Knowledge Representation using First-Order Logic

Reading: Chapter 8, 9.1-9.2

First lecture slides read: 8.1-8.2

Second lecture slides read: 8.3-8.4

Third lecture slides read: Chapter 9.1-9.2

(lecture slides spread across two class sessions)

(Please read lecture topic material before and after each lecture on that topic)

Review: KB |= S means |= (KB $\Rightarrow$ S)

- KB |= S is read "KB entails S."
  - Means "S is true in every world (model) in which KB is true."
  - Means "In the world, S follows from KB."
- KB |= S is equivalent to |= (KB $\Rightarrow$ S)
  - Means "(KB $\Rightarrow$ S) is true in every world (i.e., is valid)."
- And so: { } |= S is equivalent to |= ({ } $\Rightarrow$ S)
- So what does ({ } $\Rightarrow$ S) mean?
  - Means "True implies S."
  - Means "S is valid."
  - In Horn form, means "S is a fact." p. 256 (3rd ed.; p. 281, 2nd ed.)
- Why does { } mean True here, but False in resolution proofs?

Review: (True $\Rightarrow$ S) means "S is a fact."

- By convention,
  - The null conjunct is "syntactic sugar" for True.
  - The null disjunct is "syntactic sugar" for False.
  - Each is assigned the truth value of its identity element.
    - For conjuncts, True is the identity: $(A \wedge \text{True}) \equiv A$
    - For disjuncts, False is the identity: $(A \vee \text{False}) \equiv A$

- A KB is the conjunction of all of its sentences.
  - So in the expression: { } |= S
    - We see that { } is the null conjunct and means True.
  - The expression means "S is true in every world where True is true."
    - I.e., "S is valid."
  - Better way to think of it: { } does not exclude any worlds (models).

- In Conjunctive Normal Form each clause is a disjunct.
  - So in, say, KB = { (P Q) ($\neg$Q R) () (X Y $\neg$Z) }
    - We see that () is the null disjunct and means False.

Side Trip: Functions AND, OR, and null values (Note: These are "syntactic sugar" in logic.)

function AND(*arglist*) **returns** a truth-value
  return ANDOR(*arglist,* True)

function OR(*arglist*) **returns** a truth-value
  return ANDOR(*arglist,* False)

function ANDOR(*arglist, nullvalue*) **returns** a truth-value
  /* *nullvalue* is the identity element for the caller. */
  if (*arglist* = { })
    then return *nullvalue*
  if (FIRST(*arglist*) = NOT(*nullvalue*))
    then return NOT(*nullvalue*)
  return ANDOR(REST(*arglist*))

Review: Resolution as Implication

(OR A B C D)	->Same ->	(NOT (OR B C D)) => A
(OR ¬A E F G)	->Same ->	A => (OR E F G)
------------------------------		--
(OR B C D E F G)		(NOT (OR B C D)) => (OR E F G)
		--
		(OR B C D E F G)

Outline

- Propositional Logic is **Useful** --- but has **Limited Expressive Power**
- First Order Predicate Calculus (FOPC), or First Order Logic (FOL).
  - FOPC has greatly expanded expressive power, though still limited.
- New Ontology
  - The world consists of OBJECTS (for propositional logic, the world was facts).
  - OBJECTS have PROPERTIES and engage in RELATIONS and FUNCTIONS.
- New Syntax
  - Constants, Predicates, Functions, Properties, Quantifiers.
- New Semantics
  - Meaning of new syntax.
- Knowledge engineering in FOL
- Required Reading:
  - For today, all of Chapter 8; for next lecture, all of Chapter 9.

You will be expected to know

- FOPC syntax and semantics
  - Syntax: Sentences, predicate symbols, function symbols, constant symbols, variables, quantifiers
  - Semantics: Models, interpretations
- De Morgan's rules for quantifiers
  - connections between $\forall$ and $\exists$
- Nested quantifiers
  - Difference between "$\forall$ x $\exists$ y P(x, y)" and "$\exists$ x $\forall$ y P(x, y)"
  - $\forall$ x $\exists$ y Likes(x, y)
  - $\exists$ x $\forall$ y Likes(x, y)
- Translate simple English sentences to FOPC and back
  - $\forall$ x $\exists$ y Likes(x, y) $\Leftrightarrow$ "Everyone has someone that they like."
  - $\exists$ x $\forall$ y Likes(x, y) $\Leftrightarrow$ "There is someone who likes every person."

