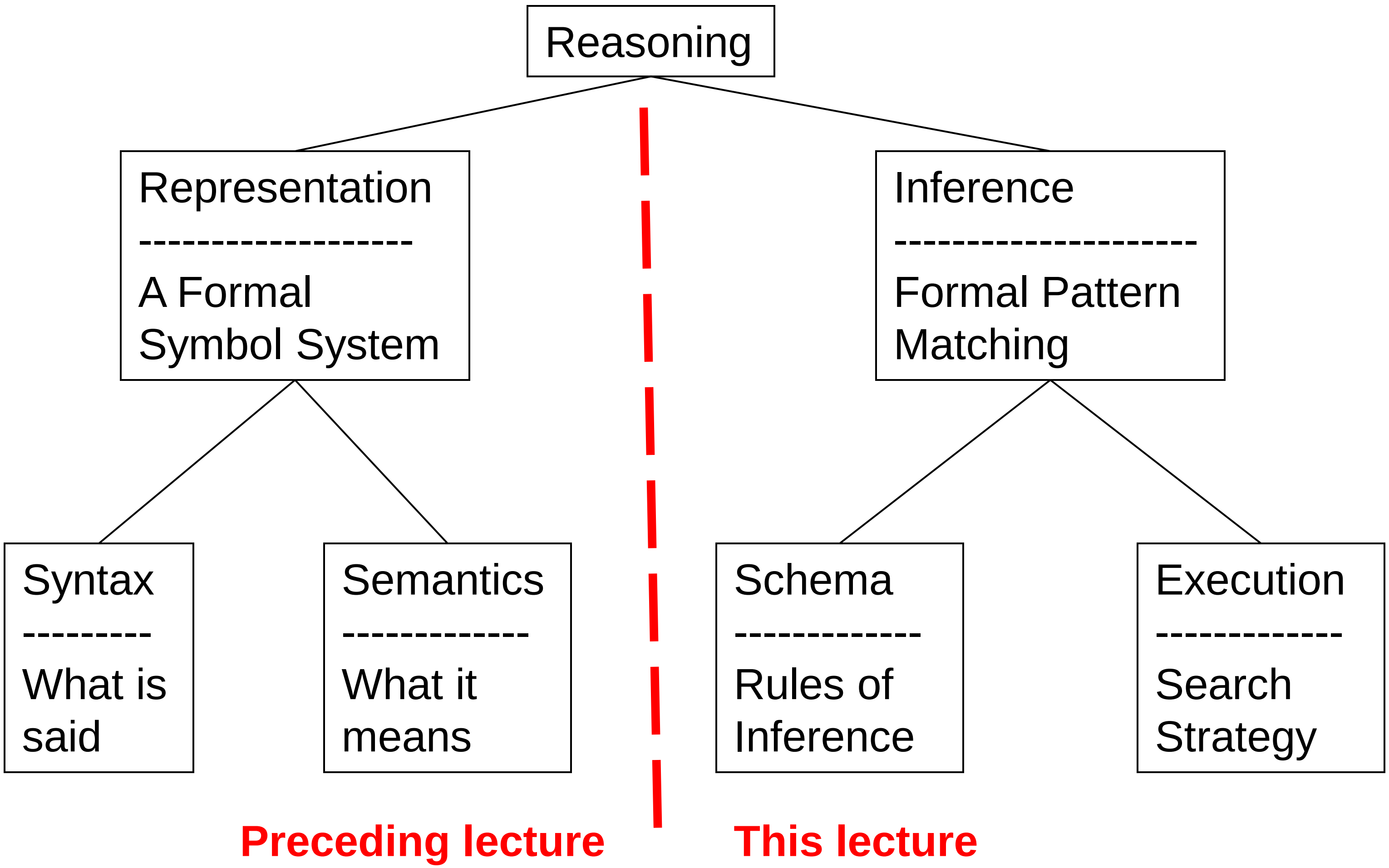
- Basic definitions
- Conjunctive Normal Form (CNF)
  - Convert a Boolean formula to CNF
- Do a short resolution proof
- ~~Do a short forward-chaining proof~~
- ~~Do a short backward-chaining proof~~
- ~~Model checking with backtracking search~~
- ~~Model checking with local search~~

Inference in Formal Symbol Systems: Ontology, Representation, Inference

- **Formal Symbol Systems**
  - **Symbols** correspond to **things/ideas** in the world
  - **Pattern matching & rewrite** corresponds to **inference**

- **Ontology:** What exists in the world?
  - What must be represented?
- **Representation:** Syntax vs. Semantics
  - What’s Said vs. What’s Meant
- **Inference:** Schema vs. Mechanism
  - Proof Steps vs. Search Strategy

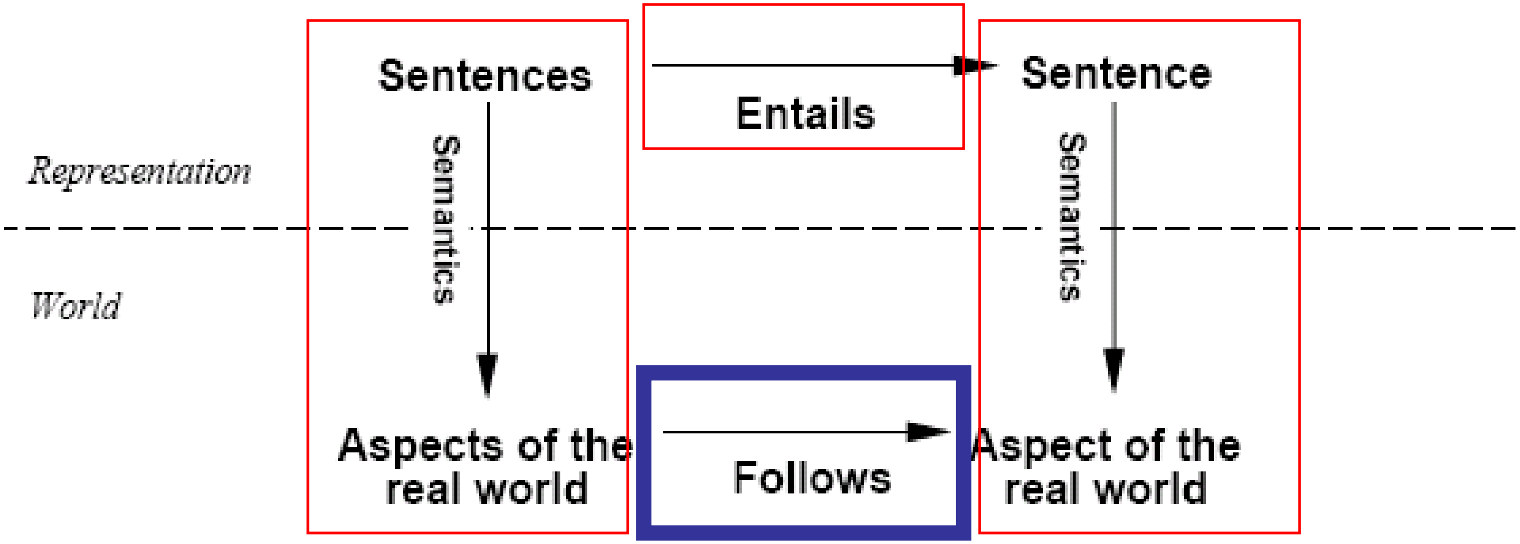
Ontology:
What kind of things exist in the world?
What do we need to describe and reason about?



Review

- Definitions:
  - Syntax, Semantics, Sentences, Propositions, Entails, Follows, Derives, Inference, Sound, Complete, Model, Satisfiable, Valid (or Tautology)
- Syntactic Transformations:
  - E.g., $(A \Rightarrow B) \Leftrightarrow (\neg A \vee B)$
- Semantic Transformations:
  - E.g., $(KB \models \alpha) \equiv (\models (KB \Rightarrow \alpha))$
- Truth Tables
  - Negation, Conjunction, Disjunction, Implication, Equivalence (Biconditional)
  - Inference by Model Enumeration

Review: Schematic perspective



If KB is true in the real world,

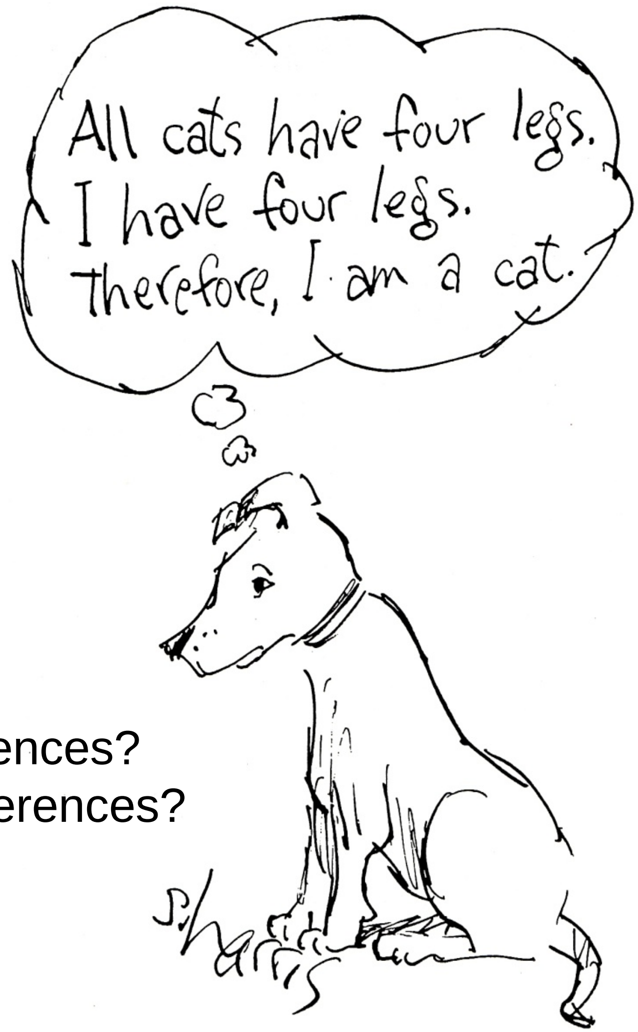
then any sentence α ***entailed*** *by KB*

is also true in the real world.

