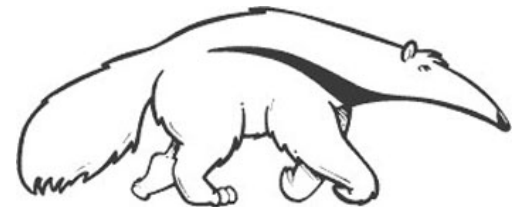# Local search algorithms

CS171, Fall Quarter, 2019
Introduction to Artificial Intelligence
Prof. Richard Lathrop

Read Beforehand: R&N 4.1-4.2, 4.6
Optional: R&N 4.3-4.5

Local search algorithms

- In many optimization problems, the path to the goal is irrelevant; the goal state itself is the solution
  - Local search: widely used for *very big* problems
  - Returns good but *not optimal* solutions
  - *Usually very slow,* but can yield good solutions if you wait

- State space = set of "complete" configurations
- Find a complete configuration satisfying constraints
  - Examples: n-Queens, VLSI layout, airline flight schedules

- Local search algorithms
  - Keep a single "current" state, or small set of states
  - Iteratively try to improve it / them
  - Very memory efficient
    - keeps only one or a few states
    - You control how much memory you use

Basic idea of local search (many variations)

// initialize to something, usually a random initial state
// alternatively, might pass in a human-generated initial state
$best\_found \leftarrow current\_state \leftarrow$ RandomState()
// now do local search
loop do
  if (tired of doing it) **then return** $best\_found$
  else

    $current\_state \leftarrow$ MakeNeighbor($current\_state$)
    if (Cost($current\_state$) < Cost($best\_found$)) **then**
      // keep best result found so far
 $best\_found \leftarrow current\_state$

You, as algorithm designer, write the functions named in red.

Typically, "tired of doing it" means that some resource limit is exceeded, e.g., number of iterations, wall clock time, CPU time, etc.
It may also mean that result improvements are small and infrequent, e.g., less than 0.1% result improvement in the last week of run time.

Example: *n*-queens

- Goal: Put *n* queens on an *n* × *n* board with no two queens on the same row, column, or diagonal

- Neighbor: move one queen to another row

- Search: go from one neighbor to the next...
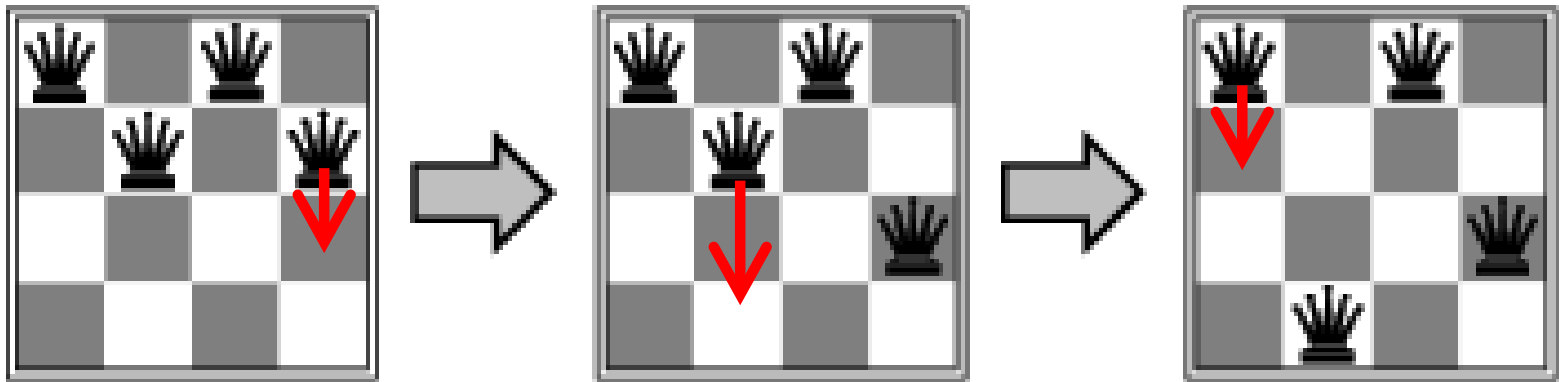
Algorithm design considerations

- How do you represent your problem?

- What is a "complete state"?

- What is your objective function?
  - How do you measure **cost** or **value** of a state?
  - Stand on your head: **cost = −value, value = −cost**

- What is a "neighbor" of a state?
  - Or, what is a "step" from one state to another?
  - How can you compute a neighbor or a step?

- Are there any constraints you can exploit?

Random restart wrapper

- We'll use stochastic local search methods
  - Return different solution for each trial & initial state

- Almost every trial hits difficulties (see sequel)
  - Most trials will not yield a good result (sad!)

- Using many random restarts improves your chances
  - Many "shots at goal" may finally get a good one

- Restart a random initial state, *many times*
  - Report the best result found across *many* trials

Random restart wrapper

best_found ← **RandomState**()   // initialize to something

// now do repeated local search
loop do
  if (tired of doing it)
    then return *best_found*
  else
    result ← LocalSearch(**RandomState**())
    if (**Cost**(*result*) < **Cost**(*best_found*))
      // keep best result found so far
   **then** *best_found* ← *result*

You, as algorithm designer, write the functions named in red.

Typically, **"tired of doing it"** means that some resource limit has been exceeded, e.g., number of iterations, wall clock time, CPU time, etc. It may also mean that result improvements are small and infrequent, e.g., less than 0.1% result improvement in the last week of run time.

Tabu search wrapper

- Add recently visited states to a tabu-list
  - Temporarily excluded from being visited again
  - Forces solver away from explored regions
  - Less likely to get stuck in local minima (hope, in principle)

- Implemented as a hash table + FIFO queue
  - Unit time cost per step; constant memory cost
  - You control how much memory is used

- RandomRestart(TabuSearch (LocalSearch()))