Common Sense Reasoning

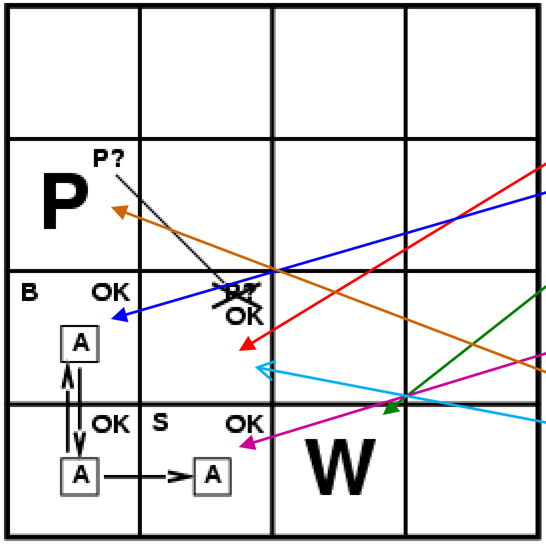
Example, adapted from Lenat

You are told: John drove to the grocery store and bought a pound of noodles, a pound of ground beef, and two pounds of tomatoes.

- Is John 3 years old?
- Is John a child?
- What will John do with the purchases?
- Did John have any money?
- Does John have less money after going to the store?
- Did John buy at least two tomatoes?
- Were the tomatoes made in the supermarket?
- Did John buy any meat?
- Is John a vegetarian?
- Will the tomatoes fit in John's car?

- Can Propositional Logic support these inferences?