So --- how do we keep it from "Just making things up." ?

Is this inference correct?

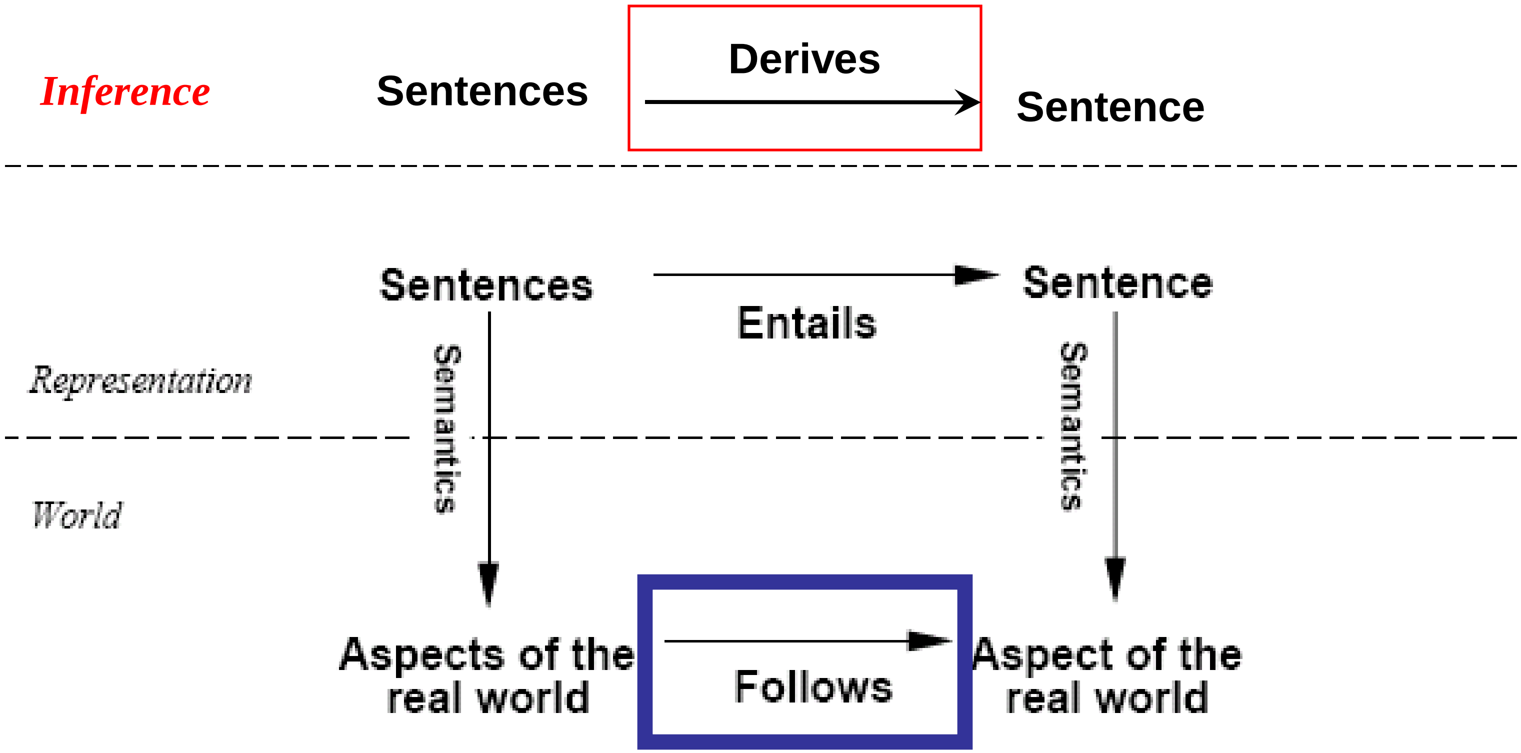
How do you know?
How can you tell?



How can we **make correct** inferences?
How can we **avoid incorrect** inferences?

"Einstein Simplified: Cartoons on Science"
by Sydney Harris, 1992,
Rutgers University Press

Schematic perspective



If KB is true in the real world,

then any sentence α ***derived*** *from KB*

by a sound inference procedure

is also true in the real world.

Logical inference

- The notion of entailment can be used for logic inference.
  - Model checking (see wumpus example):
    enumerate all possible models and check whether α is true.

- *Sound* (or *truth preserving*):
  The algorithm **only** derives entailed sentences.
  - Otherwise it just makes things up.
    i is sound iff whenever KB $\vdash_i \alpha$ *it is also true that KB* $\models \alpha$
  - *E.g., model-checking is sound*

- Complete:
  The algorithm can derive **every** entailed sentence.
    i is complete iff whenever KB $\models \alpha$ *it is also true that KB* $\vdash_i \alpha$

Proof methods

- Proof methods divide into (roughly) two kinds:

  Application of inference rules:

  Legitimate (sound) generation of new sentences from old.

  – Resolution
  – Forward & Backward chaining

  Model checking

  Searching through truth assignments.

  - Improved backtracking: Davis--Putnam-Logemann-Loveland (DPLL)
  - Heuristic search in model space: Walksat.

Conjunctive Normal Form

We'd like to prove: $\boxed{\begin{array}{l} KB \models \alpha \\ \text{equivalent to}: KB \wedge \neg\alpha \text{ unsatifiable} \end{array}}$

We first rewrite $KB \wedge \neg\alpha$ into **conjunctive normal form (CNF).**

A "conjunction of disjunctions"

literals

$\underbrace{(A \vee \neg B)}_{\text{Clause}} \wedge \underbrace{(B \vee \neg C \vee \neg D)}_{\text{Clause}}$

- Any KB can be converted into CNF.
- In fact, any KB can be converted into CNF-3 using clauses with at most 3 literals.

Example: Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$

1. Eliminate $\Leftrightarrow$, replacing $\alpha \Leftrightarrow \beta$ with $(\alpha \Rightarrow \beta) \land (\beta \Rightarrow \alpha)$.
   $(B_{1,1} \Rightarrow (P_{1,2} \lor P_{2,1})) \land ((P_{1,2} \lor P_{2,1}) \Rightarrow B_{1,1})$

2. Eliminate $\Rightarrow$, replacing $\alpha \Rightarrow \beta$ with $\neg\alpha \lor \beta$.
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg(P_{1,2} \lor P_{2,1}) \lor B_{1,1})$

3. Move $\neg$ inwards using de Morgan's rules and double-negation: $\neg(\alpha \lor \beta) = \neg\alpha \land \neg\beta$
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land ((\neg P_{1,2} \land \neg P_{2,1}) \lor B_{1,1})$

4. Apply distributive law ($\land$ over $\lor$) and flatten:
   $(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1}) \land (\neg P_{1,2} \lor B_{1,1}) \land (\neg P_{2,1} \lor B_{1,1})$

Example: Conversion to CNF

$B_{1,1} \Leftrightarrow (P_{1,2} \lor P_{2,1})$

5. KB is the conjunction of all of its sentences (all are true), so write each clause (disjunct) as a sentence in KB:

…
$(\neg B_{1,1} \lor P_{1,2} \lor P_{2,1})$
$(\neg P_{1,2} \lor B_{1,1})$
$(\neg P_{2,1} \lor B_{1,1})$
…

Resolution

- Resolution: inference rule for CNF: sound and complete! *

$(A \vee B \vee C)$

$(\neg A)$

"If A or B or C is true, but not A, then B or C must be true."

$\text{------------------}$

$\therefore (B \vee C)$

$(A \vee B \vee C)$

$(\neg A \vee D \vee E)$

"If A is false then B or C must be true, or if A is true then D or E must be true, hence since A is either true or false, B or C or D or E must be true."