Tabu search wrapper (**<u>inside</u> random restart!)**
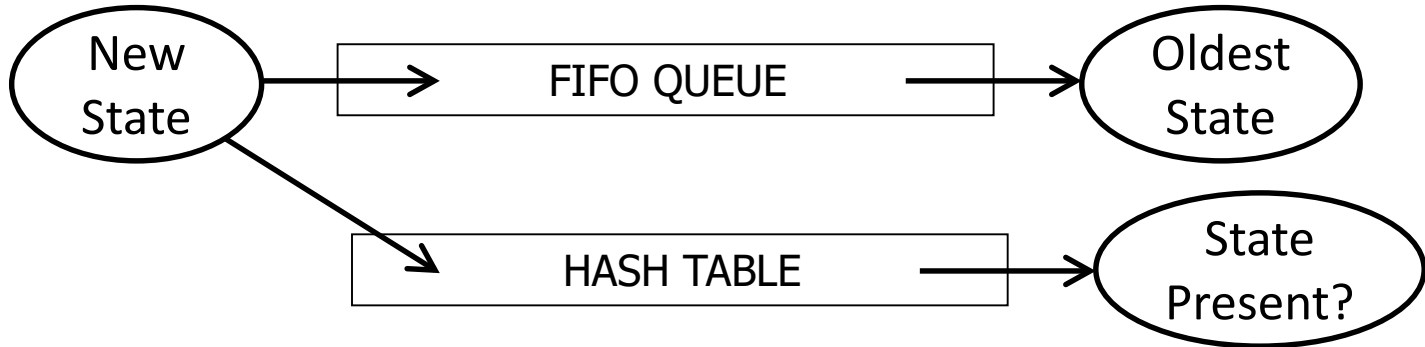
$best\_found \leftarrow current\_state \leftarrow$ **RandomState**() // initialize
loop do // now do local search
  if (tired of doing it) **then return** $best\_found$ **else**
    $neighbor \leftarrow$ **MakeNeighbor**($current\_state$)
    if ($neighbor$ is in $hash\_table$) **then** discard $neighbor$
    else push $neighbor$ onto $fifo$, pop $oldest\_state$
      remove $oldest\_state$ from $hash\_table$, insert $neighbor$
      $current\_state \leftarrow neighbor$;
      if (**Cost**($current\_state$) < **Cost**($best\_found$))
   **then** $best\_found \leftarrow current\_state$

Local search algorithms

- Hill-climbing search
  - Gradient descent in continuous state spaces
  - Can use, e.g., Newton's method to find roots
- Simulated annealing search
- Local beam search
- Genetic algorithms
- Linear Programming (for specialized problems)

Local Search Difficulties

These difficulties apply to ALL local search algorithms, and become MUCH more difficult as the search space increases to high dimensionality.

- <u>Problems:</u> depending on state, can get stuck in local maxima
  - Many other problems also endanger your success!!
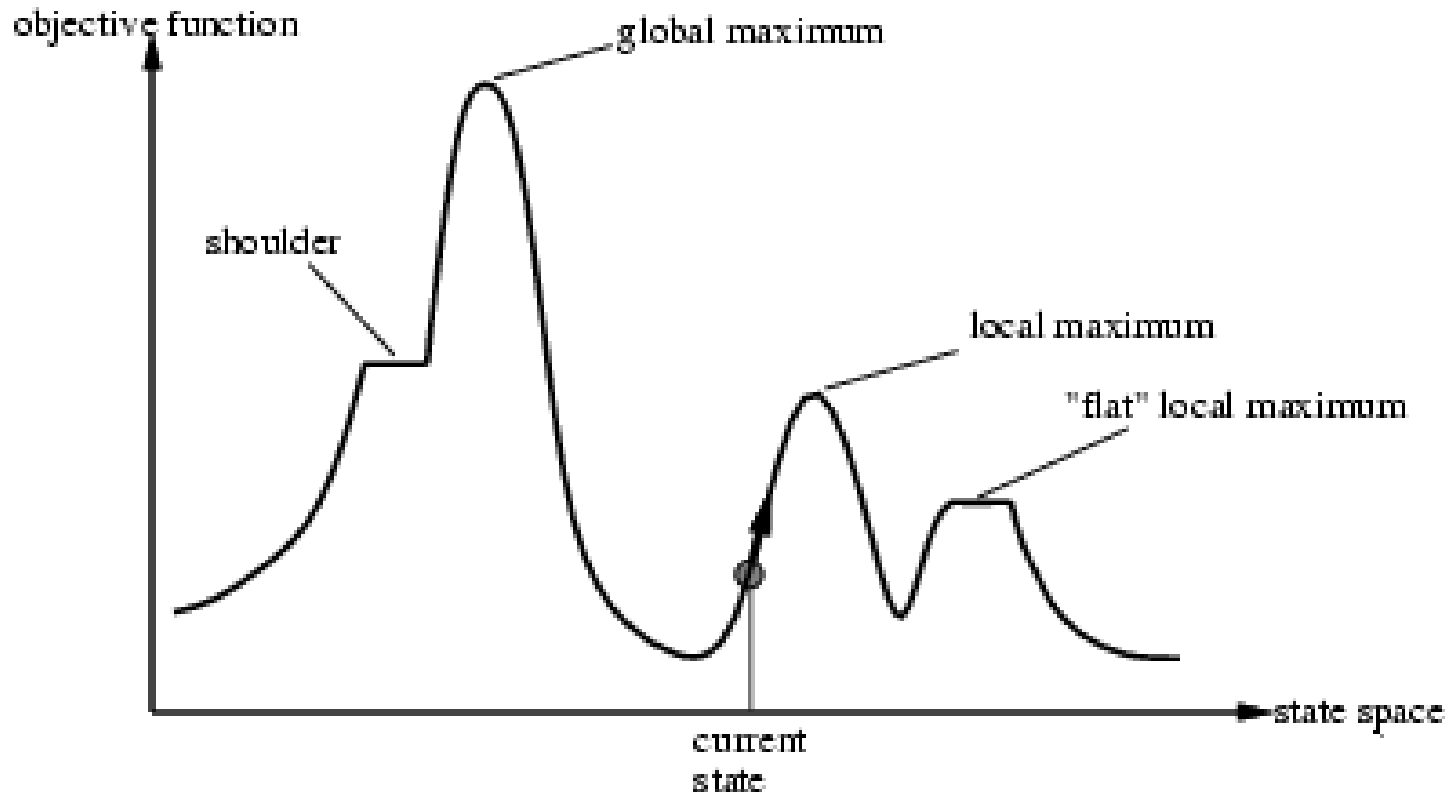
Local Search Difficulties

These difficulties apply to ALL local search algorithms, and become MUCH more difficult as the search space increases to high dimensionality.

- <u>Ridge problem:</u> Every neighbor appears to be downhill
  - But the search space has an uphill!! (worse in high dimensions)

<u>Ridge:</u>
Fold a piece of paper and hold it tilted up at an unfavorable angle to every possible search space step. Every step leads downhill; but the ridge leads uphill.

Figure 4.4 **FILES: figures/ridge.eps (Tue Nov 3 16:23:29 2009).** Illustration of why ridges cause difficulties for hill climbing. The grid of states (dark circles) is superimposed on a ridge rising from left to right, creating a sequence of local maxima that are not directly connected to each other. From each local maximum, all the available actions point downhill.