Exploring a Wumpus world

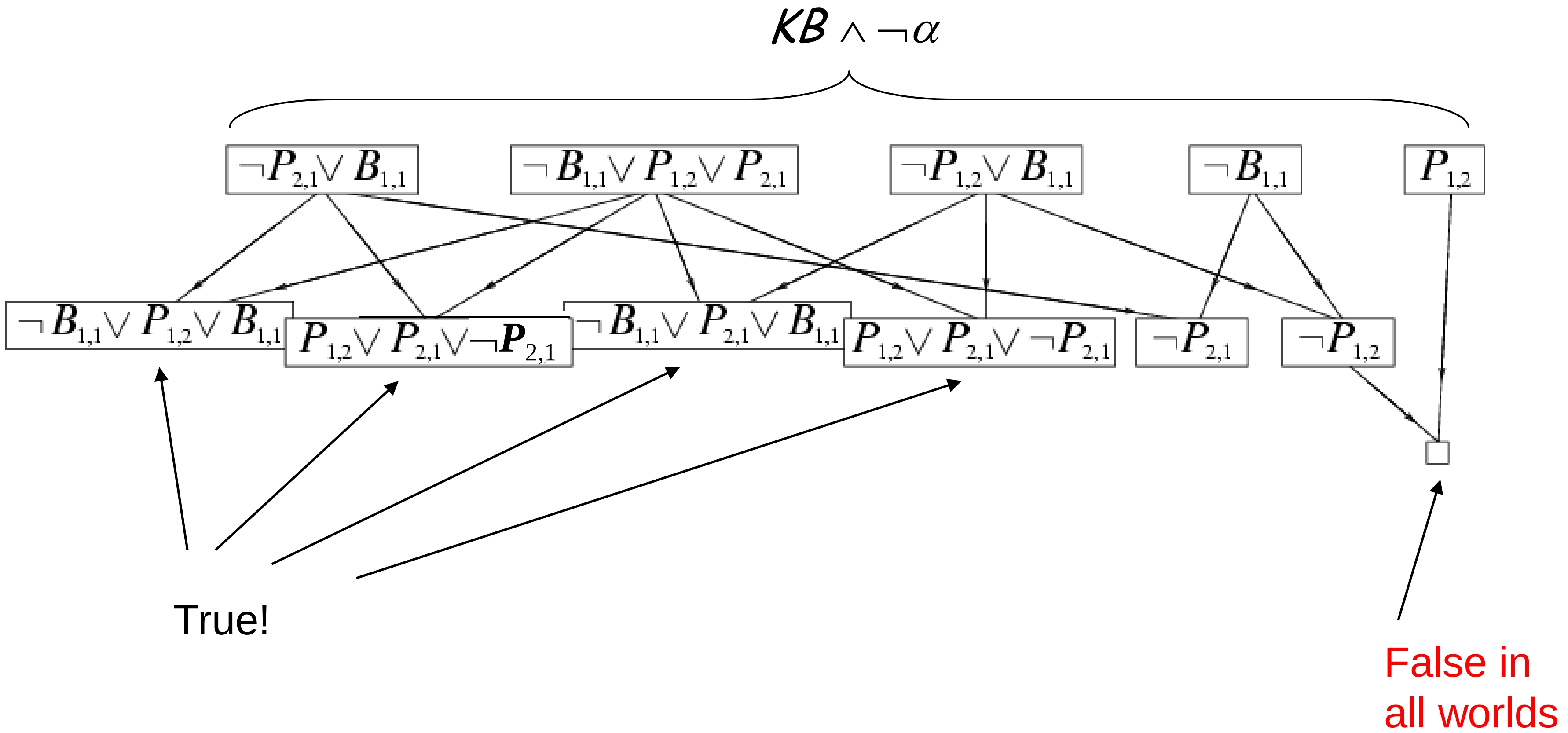


If the Wumpus were here, stench should be here. Therefore it is here.
Since, there is no breeze here, the pit must be there, and it must be OK here

We need rather sophisticated reasoning here!

Resolution example

- $KB = (B_{1,1} \Leftrightarrow (P_{1,2} \vee P_{2,1})) \wedge \neg B_{1,1}$
- $\alpha = \neg P_{1,2}$



Pros and cons of propositional logic

☺ Propositional logic is declarative

- Knowledge and inference are separate

☺ Propositional logic allows partial/disjunctive/negated information

– unlike most programming languages and databases

☺ Propositional logic is compositional:

– meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$

☺ Meaning in propositional logic is context-independent

– unlike natural language, where meaning depends on context

☹ Propositional logic has **limited expressive power**

– E.g., cannot say "Pits cause breezes in adjacent squares."
  - except by writing one sentence for each square

– Needs to refer to objects in the world,

– Needs to express general rules

First-Order Logic (FOL), also called First-Order Predicate Calculus (FOPC)

- Propositional logic assumes the world contains facts.

- First-order logic (like natural language) assumes the world contains

  - **Objects:** people, houses, numbers, colors, baseball games, wars, …
  - **Functions:** father of, best friend, one more than, plus, …
    - Function arguments are objects; function returns an object
  - **Objects generally correspond to English NOUNS**

  - **Predicates/Relations/Properties:** red, round, prime, brother of, bigger than, part of, comes between, …
    - Predicate arguments are objects; predicate returns a truth value
  - **Predicates generally correspond to English VERBS**
    - **First argument is generally the subject, the second the object**