$\therefore (B \vee C \vee D \vee E)$

$(A \vee B)$

$(\neg A \vee B)$

$\therefore (B \vee B) \equiv B$

Simplification

***** Resolution is "refutation complete" in that it can prove the truth of any entailed sentence by refutation.

Review: Resolution as Efficient Implication

(OR A B C D) ->Same -> (NOT (OR B C D)) => A
(OR ¬A E F G) ->Same -> A => (OR E F G)
----------------------------- ---
(OR B C D E F G) (NOT (OR B C D)) => (OR E F G)

(OR B C D E F G)

Resolution Algorithm

- The resolution algorithm tries to prove: $KB \models \alpha$ *equivalent to* $KB \wedge \neg\alpha$ *unsatisfiable*

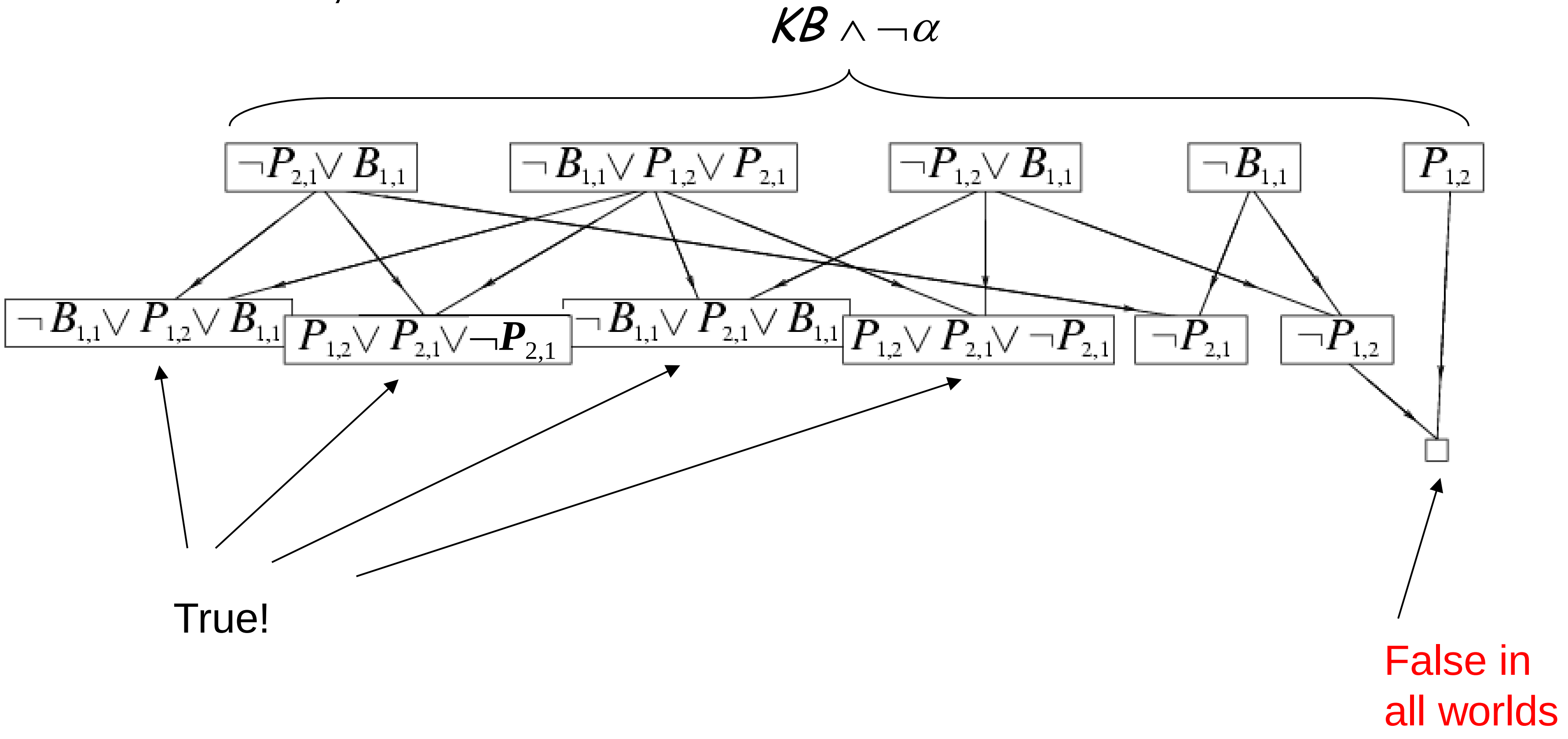
- Generate all new sentences from KB and the (negated) query.
- One of two things can happen:

1. We find $P \wedge \neg P$ which is unsatisfiable. I.e. we can entail the query.

2. We find no contradiction: there is a model that satisfies the sentence $KB \wedge \neg\alpha$ (non-trivial) and hence we cannot entail the query.

Resolution example

- $KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$



Try it Yourselves

- 7.9 page 238: (Adapted from Barwise and Etchemendy (1993).) If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.

- *Derive the KB in normal form.*
- *Prove: Horned, Prove: Magical.*

Exposes useful constraints

- **"You can't learn what you can't represent."** --- G. Sussman
- **In logic:** *If the unicorn is mythical, then it is immortal, but if it is not mythical, then it is a mortal mammal. If the unicorn is either immortal or a mammal, then it is horned. The unicorn is magical if it is horned.*

  Prove that the unicorn is both magical and horned.
- **A good representation makes this problem easy:**

$$(\neg Y \vee \neg R) \wedge (Y \vee R) \wedge (Y \vee M) \wedge (R \vee H) \wedge (\neg M \vee H) \wedge (\neg H \vee G)$$

Horn Clauses

- Resolution can be exponential in space and time.

- If we can reduce all clauses to "Horn clauses" resolution is linear in space and time

A clause with at most 1 positive literal.
e.g. $A \vee \neg B \vee \neg C$
- Every Horn clause can be rewritten as an implication with a conjunction of positive literals in the premises and a single positive literal as a conclusion.

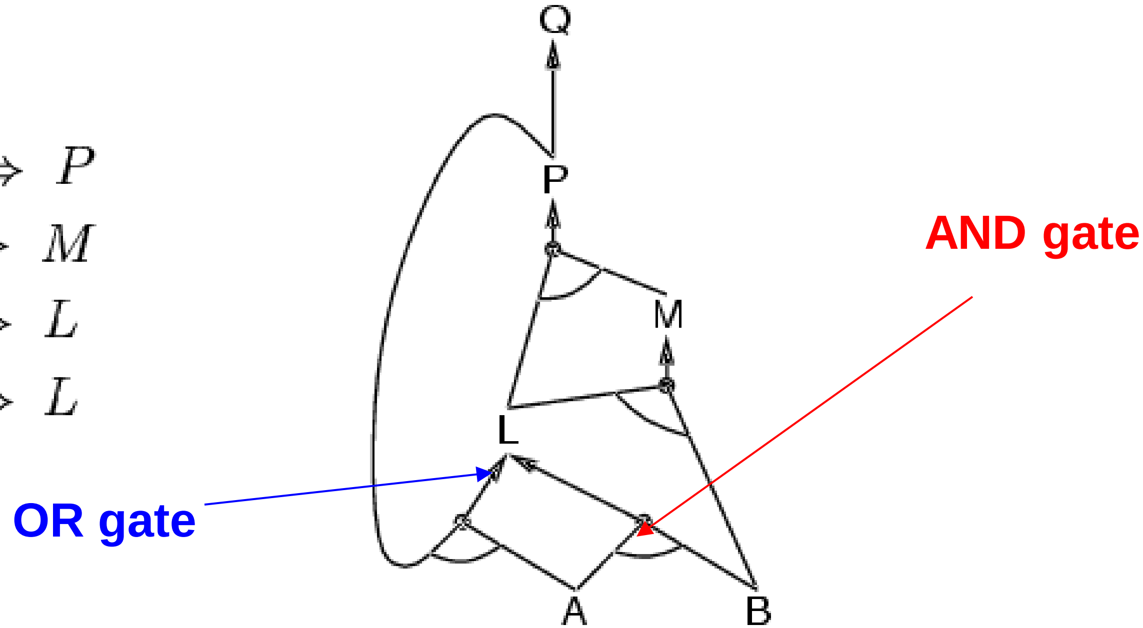
e.g. $B \wedge C \Rightarrow A$
- 1 positive literal: definite clause
- 0 positive literals: integrity constraint:
- e.g. $(\neg A \vee \neg B) \equiv (A \wedge B \Rightarrow False)$
- 0 negative literals: fact
- Forward Chaining and Backward chaining are sound and complete with Horn clauses and run linear in space and time.

Forward chaining (FC)

- Idea: fire any rule whose premises are satisfied in the *KB*, add its conclusion to the *KB*, until query is found.

