First Order Logic and Probability & Bayesian Nets

Thur, June 30, 2016

Review: First Order Logic

Terms

- Function: LeftLeg(x)
- Constant: King(x)

Atomic Sentences

- P(x,y) reads: x is a P of y
- Eg: Teacher(x,y) : x is a teacher of y

Logical Connectives

- $\Leftrightarrow$ (biconditional), $\Rightarrow$ (implication), $\wedge$ (and), $\vee$ (or), $\neg$ (negation)

Quantifiers

- Universal Quantifier $\forall$
- Existential Quantifier $\exists$

Review: First Order Logic

We **could** replace ∀ with ∃, or vise-versa. (De Morgan's Laws)

$$\forall x\, P(x) \equiv \neg \exists x\, \neg P(x)$$
$$\exists x\, P(x) \equiv \neg \forall x\, \neg P(x)$$

which is similar to ∧ ∨ relations:

$$(P \lor Q) \equiv \neg(\neg P \land \neg Q)$$
$$(P \land Q) \equiv \neg(\neg P \lor \neg Q)$$

First Order Logic: Problems

For the English sentence below, find the best FOL sentence. Then translate other FOL sentences to English sentences.

All students are persons

(A) $\forall x\ Student(x) \land Person(x)$
(B) $\forall x\ Student(x) \Rightarrow Person(x)$
(C) $\exists x\ Student(x) \land Person(x)$

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

All students are persons - (B)

(A) $\forall x\ Student(x) \wedge Person(x)$

(B) $\boldsymbol{\forall x\ Student(x) \Rightarrow Person(x)}$

(C) $\exists x\ Student(x) \wedge Person(x)$

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

All students are persons - (B)

(A) $\forall x\ Student(x) \land Person(x)$

Every object is a student and is a person.

(B) $\forall x\ Student(x) \Rightarrow Person(x)$

(C) $\exists x\ Student(x) \land Person(x)$

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

All students are persons - (B)

(A) $\forall x \; Student(x) \land Person(x)$

Every object is a student and is a person.

(B) $\boldsymbol{\forall x \; Student(x) \Rightarrow Person(x)}$

(C) $\exists x \; Student(x) \land Person(x)$

Some students are persons.

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

Every student attends all lectures.

(A) $\forall x \; \forall y \; [Student(x) \land Lecture(y)] \Rightarrow attend(x, y)$

(B) $\forall x \; \forall y \; Student(x) \Rightarrow [Lecture(y) \land attend(x, y)]$

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

Every student attends all lectures. (A)

(A) $\forall x\ \forall y\ [Student(x) \land Lecture(y)] \Rightarrow attend(x, y)$

(B) $\forall x\ \forall y\ Student(x) \Rightarrow [Lecture(y) \land attend(x, y)]$

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(B) $\forall x\ \forall y\ Student(x) \Rightarrow [Lecture(y) \land attend(x, y)]$

For every student, every object is a lecture and the student attends it.

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

Some students attend all lectures.

(A) $\exists x \; \forall y \; [Student(x) \land Lecture(y)] \Rightarrow attend(x, y)$

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First Order Logic: Problems

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(B) $\exists x \; \forall y \; Lecture(y) \Rightarrow [\, Student(x) \land attend(x, y)]$

First Order Logic: Problems

For the english sentence below, find the best FOL sentence. Then translate other FOL sentences to english sentences.

Some students attend all lectures. (B)

(A) $\exists x\ \forall y\ [Student(x) \land Lecture(y)] \Rightarrow attend(x, y)$

There exist an object, that if the object is a student, it will attend all lectures.

(B) $\exists x\ \forall y\ Lecture(y) \Rightarrow [\ Student(x) \land attend(x, y)]$

First Order Logic: Problems

An interesting observation:

Some persons are students.

$\exists x\ [Person(x) \land Student(x)]$

All persons are students.

$\forall x\ Person(x) \Rightarrow Student(x)$

$\forall$ is often associated with: $\forall x\ P(x) \Rightarrow Q(x)$

$\exists$ is often associated with: $\exists x\ P(x) \land Q(x)$

Probability: Review

Independence and Conditional Independence.

Independence:

Alice and Charlie work for different companies. The time Alice goes home after work is independent of the time Charlie goes home.

Conditional Independence:

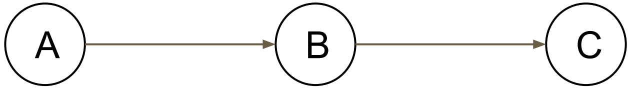
Alice and Bob work for the same boss. The time Alice goes home depends on the time Bob goes home;

But given the time the boss leaves the office, the time Alice goes to home is independent of the time Bob goes home.

Probability: Review

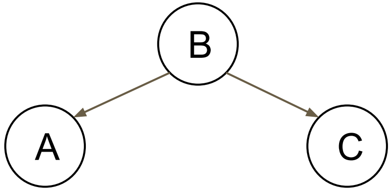
Conditional Independence (with graphical model representation)

In both models, A and C are conditionally independent given B



Ex: What people wear depends on the season; but given the temperature, what people wear is independent of the of season.