Hill-climbing search

You must shift effortlessly between maximizing value and minimizing cost

"…like trying to find the top of Mount Everest in a thick fog while suffering from amnesia"

```
function HILL-CLIMBING( problem) returns a state that is a local maximum
    inputs: problem, a problem
    local variables: current, a node
                     neighbor, a node

    current ← MAKE-NODE(INITIAL-STATE[problem])
    loop do
        neighbor ← a highest-valued successor of current
        if VALUE[neighbor] ≤ VALUE[current] then return STATE[current]
        current ← neighbor
```

Equivalently:
"…a lowest-cost successor…"

Equivalently: "if **COST**[neighbor] ≥ **COST**[current] then …"

Example: Hill-climbing, 8-queens

h = # of pairs of queens that are attacking each other, either directly or indirectly (cost)
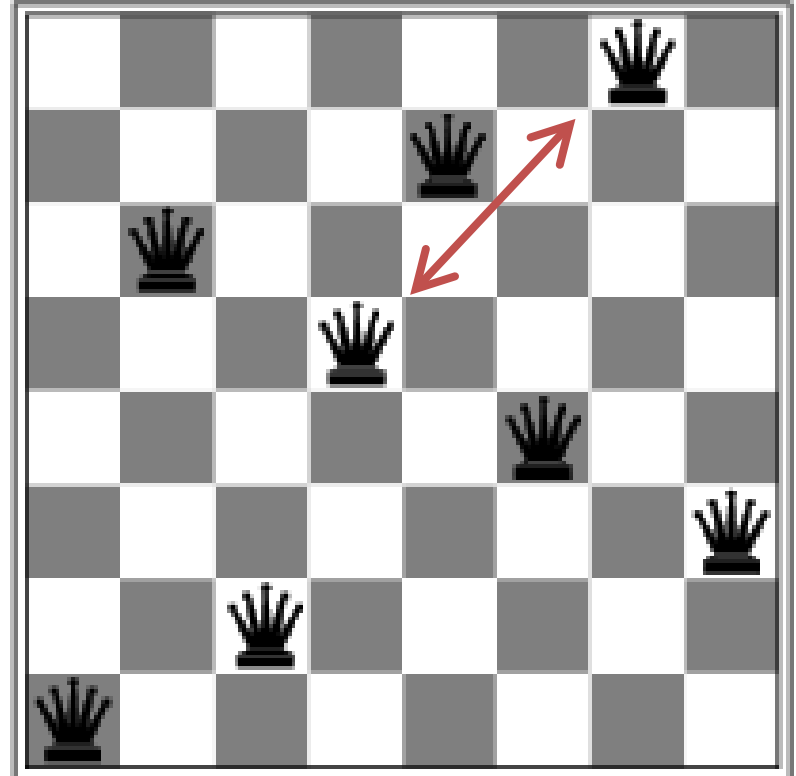
h=17 for this state

Each number indicates *h* if we move a queen in its column to that square

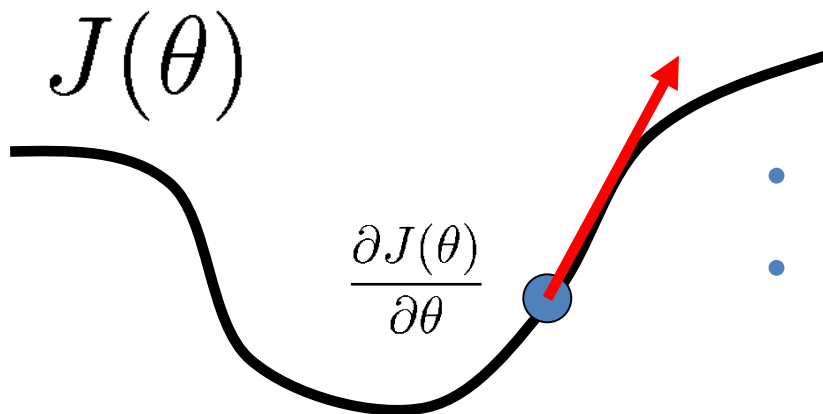12 (boxed) = best *h* among all neighors; select one randomly

Example: Hill-climbing, 8-queens

- A local minimum with h=1

- All one-step neighbors have higher h values

- What can you do to get out of this local minimum?

Gradient descent

- Hill-climbing in continuous state spaces

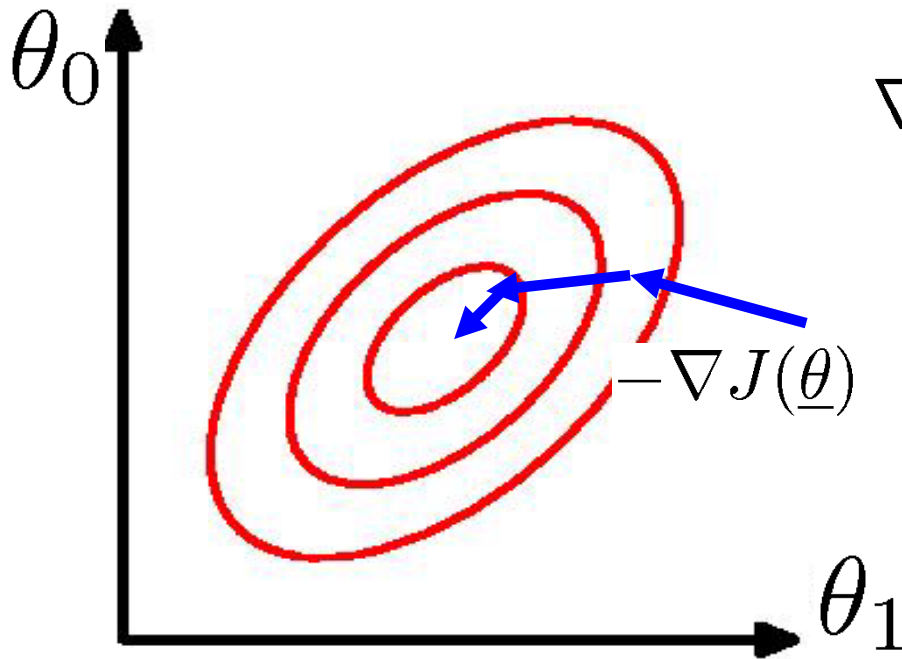- Denote "state" as θ, a vector of parameters

- Denote cost as J(θ)

$$J(\theta)$$

$$\frac{\partial J(\theta)}{\partial \theta}$$

- How to change θ to improve J(θ)?
- Choose a direction in which J(θ) is decreasing
- Derivative $\frac{\partial J(\theta)}{\partial \theta}$

- Positive => increasing cost
- Negative => decreasing cost

The curly D means to take a derivative while holding all other variables constant. You are not responsible for multivariate calculus, but gradient descent is a very important method, so it is presented.