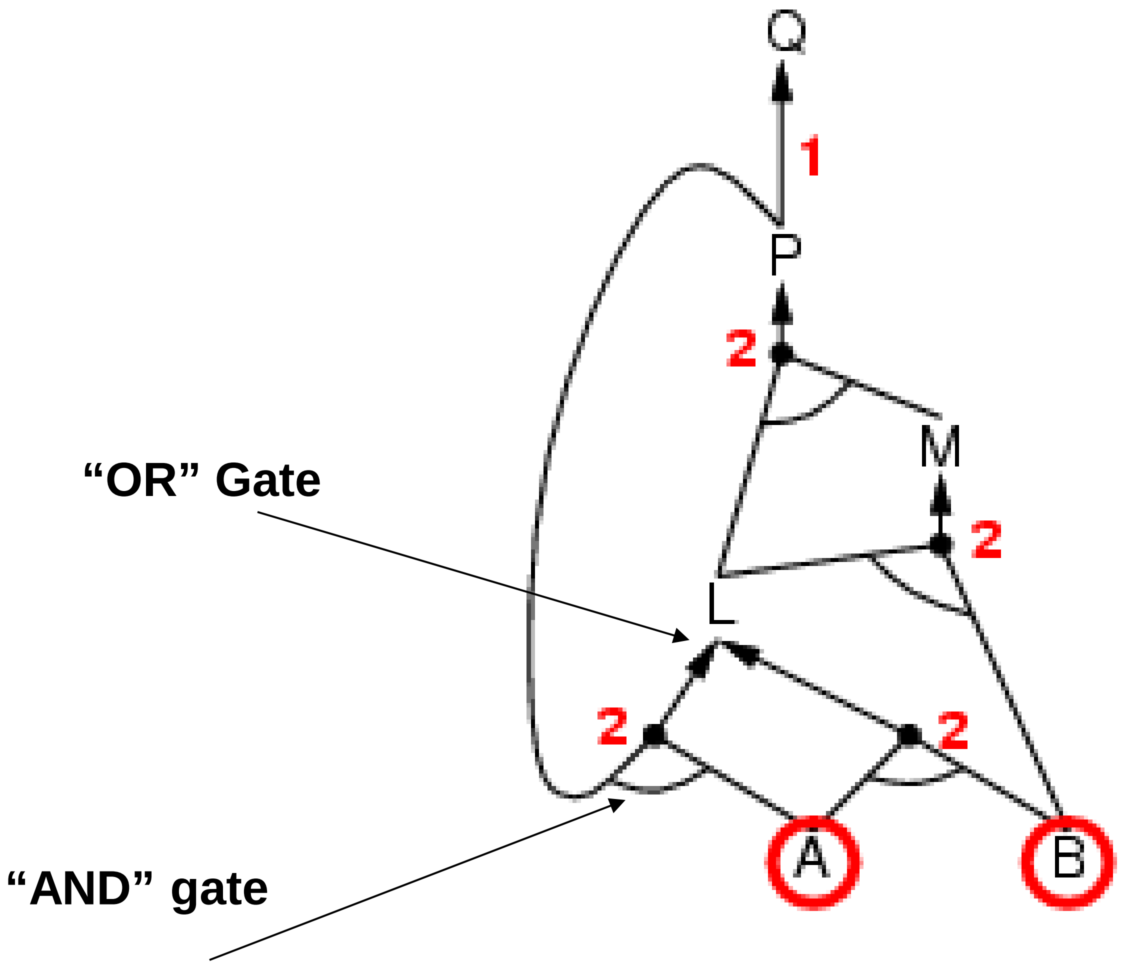
- This proves that $KB \Rightarrow Q$ is true in all possible worlds (i.e. trivial), and hence it proves entailment.

$$
\begin{aligned}
P &\Rightarrow Q \\
L \wedge M &\Rightarrow P \\
B \wedge L &\Rightarrow M \\
A \wedge P &\Rightarrow L \\
A \wedge B &\Rightarrow L \\
A & \\
B &
\end{aligned}
$$