Aside: First-Order Logic (FOL) vs. Second-Order Logic

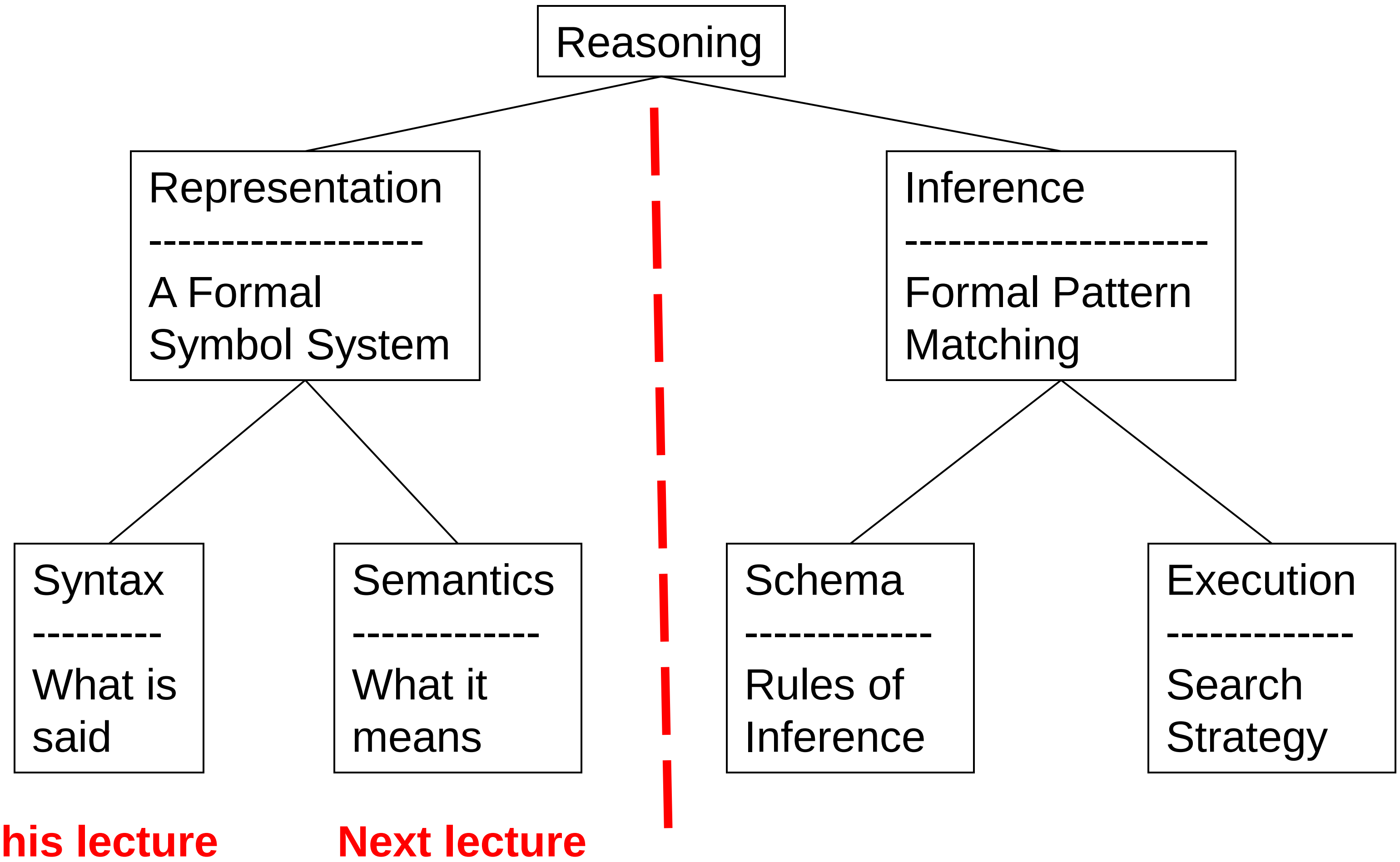
- First Order Logic (FOL) allows variables and general rules
  - "First order" because quantified variables represent objects.
  - "Predicate Calculus" because it quantifies over predicates on objects.
    - E.g., "Integral Calculus" quantifies over functions on numbers.
- Aside: Second Order logic
  - "Second order" because quantified variables can also represent predicates and functions.
    - E.g., can define "Transitive Relation," which is beyond FOPC.
- Aside: In FOL we can state that a relationship is transitive
  - E.g., BrotherOf is a transitive relationship
  - $\forall$ x, y, z BrotherOf(x,y) $\wedge$ BrotherOf(y,z) => BrotherOf(x,z)
- Aside: In Second Order logic we can define "Transitive"
  - $\forall$ P, x, y, z Transitive(P) $\Leftrightarrow$ (P(x,y) $\wedge$ P(y,z) => P(x,z))
  - Then we can state directly, Transitive(BrotherOf)

FOL (or FOPC) Ontology:

What kind of things exist in the world?

What do we need to describe and reason about?

Objects --- with their relations, functions, predicates, properties, and general rules.



Syntax of FOL: Basic elements

- Constants KingJohn, 2, UCI,...
- Predicates Brother, >,...
- Functions Sqrt, LeftLegOf,...
- Variables x, y, a, b,...
- Connectives $\neg, \Rightarrow, \wedge, \vee, \Leftrightarrow$
- Equality =
- Quantifiers $\forall, \exists$

Syntax of FOL: Basic syntax elements are symbols

- **Constant** Symbols (correspond to English nouns)
  - Stand for objects in the world.
    - E.g., KingJohn, 2, UCI, ...