Gradient descent

Hill-climbing in continuous spaces

- Gradient vector

$$\nabla J(\underline{\theta}) = \left[\frac{\partial J(\underline{\theta})}{\partial \theta_0} \quad \frac{\partial J(\underline{\theta})}{\partial \theta_1} \quad \ldots \right]$$
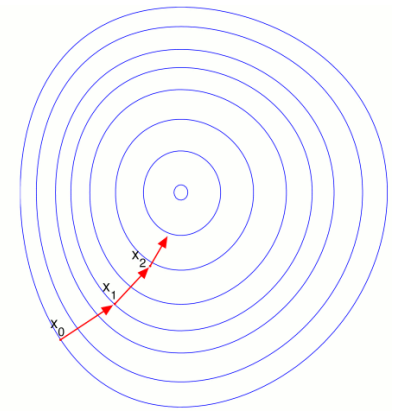
θ_0

$-\nabla J(\underline{\theta})$

θ_1

Gradient = direction of steepest ascent

Negative gradient = steepest descent

Gradient descent

Hill-climbing in continuous spaces

Gradient = the most direct direction up-hill in the objective (cost) function, so its negative minimizes the cost function.

* Assume we have some cost-function: $J(x_1, x_2, \ldots, x_n)$ and we want minimize over continuous variables $x_1, x_2, .., x_n$

1. Compute the *gradient* : $\dfrac{\partial}{\partial x_i} J(x_1, \ldots, x_n)$ $\qquad \forall i$

2. Take a small step downhill in the direction of the gradient:

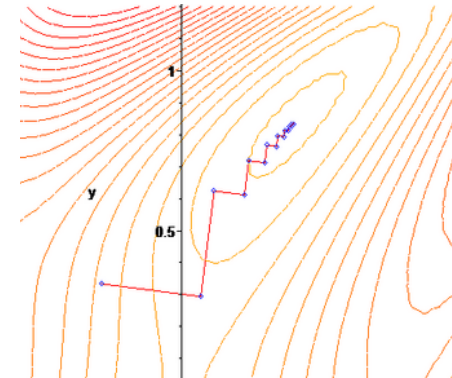$$x'_i = x_i - \lambda \frac{\partial}{\partial x_i} J(x_1, \ldots, x_n)$$

3. Check if $J(x'_1, \ldots, x'_n) < J(x_1, \ldots, x_n)$

(or, Armijo rule, etc.)

4. If true then accept move, if not "reject".

(decrease step size, etc.)

5. Repeat.

Gradient descent

Hill-climbing in continuous spaces

- How do I determine the gradient?
  - Derive formula using multivariate calculus.
  - Ask a mathematician or a domain expert.
  - Do a literature search.

- Variations of gradient descent can improve performance for this or that special case.
  - See Numerical Recipes in C (and in other languages) by Press, Teukolsky, Vetterling, and Flannery.
  - Simulated Annealing, Linear Programming too

- Works well in smooth spaces; poorly in rough.

Newton's method

- Want to find the roots of f(x)
  - "Root": value of x for which f(x)=0

- Initialize to *some* point x_n
- Compute the tangent at x_n & compute x_{n+1} = where it crosses x-axis
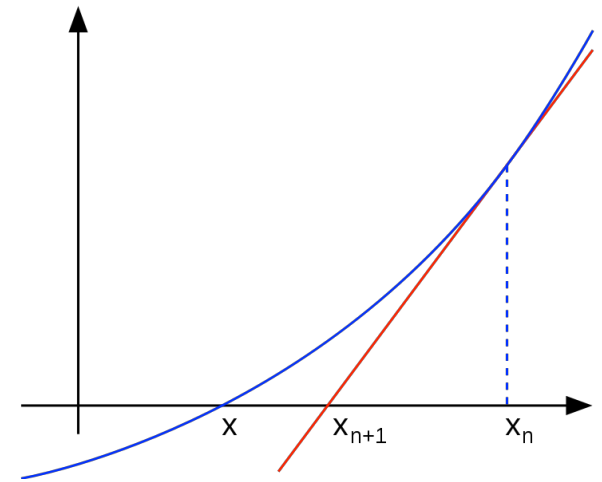
Newton's method

- Want to find the roots of f(x)
  - "Root": value of x for which f(x)=0

- Initialize to *some* point x_n
- Compute the tangent at x_n & compute x_{n+1} = where it crosses x-axis
- Repeat for x_{n+1}

  - Does not always converge; sometimes unstable
  - If converges, usually very fast
  - Works well for smooth, non-pathological functions; accurate linearization
  - Works poorly for wiggly, ill-behaved functions; tangent is a poor guide to root
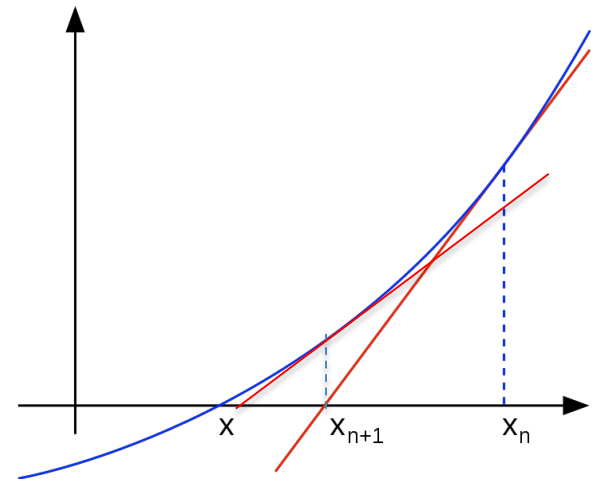
Simulated annealing (Physics!)

- Idea: escape local maxima by allowing some "bad" moves but gradually decrease their frequency

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
    inputs: problem, a problem
            schedule, a mapping from time to "temperature"
    local variables: current, a node
                     next, a node
                     T, a "temperature" controlling prob. of downward steps