A: Season
B: Temperature
C: Outfit



Ex: Alice and Bob work for the same boss. Given the time the boss leaves, the time Alice goes to home is independent of the time Bob goes home.

A: The time Alice goes home
B: The time the boss leaves
C: The time Bob goes home

Probability: Problems

You have a drawer full of 4, 8 and 12-sided dice. They are in proportion 2:1:1. Your friend picks one at random and rolls it once getting a 5.

(a) What is the probability your friend picked the 8-sided die?

(b) (i) What is the probability the next roll will be a 5?
(ii) What is the probability the next roll will be a 10?

Probability: Problems

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Before we start, write down "obvious" clues:

P(Pick 4) = 2/(2+1+1) = 1/2 ; P(Pick 8) = 1/4 ; P(pick 12) = 1/4

(a) P(pick 8 | roll 5)
(b) (i) P(roll 5 next | roll 5)
(ii) P(roll 10 next | roll 5)

Probability: Problems

You have a drawer full of 4, 8 and 12-sided dice. They are in proportion 2:1:1. Your friend picks one at random and rolls it once getting a 5.

Notation:
P(px) : Probability of picking x-sided dice
P(rx) : Probability of rolling x

(a) *What is the probability your friend picked the 8-sided die?*

P(p8|r5) = P(p8, r5)/P(r5) = P(r5|p8)P(p8)/P(r5)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

P(r5|p8) = ⅛
P(p8) = ¼
P(r5) = P(p4, r5) + P(p8, r5) + P(p12, r5)

$$P(A) = \sum_B P(A,B)$$

= P(r5|p4)P(p4) + P(r5|p8)P(p8) + P(r5|p12)P(p12)

$$P(A,B) = P(B|A)P(A)$$

= 0 + (1/8 x 1/4) + (1/12 x 1/4)

P(p8|r5) = **(1/8 x 1/4) / [(1/8 x 1/4) + (1/12 x 1/4)]** = 0.6

Probability: Problems

You have a drawer full of 4, 8 and 12-sided dice. They are in proportion 2:1:1. Your friend picks one at random and rolls it once getting a 5.

(b) *(i) What is the probability the next roll will be a 5?*

$$P(r5^*|r5) = P(p4,r5^*|r5) + P(p8,r5^*|r5) + P(p12,r5^*|r5)$$
$$= P(p4|r5)\ P(r5^*|p4, r5) + P(p8|r5)\ P(r5^*|p8, r5) + P(p12|r5)\ P(r5^*|p12, r5)$$

Note: Give what dice is picked, the last roll is independent of the new roll.

Thus: $P(r5^*|p4, r5) = P(r5^*|p4)$

$$P(r5^*|r5) = P(p4|r5)\ P(r5^*|p4) + P(p8|r5)\ P(r5^*|p8) + P(p12|r5)\ P(r5^*|p12)$$
$$= 0 + 0.6 \times 1/8 + 0.4 \times 1/12$$
$$= 0.1083$$

Probability: Problems

You have a drawer full of 4, 8 and 12-sided dice. They are in proportion 2:1:1. Your friend picks one at random and rolls it once getting a 5.

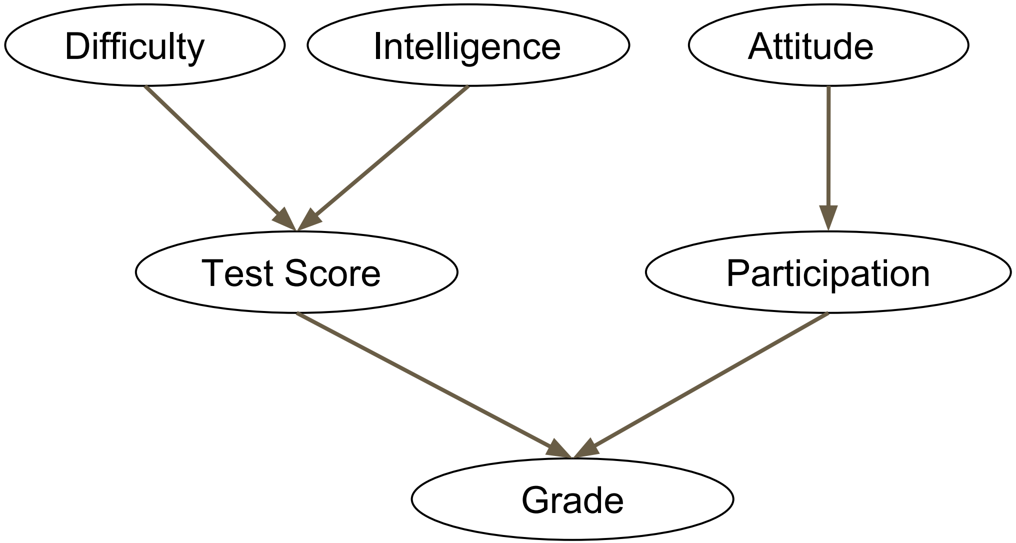
(b) *(ii) What is the probability the next roll will be a 10?*