- **Predicate** Symbols (correspond to English verbs)
  - Stand for relations (maps a tuple of objects to a **truth-value**)
    - E.g., Brother(Richard, John), greater_than(3,2), ...
  - P(x, y) is usually read as “x is P of y.”
    - E.g., Mother(Ann, Sue) is usually “Ann is Mother of Sue.”

- **Function** Symbols (correspond to English nouns)
  - Stand for functions (maps a tuple of objects to an **object**)
    - E.g., Sqrt(3), LeftLegOf(John), ...

- **Model** (world) = set of domain objects, relations, functions
- **Interpretation** maps symbols onto the model (world)
  - Very many interpretations are possible for each KB and world!
  - Job of the KB is to rule out models inconsistent with our knowledge.

Syntax : Relations, Predicates, Properties, Functions

- Mathematically, all the Relations, Predicates, Properties, and Functions CAN BE represented simply as sets of *m*-tuples of objects:

- Let *W* be the set of objects in the world.

- Let $W^m = W \times W \times \ldots (m \text{ times}) \ldots \times W$
  - The set of all possible *m*-tuples of objects from the world

- An ***m*-ary Relation** is a subset of W^m.
  - Example: Let *W* = *{John, Sue, Bill}*
  - Then W^2 = *{<John, John>, <John, Sue>, …, <Sue, Sue>}*
  - E.g., *MarriedTo* = *{<John, Sue>, <Sue, John>}*
  - E.g., *FatherOf* = *{<John, Bill>}*

- Analogous to a constraint in CSPs
  - The constraint lists the *m*-tuples that satisfy it.
  - The relation lists the *m*-tuples that participate in it.

Syntax : Relations, Predicates, Properties, Functions

- A **Predicate** is a list of *m*-tuples making the predicate true.
  - E.g., PrimeFactorOf = {<2,4>, <2,6>, <3,6>, <2,8>, <3,9>, ...}
  - This is the same as an *m*-ary Relation.
  - Predicates (and properties) generally correspond to English verbs.

- A **Property** lists the m-tuples that have the property.
  - Formally, it is a predicate that is true of tuples having that property.
  - E.g., IsRed = {<Ball-5>, <Toy-7>, <Car-11>, ...}
  - This is the same as an *m*-ary Relation.

- A **Function** CAN BE represented as an *m*-ary relation
  - the first *(m-1)* objects are the arguments and the m^{th} is the value.
  - E.g., Square = *{<1, 1>, <2, 4>, <3, 9>, <4, 16>, ...}*

- An **Object** CAN BE represented as a function of zero arguments that returns the object.
  - This is just a 1-ary relationship.

Syntax of FOL: Terms

- **Term** = logical expression that **refers to an object**

- **There are two kinds of terms:**

  - **Constant Symbols** stand for (or name) objects:
    - E.g., KingJohn, 2, UCI, Wumpus, ...

  - **Function Symbols** map tuples of objects to an object:
    - E.g., LeftLeg(KingJohn), Mother(Mary), Sqrt(x)
    - This is nothing but a complicated kind of name
      - No “subroutine” call, no “return value”

Syntax of FOL: Atomic Sentences

- **Atomic Sentences** state facts (logical truth values).
  - An **atomic sentence** is a Predicate symbol, optionally followed by a parenthesized list of any argument terms
  - E.g., *Married(Father(Richard), Mother(John))*
  - An **atomic sentence** asserts that some relationship (some predicate) holds among the objects that are its arguments.

- An **Atomic Sentence is true** in a given model if the relation referred to by the predicate symbol holds among the objects (terms) referred to by the arguments.

Syntax of FOL: Atomic Sentences

- Atomic sentences in logic state facts that are true or false.

- Properties and *m*-ary relations do just that:

  LargerThan(2, 3) is false.

  BrotherOf(Mary, Pete) is false.

  Married(Father(Richard), Mother(John)) could be true or false.

  Properties and *m*-ary relations are Predicates that are true or false.

- Note: Functions refer to objects, do not state facts, and form no sentence:
  - Brother(Pete) refers to John (his brother) and is neither true nor false.
  - Plus(2, 3) refers to the number 5 and is neither true nor false.