    current ← MAKE-NODE(INITIAL-STATE[problem])
    for t ← 1 to ∞ do
        T ← schedule[t]
        if T = 0 then return current
        next ← a randomly selected successor of current
        ΔE ← VALUE[next] − VALUE[current]
        if ΔE > 0 then current ← next
        else current ← next only with probability e^{ΔE/T}
```

Improvement: Track the BestResultFoundSoFar. Here, this slide follows Fig. 4.5 of the textbook, which is simplified.

Typical annealing schedule

- Usually use a decaying exponential
- Axis values scaled to fit problem characteristics
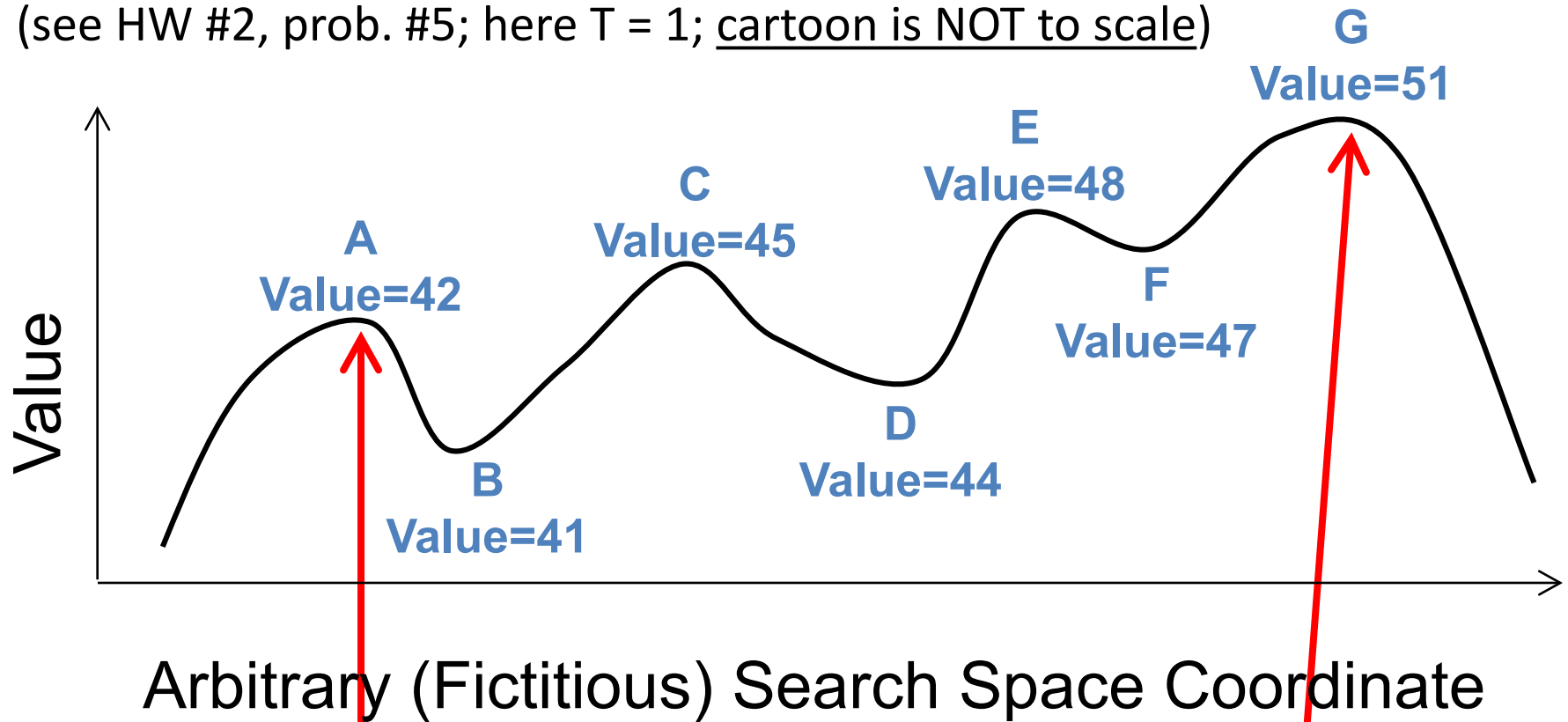
Probability(accept worse successor)

- Decreases as temperature T decreases
- Increases as |Δ E| decreases
- Sometimes, step size also decreases with T

(accept very bad moves early on; later, mainly accept "not very much worse")

$e^{\Delta E / T}$		Temperature T	
		High	**Low**
|ΔE |	**High**	Medium	Low
	Low	High	Medium

$next \leftarrow$ a randomly selected successor of $current$

$\Delta E \leftarrow \text{VALUE}[next] - \text{VALUE}[current]$

if $\Delta E > 0$ then $current \leftarrow next$

else $current \leftarrow next$ only with probability $e^{\Delta E/T}$

Temperature

time ⟶
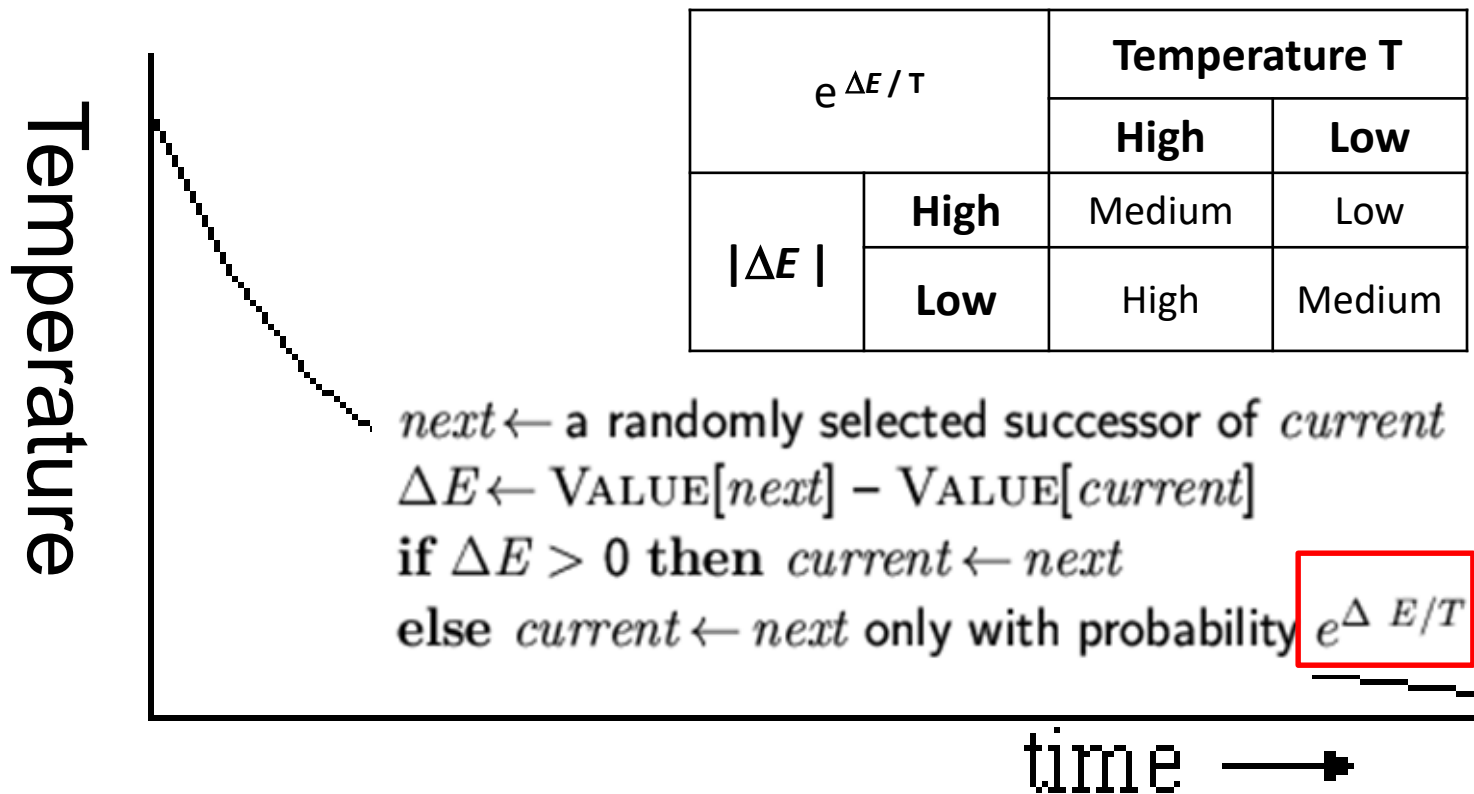
Goal: "ratchet up" a bumpy slope

(see HW #2, prob. #5; here T = 1; <u>cartoon is NOT to scale</u>)