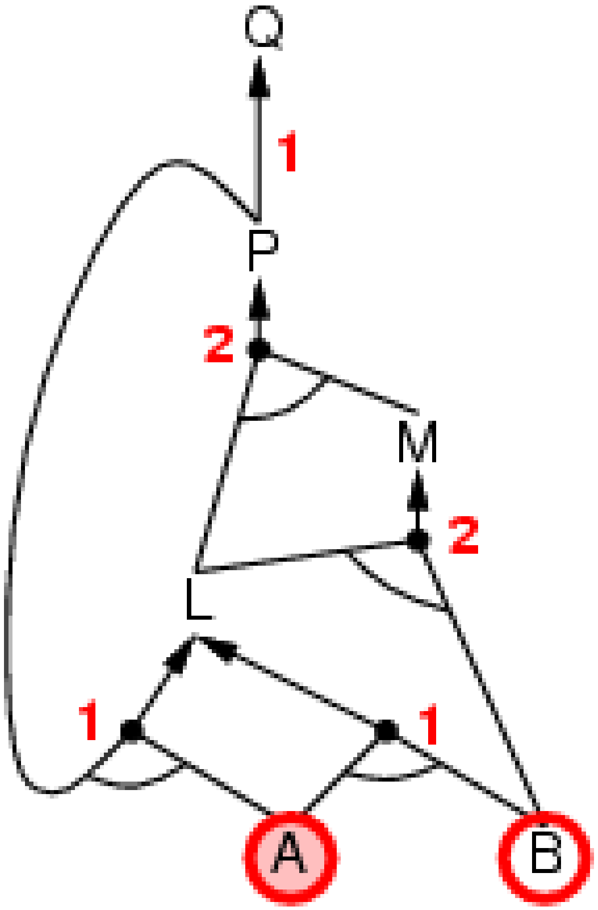


- Forward chaining is sound and complete for Horn KB

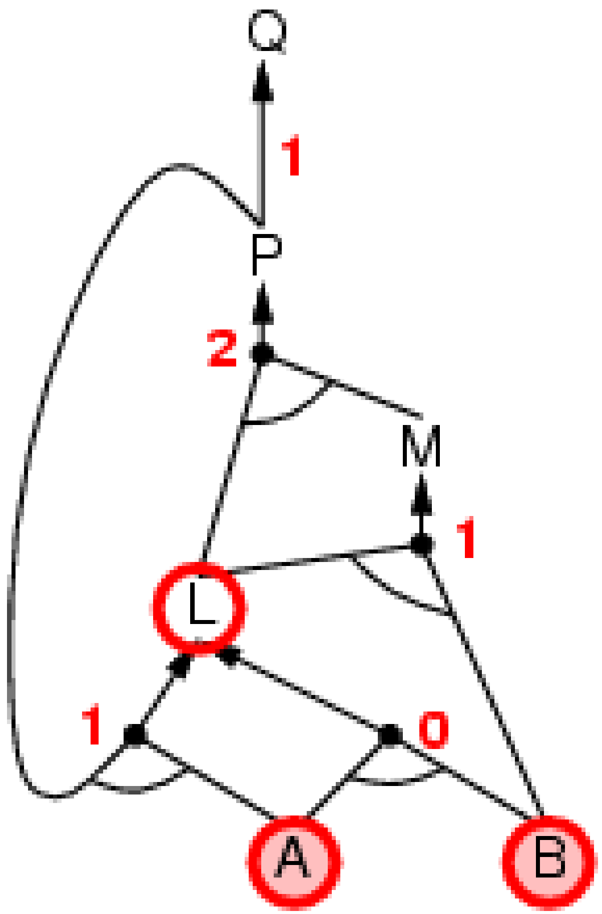
Forward chaining example



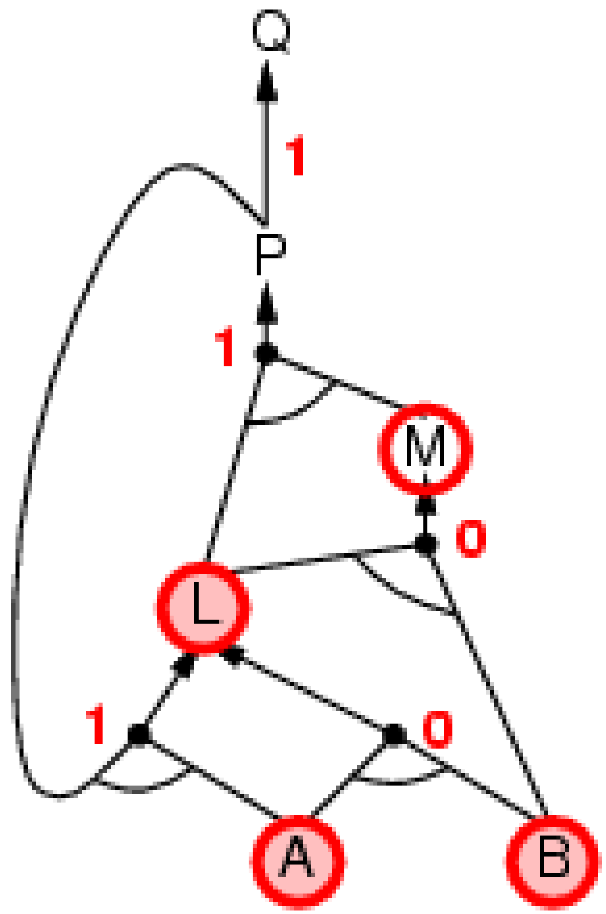
Forward chaining example



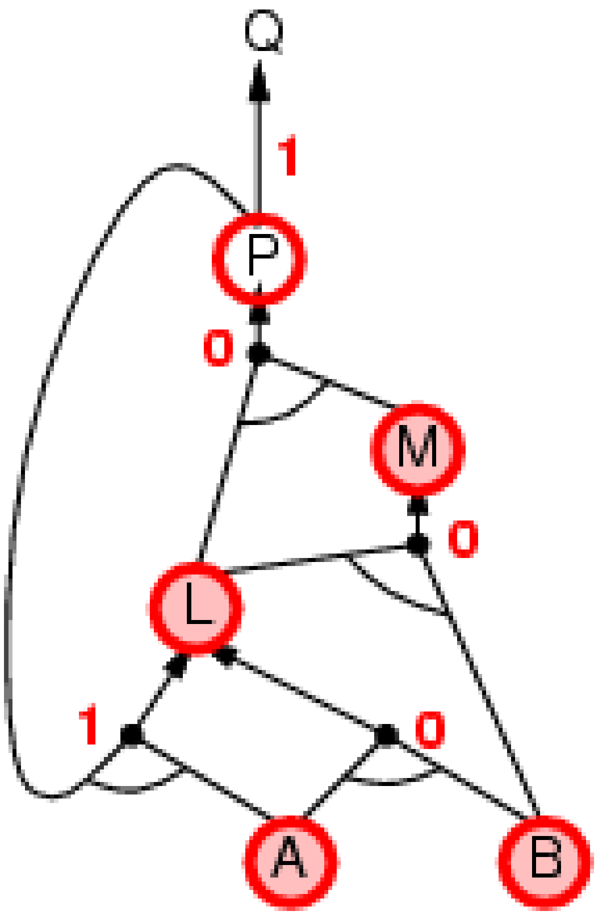
Forward chaining example



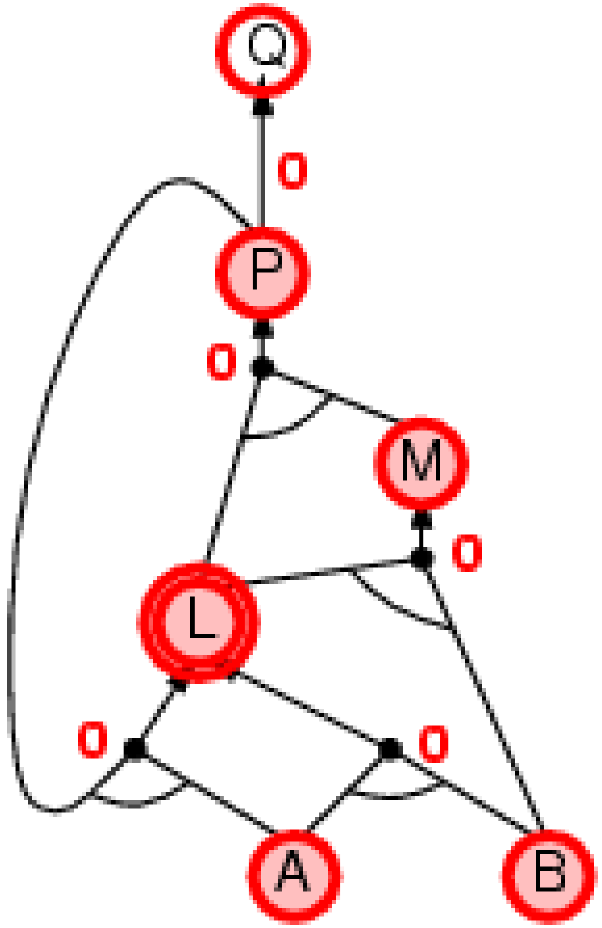
Forward chaining example



Forward chaining example



Forward chaining example



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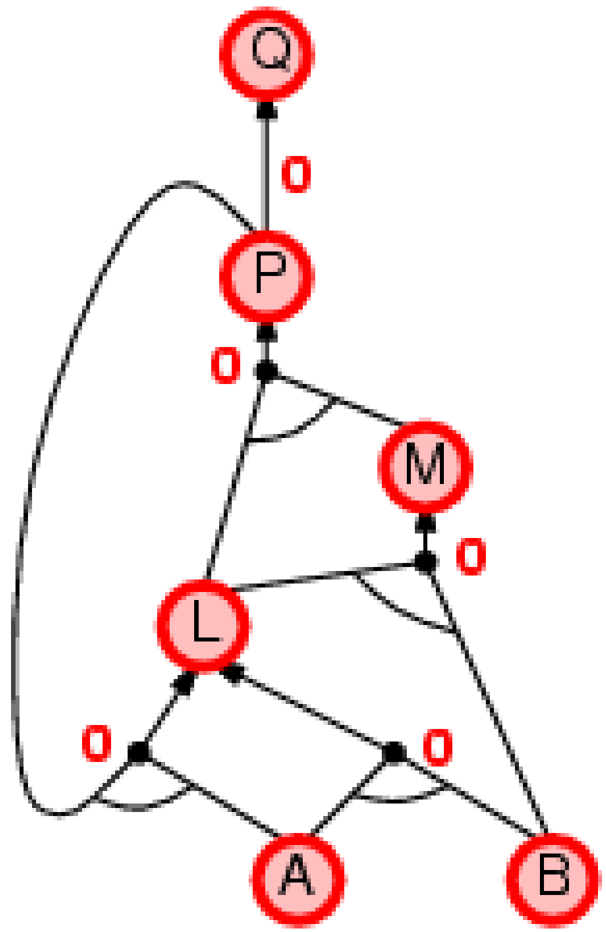


Backward chaining (BC)

Idea: work backwards from the query *q*

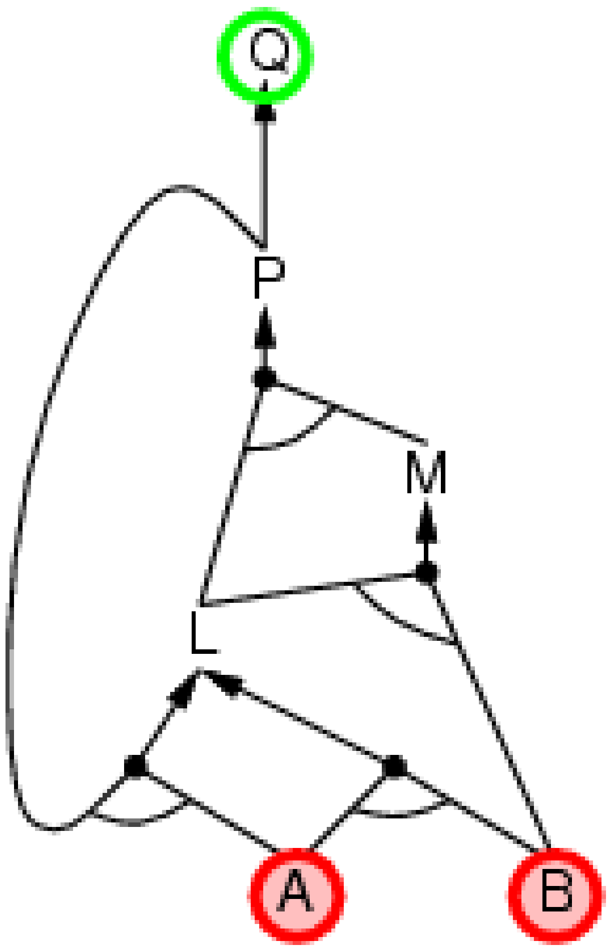
- check if *q* is known already, or
- prove by BC all premises of some rule concluding *q*
- Hence BC maintains a stack of sub-goals that need to be proved to get to q.