- BrotherOf(Pete, Brother(Pete)) is True.

Binary relation
is a truth value.

Function refers to John, an object in the
world, i.e., John is Pete's brother.
(Works well iff John is Pete's only brother.)

Syntax of FOL: Connectives & Complex Sentences

- **Complex Sentences** are formed in the same way, and are formed using the same logical connectives, as we already know from propositional logic

- The **Logical Connectives**:
  - $\Leftrightarrow$ biconditional
  - $\Rightarrow$ implication
  - $\wedge$ and
  - $\vee$ or
  - $\neg$ negation

- **Semantics** for these logical connectives are the same as we already know from propositional logic.

Complex Sentences

- We make complex sentences with connectives (just like in propositional logic).

property

$$\neg Brother(LeftLeg(Richard), John) \lor (Democrat(Bush))$$

binary relation

function

objects

connectives

Examples

- Brother(Richard, John) $\wedge$ Brother(John, Richard)

- King(Richard) $\vee$ King(John)

- King(John) => $\neg$ King(Richard)

- LessThan(Plus(1,2) ,4) $\wedge$ GreaterThan(1,2)

(Semantics are the same as in propositional logic)

Syntax of FOL: Variables

- **Variables** range over objects in the world.
- A **variable** is like a **term** because it represents an object.
- A **variable** may be used wherever a **term** may be used.
  - **Variables** may be arguments to functions and predicates.
- (A **term with NO variables** is called a **ground term**.)
- (A **variable not bound by a quantifier** is called **free**.)

Syntax of FOL: Logical Quantifiers

- There are two **Logical Quantifiers:**
  - **Universal:** $\forall$ x P(x) means "For all x, P(x)."
    - The "upside-down A" reminds you of "ALL."
  - **Existential:** $\exists$ x P(x) means "There exists x such that, P(x)."
    - The "upside-down E" reminds you of "EXISTS."

- Syntactic "sugar" --- we really only need one quantifier.
  - $\forall$ x P(x) $\equiv \neg\exists$ x $\neg$P(x)
  - $\exists$ x P(x) $\equiv \neg\forall$ x $\neg$P(x)
  - You can ALWAYS convert one quantifier to the other.

- **RULES:** $\forall \equiv \neg\exists\neg$ and $\exists \equiv \neg\forall\neg$

- **RULE:** To move negation "in" across a quantifier,
  change the quantifier to "the other quantifier"
  and negate the predicate on "the other side."
  - $\neg\forall$ x P(x) $\equiv \exists$ x $\neg$P(x)
  - $\neg\exists$ x P(x) $\equiv \forall$ x $\neg$P(x)

Universal Quantification $\forall$

- $\forall$ means “for all”

- Allows us to make statements about all objects that have certain properties

- Can now state general rules:

  $\forall$ x King(x) => Person(x) “All kings are persons.”

  $\forall$ x Person(x) => HasHead(x) “Every person has a head.”

  $\forall$ i Integer(i) => Integer(plus(i,1)) “If i is an integer then i+1 is an integer.”

  Note that
  $\forall$ x King(x) $\wedge$ Person(x) is not correct!
  This would imply that all objects x are Kings and are People

  $\forall$ x King(x) => Person(x) is the correct way to say this

  Note that => is the natural connective to use with $\forall$.

Universal Quantification $\forall$

- Universal quantification is equivalent to:
  - Conjunction of all sentences obtained by substitution of an object for the quantified variable.

- All Cats are Mammals.
  - $\forall x\ Cat(x) \Rightarrow Mammal(x)$

- Conjunction of all sentences obtained by substitution of an object for the quantified variable:

  Cat(Spot) $\Rightarrow$ *Mammal(Spot)* $\wedge$

  Cat(Rick) $\Rightarrow$ *Mammal(Rick)* $\wedge$

  Cat(LAX) $\Rightarrow$ *Mammal(LAX)* $\wedge$

  Cat(Shayama) $\Rightarrow$ *Mammal(Shayama)* $\wedge$

  Cat(France) $\Rightarrow$ *Mammal(France)* $\wedge$

  Cat(Felix) $\Rightarrow$ *Mammal(Felix)* $\wedge$

  …

Existential Quantification $\exists$

- $\exists$ x means “there exists an x such that....” (at least one object x)
- Allows us to make statements about some object without naming it
- Examples:

$\exists$ x King(x) “Some object is a king.”