This is an illustrative *cartoon*…

Goal: "ratchet up" a jagged slope

A
Value=42
$\Delta E(AB)=-1$
$P(AB) \approx .37$

B
Value=41
$\Delta E(BA)=1$
$\Delta E(BC)=4$
$P(BA)=1$
$P(BC)=1$

C
Value=45
$\Delta E(CB)=-4$
$\Delta E(CD)=-1$
$P(CB) \approx .018$
$P(CD) \approx .37$

D
Value=44
$\Delta E(DC)=1$
$\Delta E(DE)=4$
$P(DC)=1$
$P(DE)=1$

E
Value=48
$\Delta E(ED)=-4$
$\Delta E(EF)=-1$
$P(ED) \approx .018$
$P(EF) \approx .37$

F
Value=47
$\Delta E(FE)=1$
$\Delta E(FG)=4$
$P(FE)=1$
$P(FG)=1$

G
Value=51
$\Delta E(GF)=-4$
$P(GF) \approx .018$
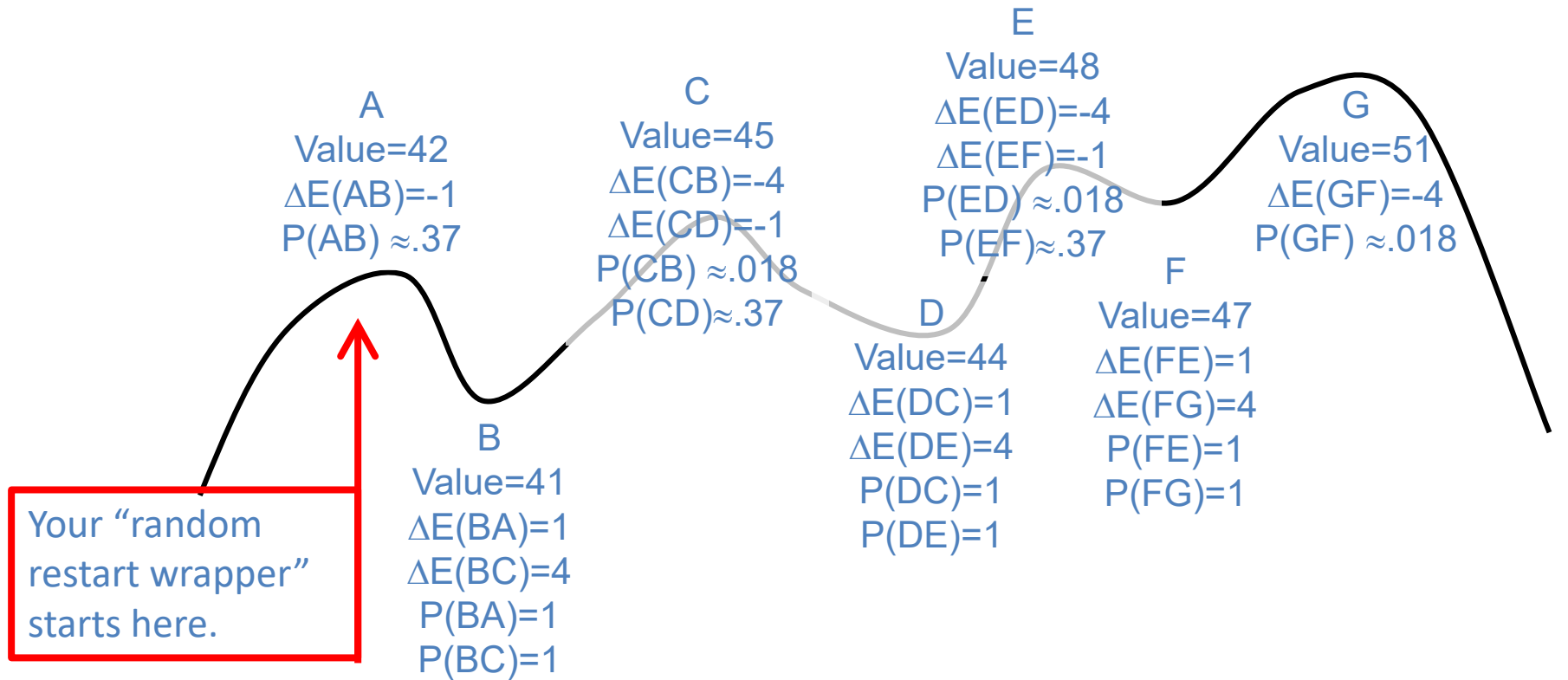
Your "random restart wrapper" starts here.

x	-1	-4
e^x	$\approx .37$	$\approx .018$

This is an illustrative *cartoon*…

From A you will accept a move to B with $P(AB) \approx .37$.
From B you are equally likely to go to A or to C.
From C you are $\approx$20X more likely to go to D than to B.
From D you are equally likely to go to C or to E.
From E you are $\approx$20X more likely to go to F than to D.
From F you are equally likely to go to E or to G.
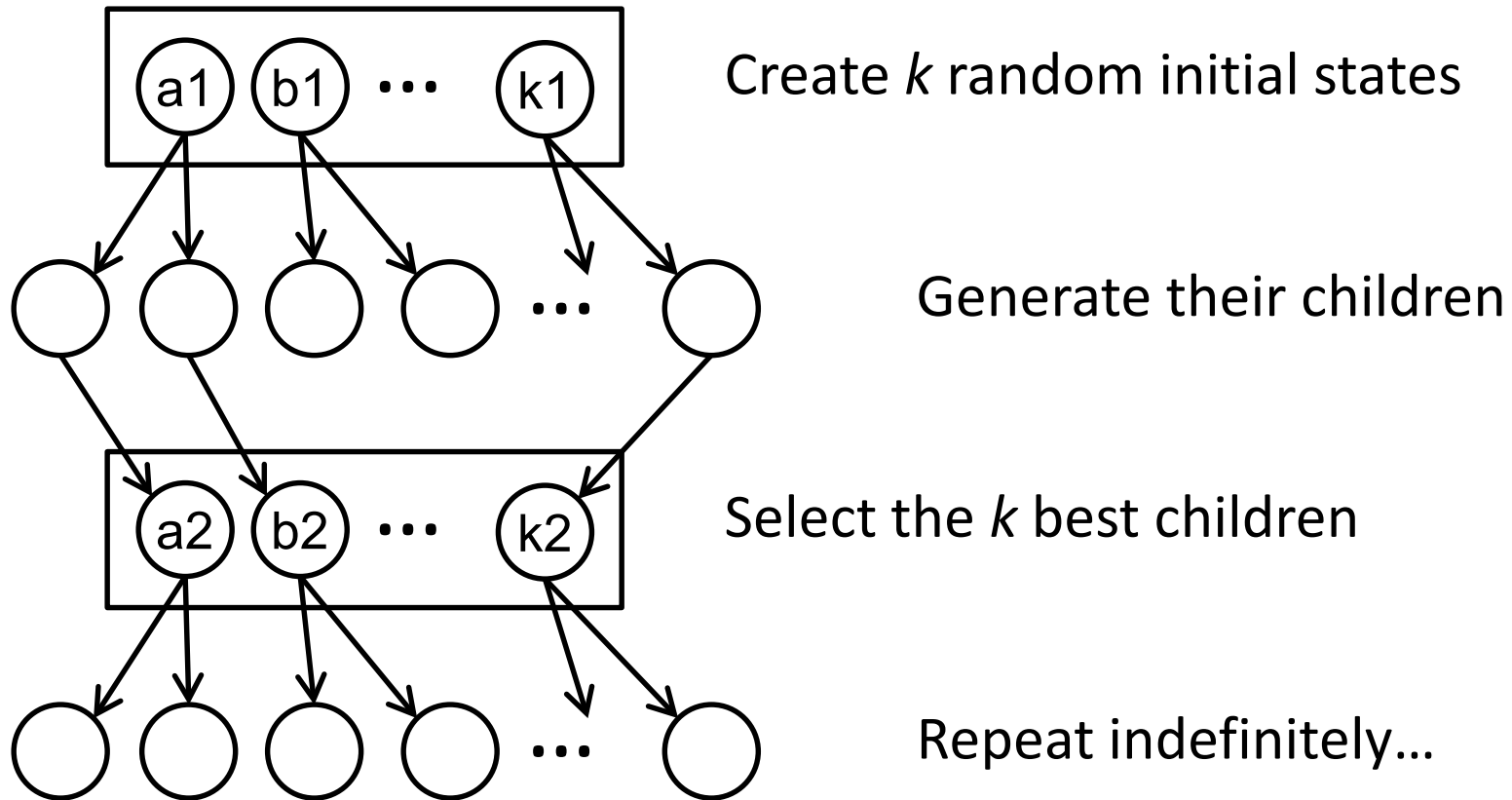Remember best point you ever found (G or neighbor?).

Properties of simulated annealing

- One can prove:
  - If T decreases slowly enough, then simulated annealing search will find a global optimum with probability approaching 1