Avoid loops: check if new sub-goal is already on the goal stack

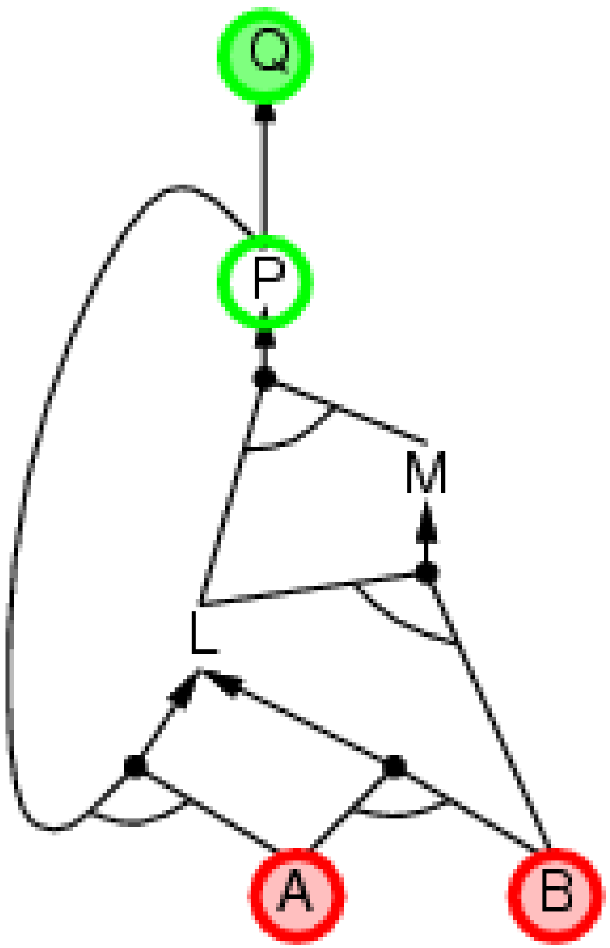
Avoid repeated work: check if new sub-goal

1. has already been proved true, or
2. has already failed

Backward chaining example



Backward chaining example



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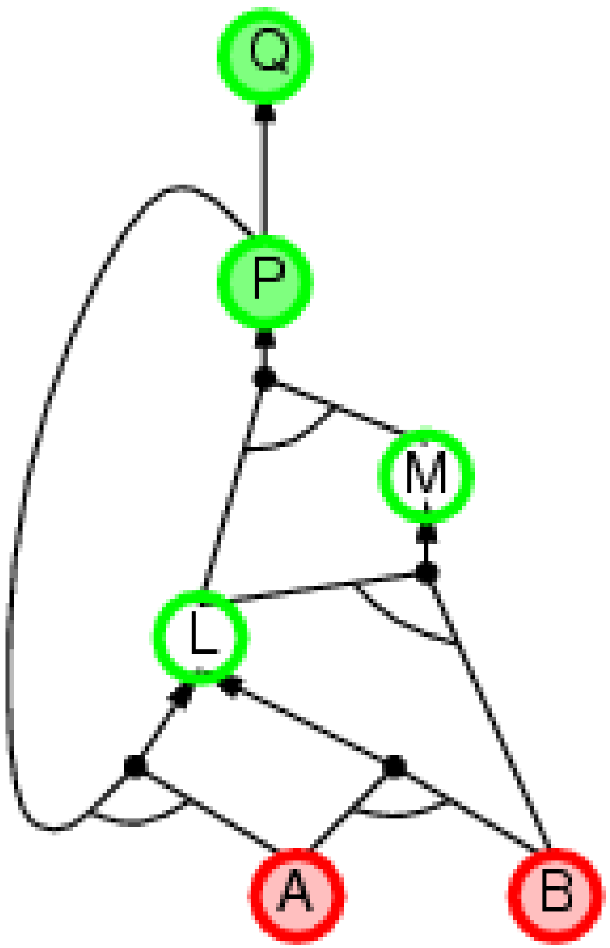
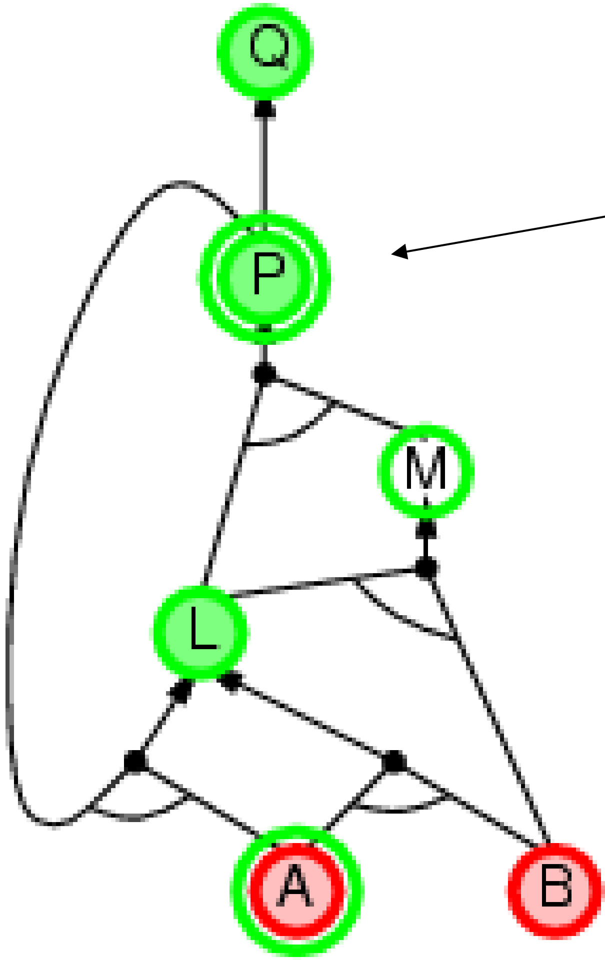
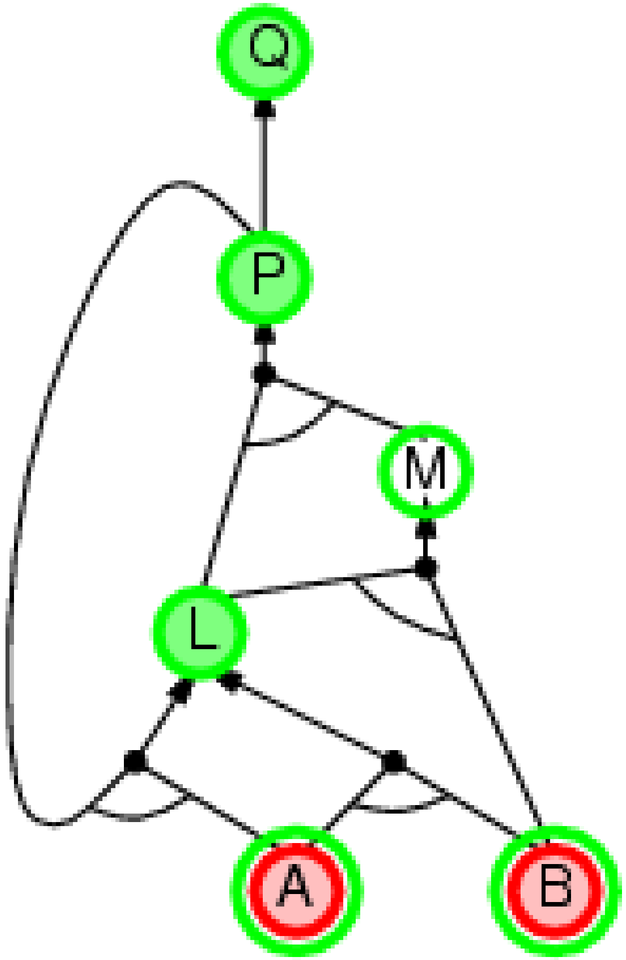


Backward chaining example



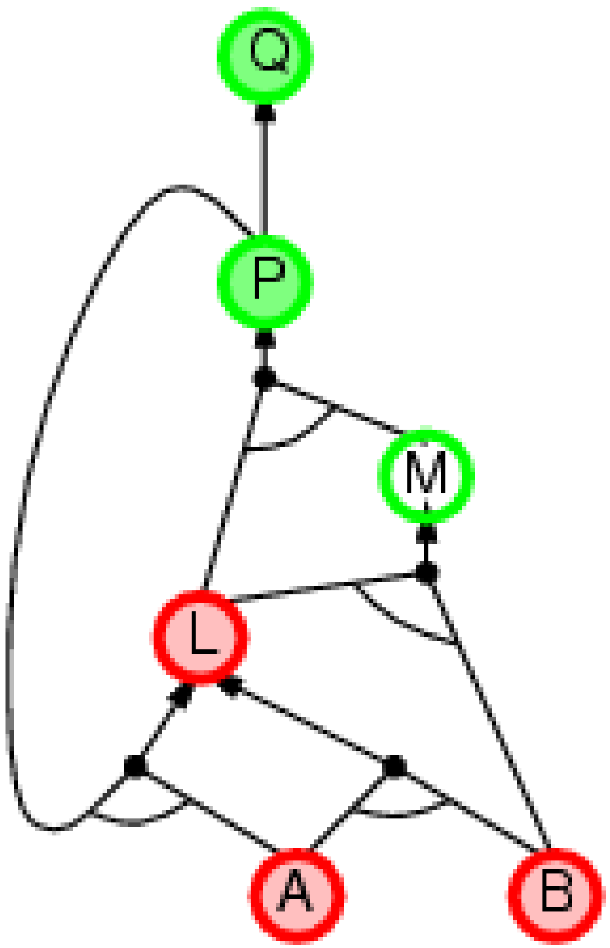
we need P to prove
L and L to prove P.