$$P(r5^*|r5) = P(p4|r5)\,P(r10^*|p4) + P(p8|r5)\,P(r10^*|p8) + P(p12|r5)\,P(r10^*|p12)$$

$$= 0 + 0 + 0.4 \times 1/12$$

$$= 0.0333$$

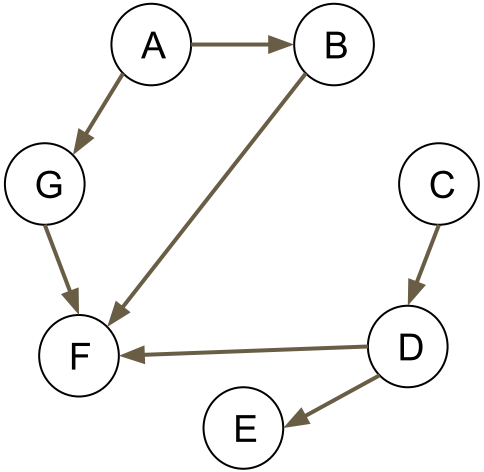
Bayesian Networks: Example



The joint probability can be factored as:

$$P(D, I, A, T, P, G) = P(D)\, P(I)\, P(T|D, I)\, P(A)\, P(P|A)\, P(G|T, P)$$

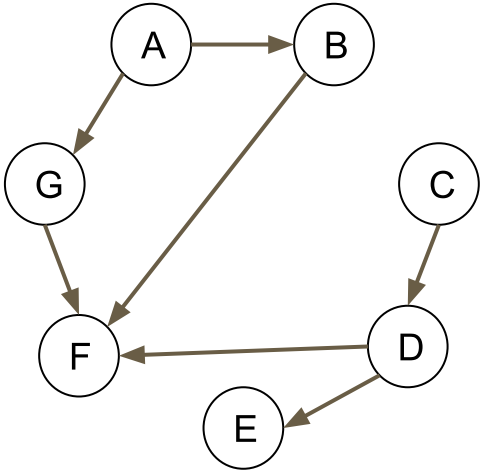
Bayesian Networks: Problems



Factor the joint probability

$P(A, B, C, D, E, F, G)$

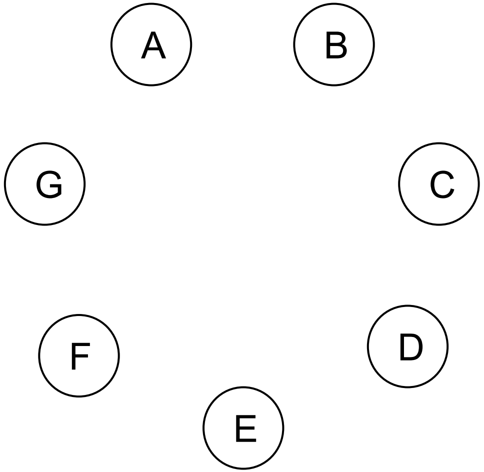
Bayesian Networks: Problems



Factor the joint probability

$$P(A, B, C, D, E, F, G) = P(A)\,P(B|A)\,P(G|A)\,P(C)\,P(D|C)\,P(E|D)\,P(F|G, B, D)$$

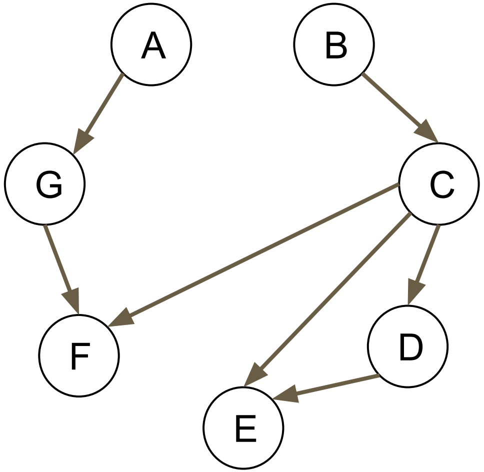
Bayesian Networks: Problems



Draw the Bayesian network corresponding to the factored conditional probability

$$P(A, B, C, D, E, F, G) = P(A)\,P(B)\,P(G|A)\,P(C|B)\,P(D|C)\,P(E|C, D)\,P(F|G, C)$$

Bayesian Networks: Problems



Draw the Bayesian network corresponding to the factored conditional probability

$$P(A, B, C, D, E, F, G) = P(A)\, P(B)\, P(G|A)\, P(C|B)\, P(D|C)\, P(E|C, D)\, P(F|G, C)$$

(There was a question asked in class, whether we should always omit the C → E link because C → D → E already exists.

The answer is no. C can be both an indirect cause of E through D, and a direct cause of E.)