$\exists$ x Lives_in(John, Castle(x)) “John lives in somebody’s castle.”

$\exists$ i Integer(i) $\wedge$ GreaterThan(i,0) “Some integer is greater than zero.”

Note that $\wedge$ is the natural connective to use with $\exists$

(And note that => is the natural connective to use with $\forall$)

Existential Quantification ∃

- Existential quantification is equivalent to:
  - Disjunction of all sentences obtained by substitution of an object for the quantified variable.

- Spot has a sister who is a cat.
  - $\exists x$ Sister(x, Spot) $\wedge$ Cat(x)

- Disjunction of all sentences obtained by substitution of an object for the quantified variable:

  Sister(Spot, Spot) $\wedge$ *Cat(Spot)* $\vee$

  Sister(Rick, Spot) $\wedge$ *Cat(Rick)* $\vee$

  Sister(LAX, Spot) $\wedge$ *Cat(LAX)* $\vee$

  Sister(Shayama, Spot) $\wedge$ *Cat(Shayama)* $\vee$

  Sister(France, Spot) $\wedge$ *Cat(France)* $\vee$

  Sister(Felix, Spot) $\wedge$ *Cat(Felix)* $\vee$

  ...

Combining Quantifiers --- Order (Scope)

The order of “unlike” quantifiers is important.

$\forall$ x $\exists$ y Loves(x,y)

- For everyone (“all x”) there is someone (“exists y”) whom they love

$\exists$ y $\forall$ x Loves(x,y)

- there is someone (“exists y”) whom everyone loves (“all x”)

Clearer with parentheses: $\exists$ y ($\forall$ x Loves(x,y))

The order of “like” quantifiers does not matter.

$$\forall x\ \forall y\ P(x, y) \equiv \forall y\ \forall x\ P(x, y)$$

$$\exists x\ \exists y\ P(x, y) \equiv \exists y\ \exists x\ P(x, y)$$

Connections between Quantifiers

- Asserting that all x have property P is the same as asserting that does not exist any x that does not have the property P

$\forall x \ \text{Likes}(x, \text{CS-171 class}) \Leftrightarrow \neg \exists x \ \neg \text{Likes}(x, \text{CS-171 class})$

- Asserting that there exists an x with property P is the same as asserting that not all x do not have the property P

$\exists x \ \text{Likes}(x, \text{IceCream}) \Leftrightarrow \neg \forall x \ \neg \text{Likes}(x, \text{IceCream})$

In effect:

- $\forall$ is a conjunction over the universe of objects
- $\exists$ is a disjunction over the universe of objects

Thus, DeMorgan's rules can be applied

De Morgan's Law for Quantifiers

De Morgan's Rule

$$P \wedge Q \equiv \neg(\neg P \vee \neg Q)$$

$$P \vee Q \equiv \neg(\neg P \wedge \neg Q)$$

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

$$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$

Generalized De Morgan's Rule

$$\forall x\, P \equiv \neg \exists x\, (\neg P)$$

$$\exists x\, P \equiv \neg \forall x\, (\neg P)$$

$$\neg \forall x\, P \equiv \exists x\, (\neg P)$$

$$\neg \exists x\, P \equiv \forall x\, (\neg P)$$

Rule is simple: if you bring a negation inside a disjunction or a conjunction, always switch between them (or →and, and → or).

Equality

- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object

- E.g., definition of *Sibling* in terms of *Parent*:

$$\forall x,y\ Sibling(x,y) \Leftrightarrow$$
$$[\neg(x = y) \wedge$$
$$\exists m,f\ \neg (m = f) \wedge Parent(m,x) \wedge Parent(f,x)$$
$$\wedge\ Parent(m,y) \wedge Parent(f,y)]$$

Equality can make reasoning much more difficult!

(See R&N, section 9.5.5, page 353)

You may not know when two objects are equal.

E.g., Ancients did not know (MorningStar = EveningStar = Venus)

You may have to prove x = y before proceeding

E.g., a resolution prover may not know 2+1 is the same as 1+2

FOL (or FOPC) Ontology:

What kind of things exist in the world?

What do we need to describe and reason about?

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