Backward chaining example



As soon as you can move forward, do so.

Backward chaining example



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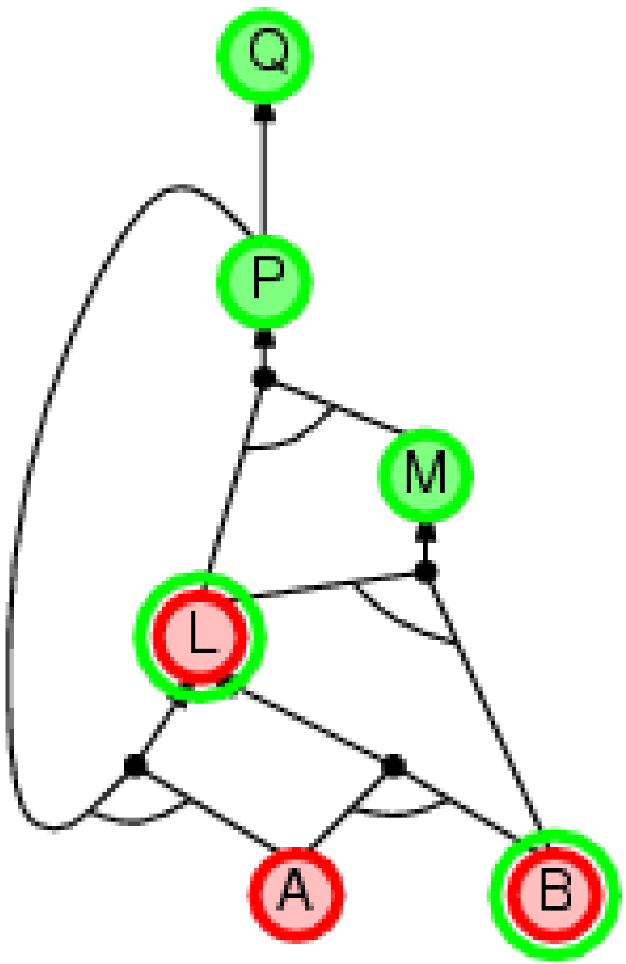

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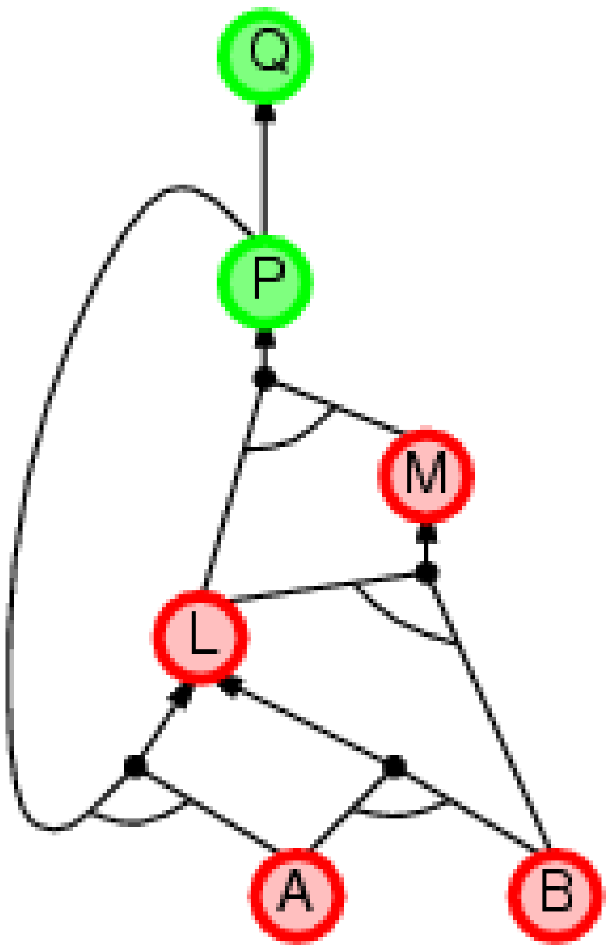

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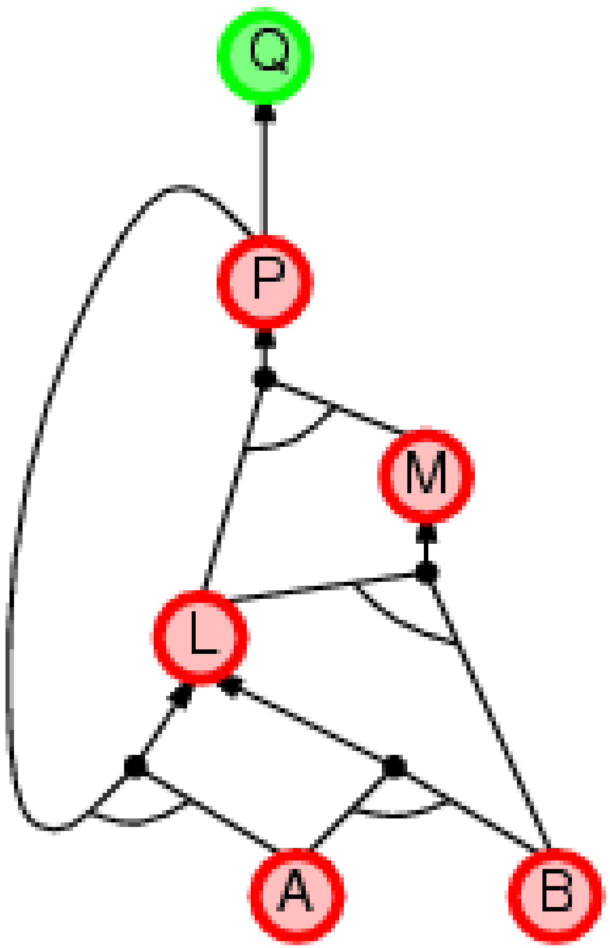

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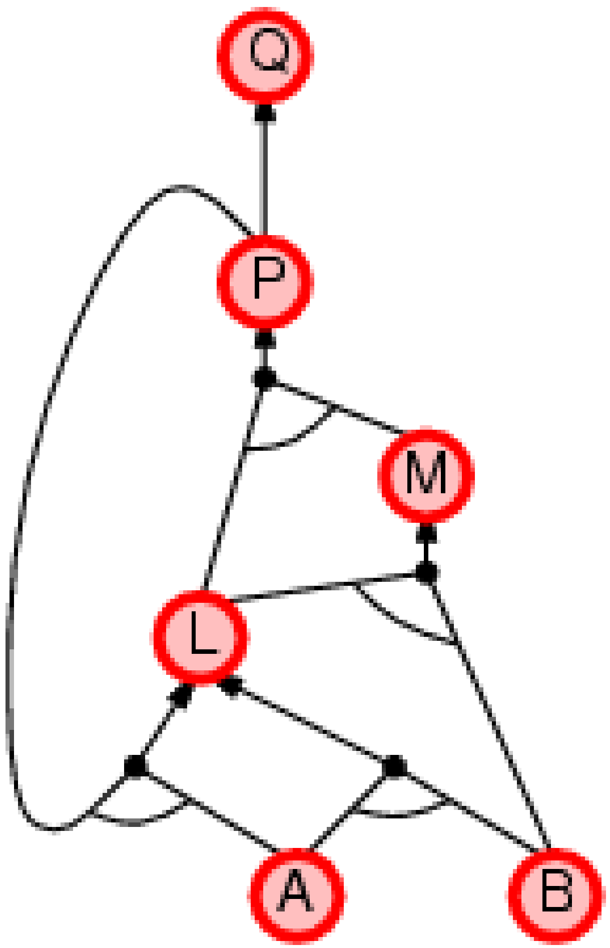


Forward vs. backward chaining

- FC is data-driven, automatic, unconscious processing,
  - e.g., object recognition, routine decisions
- May do lots of work that is irrelevant to the goal
- BC is goal-driven, appropriate for problem-solving,
  - e.g., Where are my keys? How do I get into a PhD program?
- Complexity of BC can be much less than linear in size of KB

Model Checking

Two families of efficient algorithms:

- Complete backtracking search algorithms:
  - E.g., DPLL algorithm
- Incomplete local search algorithms
  - E.g., `WalkSAT` algorithm

The DPLL algorithm

Determine if an input propositional logic sentence (in CNF) is satisfiable. This is just backtracking search for a CSP.

Improvements:

1. Early termination

   A clause is true if any literal is true.

   A sentence is false if any clause is false.

2. Pure symbol heuristic

   Pure symbol: always appears with the same "sign" in all clauses.

   e.g., In the three clauses $(A \vee \neg B)$, $(\neg B \vee \neg C)$, $(C \vee A)$, A and B are pure, C is impure.

   Make a pure symbol literal true. (if there is a model for S, then making a pure symbol true is also a model).

3 Unit clause heuristic

   Unit clause: only one literal in the clause

   The only literal in a unit clause must be true.

   Note: literals can become a pure symbol or a unit clause when other literals obtain truth values. e.g.

$$(A \cancel{\vee True}) \wedge (\neg A \vee B)$$
$$A = pure$$

The `WalkSAT` algorithm

- Incomplete, local search algorithm
- Evaluation function: The min-conflict heuristic of minimizing the number of unsatisfied clauses
- Balance between greediness and randomness

Walksat Procedure

Start with random initial assignment.

Pick a random unsatisfied clause.

Select and flip a variable from that clause:

- With probability p, pick a random variable.
- With probability 1-p, pick greedily
  - a variable that minimizes the number of unsatisfied clauses

Repeat to predefined maximum number flips;
if no solution found, restart.

Hard satisfiability problems

- Consider *random* 3-CNF sentences. e.g.,

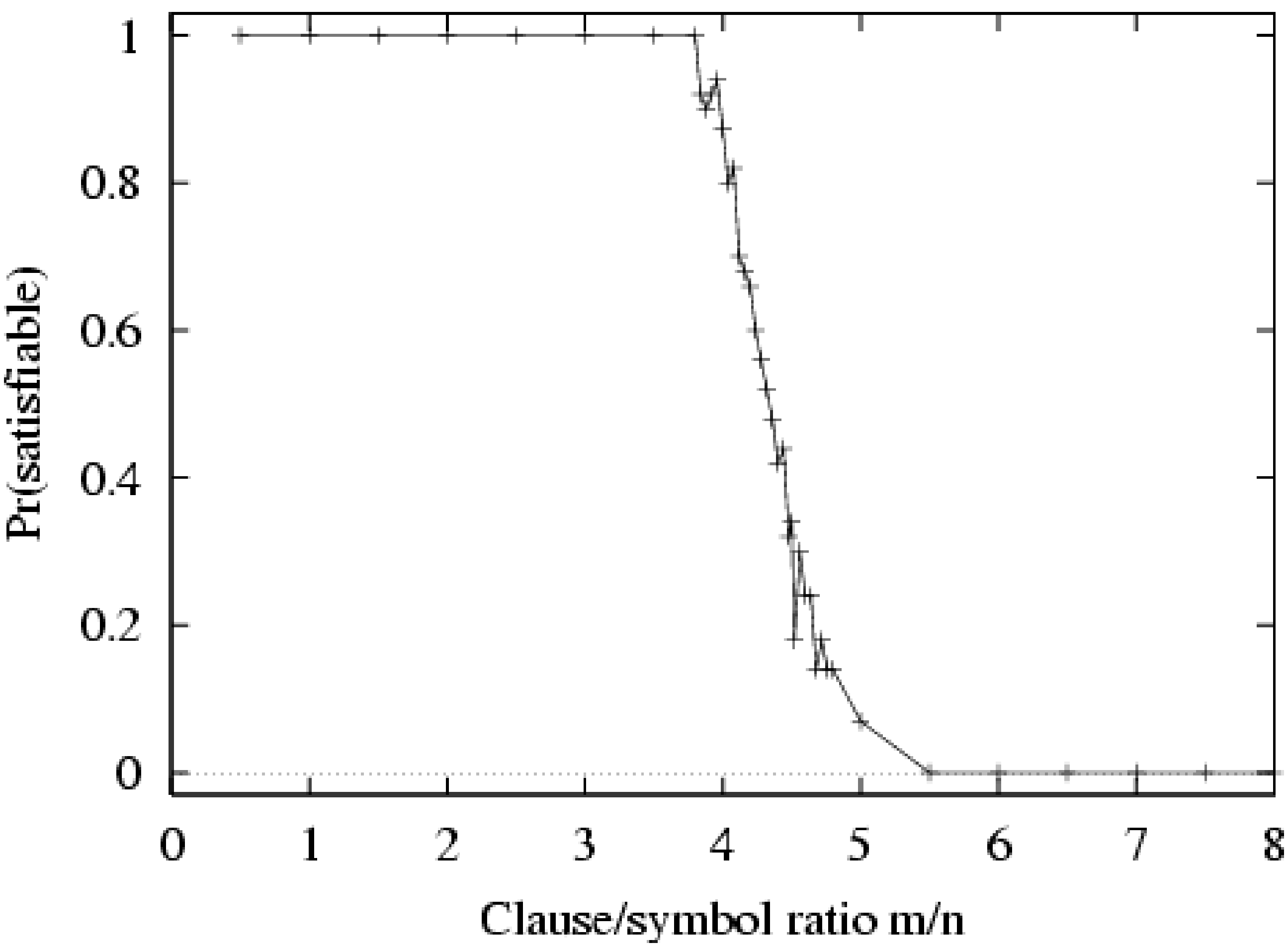
  $(\neg D \lor \neg B \lor C) \land (B \lor \neg A \lor \neg C) \land (\neg C \lor \neg B \lor E) \land (E \lor \neg D \lor B) \land (B \lor E \lor \neg C)$

  m = number of clauses (5)

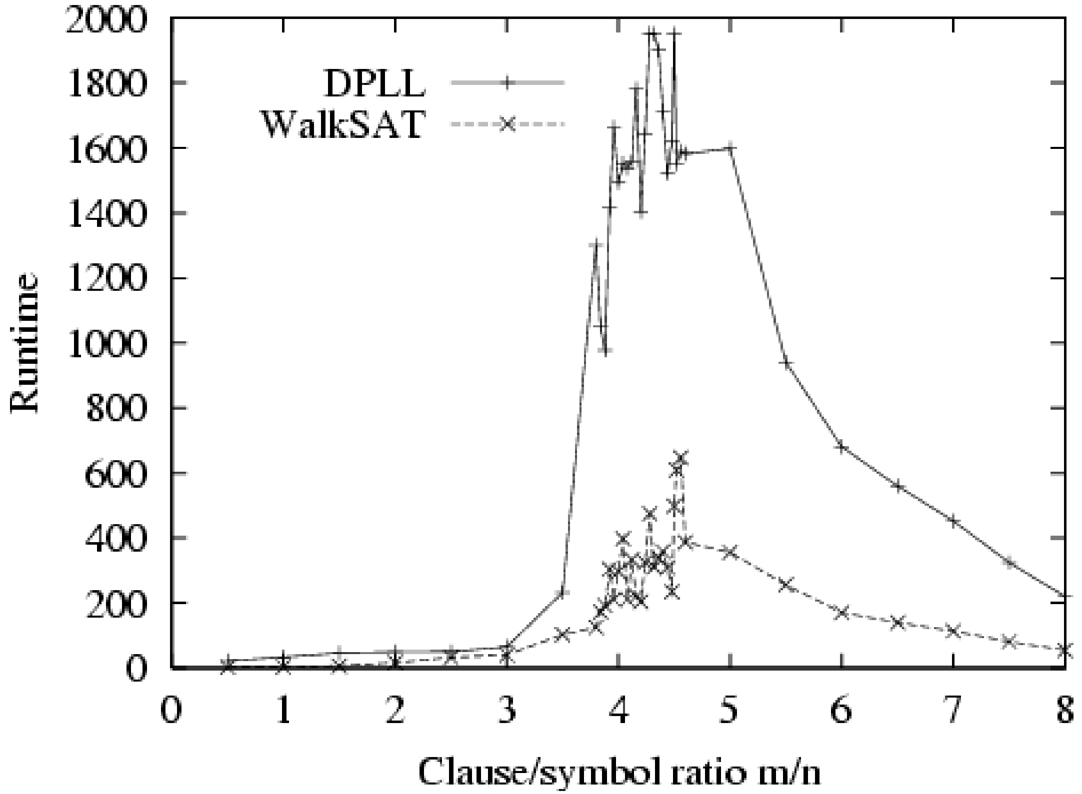
  n = number of symbols (5)

  – Hard problems seem to cluster near $m/n = 4.3$ (critical point)

Hard satisfiability problems



Hard satisfiability problems



- Median runtime for 100 satisfiable random 3-CNF sentences, $n = 50$

Common Sense Reasoning

Example, adapted from Lenat

You are told: John drove to the grocery store and bought a pound of noodles, a pound of ground beef, and two pounds of tomatoes.

- Is John 3 years old?
- Is John a child?
- What will John do with the purchases?
- Did John have any money?
- Does John have less money after going to the store?
- Did John buy at least two tomatoes?
- Were the tomatoes made in the supermarket?
- Did John buy any meat?
- Is John a vegetarian?
- Will the tomatoes fit in John's car?

- Can Propositional Logic support these inferences?

Summary

- Logical agents apply inference to a knowledge base to derive new information and make decisions
- Basic concepts of logic:
  - syntax: formal structure of sentences
  - semantics: truth of sentences wrt models
  - entailment: necessary truth of one sentence given another
  - inference: deriving sentences from other sentences
  - soundness: derivations produce only entailed sentences
  - completeness: derivations can produce all entailed sentences
- Resolution is complete for propositional logic.
  Forward and backward chaining are linear-time, complete for Horn clauses
- Propositional logic lacks expressive power