  - Unfortunately this can take a VERY VERY long time
  - Note: in any finite search space, random guessing also will find a global optimum with probability approaching 1
  - So, ultimately this is a very weak claim

- Often works very well in practice
  - But usually VERY VERY slow

- Widely used in VLSI layout, airline scheduling, etc.

Local beam search

- Keep track of *k* states rather than just one

- Start with *k* randomly generated states

- At each iteration, all the successors of all *k* states are generated

- If any one is a goal state, stop; else select the *k* best successors from the complete list and repeat.

- Concentrates search effort in areas believed to be fruitful
  - May lose diversity as search progresses, resulting in wasted effort

Local beam search

Create *k* random initial states

Generate their children

Select the *k* best children

Repeat indefinitely…

Is it better than simply running *k* searches?
Maybe…??

Genetic algorithms (**Darwin!!**)

- A state = a string over a finite alphabet (an **individual**)
  - A successor state is generated by combining two parent states

- Start with *k* randomly generated states (a **population**)

- **Fitness** function (= our heuristic objective function).
  - Higher fitness values for better states.

- **Select** individuals for next generation based on fitness
  - P(individual in next gen.) = individual fitness/total population fitness

- **Crossover** fit parents to yield next generation (**offspring**)

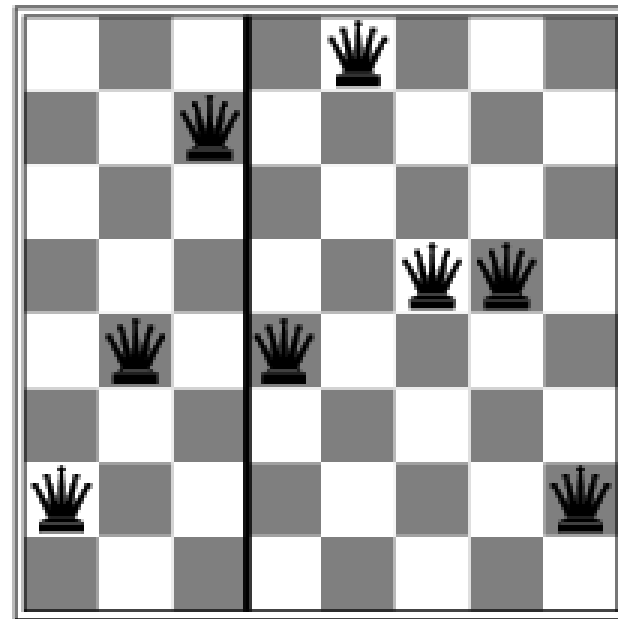- **Mutate** the offspring randomly with some low probability

Example: N-Queens

Each "individual" is a vector of each queen's row
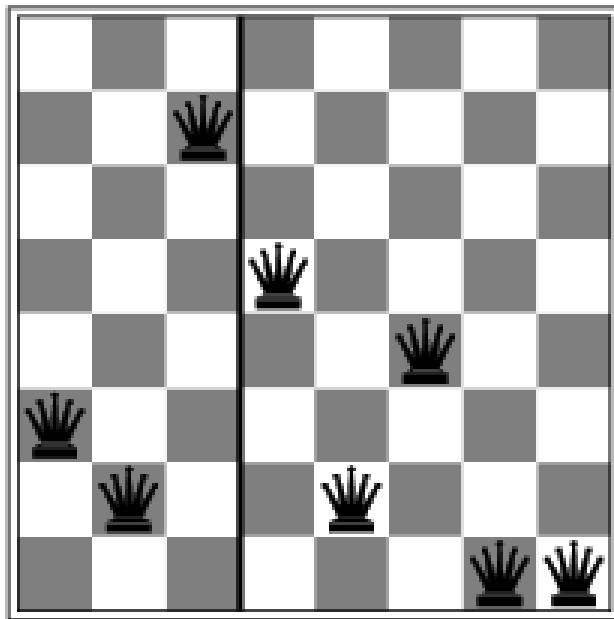
3	2	7	5	2	4	1	1

+

2	4	7	4	8	5	5	2

Example: N-Queens

Resulting individuals after cross-over
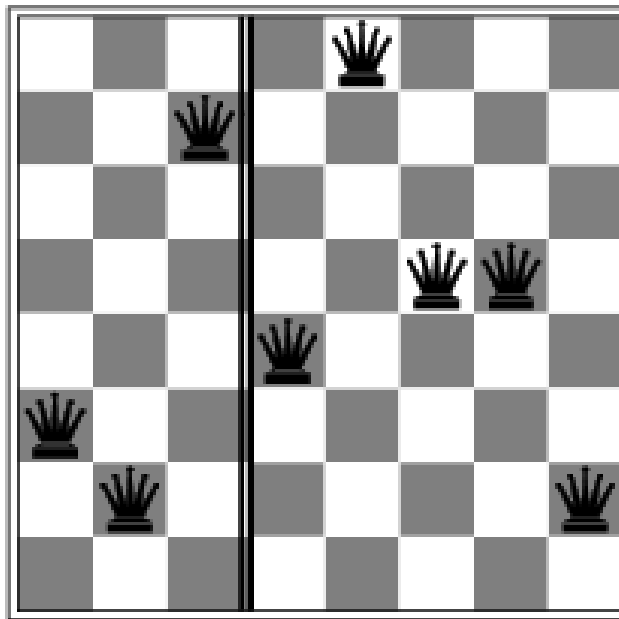
3	2	7	4	8	5	5	2
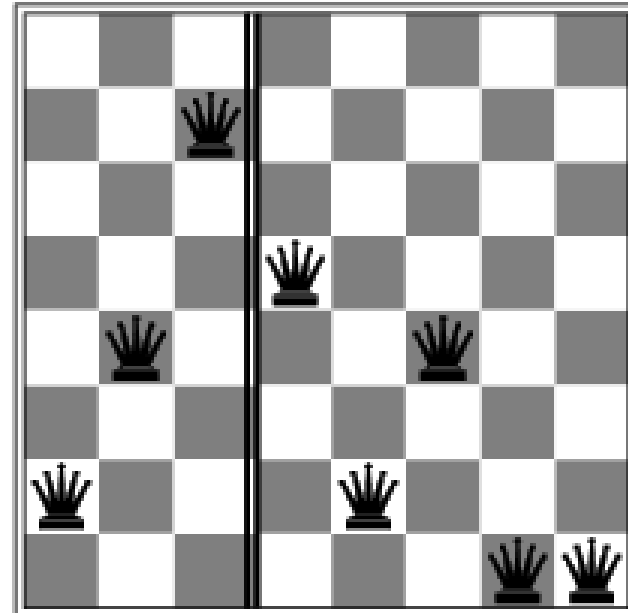
2	4	7	5	2	4	1	1

Genetic algorithms

|				
24748552	24 31%	32752411	32748552	3274815 2
32752411	23 29%	24748552	24752411	24752411
24415124	20 26%	32752411	32752124	3225 2124
32543213	11 14%	24415124	24415411	2441541 7
(a) Initial Population	(b) Fitness Function	(c) Selection	(d) Cross-Over	(e) Mutation
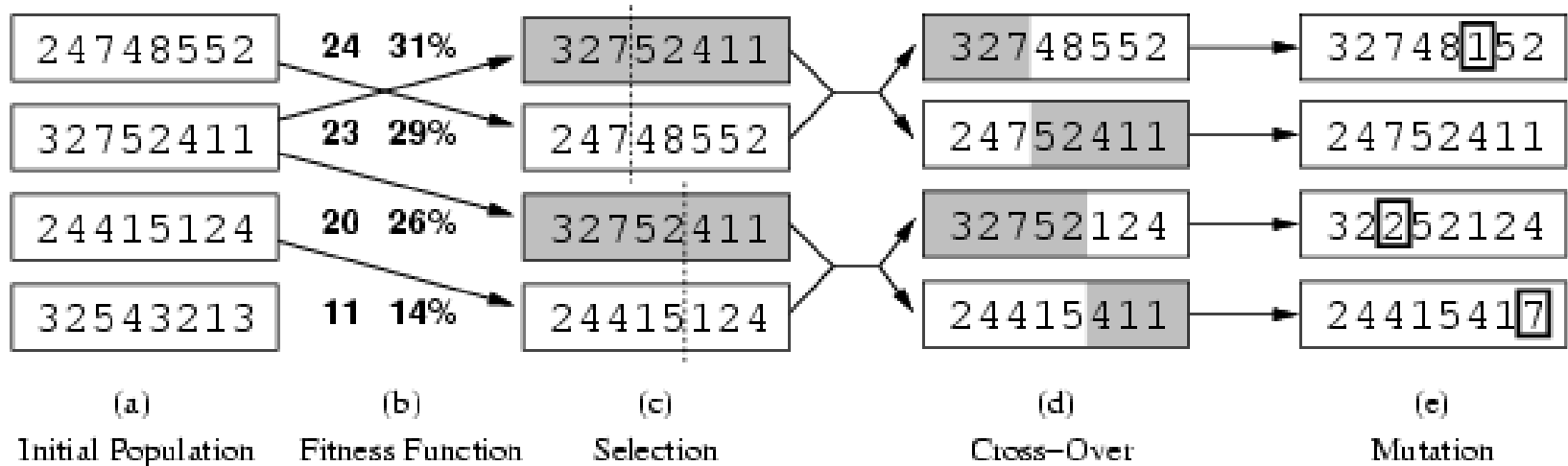
- Fitness function (value): number of non-attacking pairs of queens (min = 0, max = 8 × 7/2 = 28)
- 24/(24+23+20+11) = 31%
- 23/(24+23+20+11) = 29%; etc.

	24 31%	32752411	32748552	32748**1**52
24748552				
32752411	23 29%	24748552	24752411	24752411
24415124	20 26%	32752411	32752124	32**2**52124
32543213	11 14%	24415124	24415411	244154**7**

(a)	(b)	(c)	(d)	(e)
Initial Population	Fitness Function	Selection	Cross-Over	Mutation
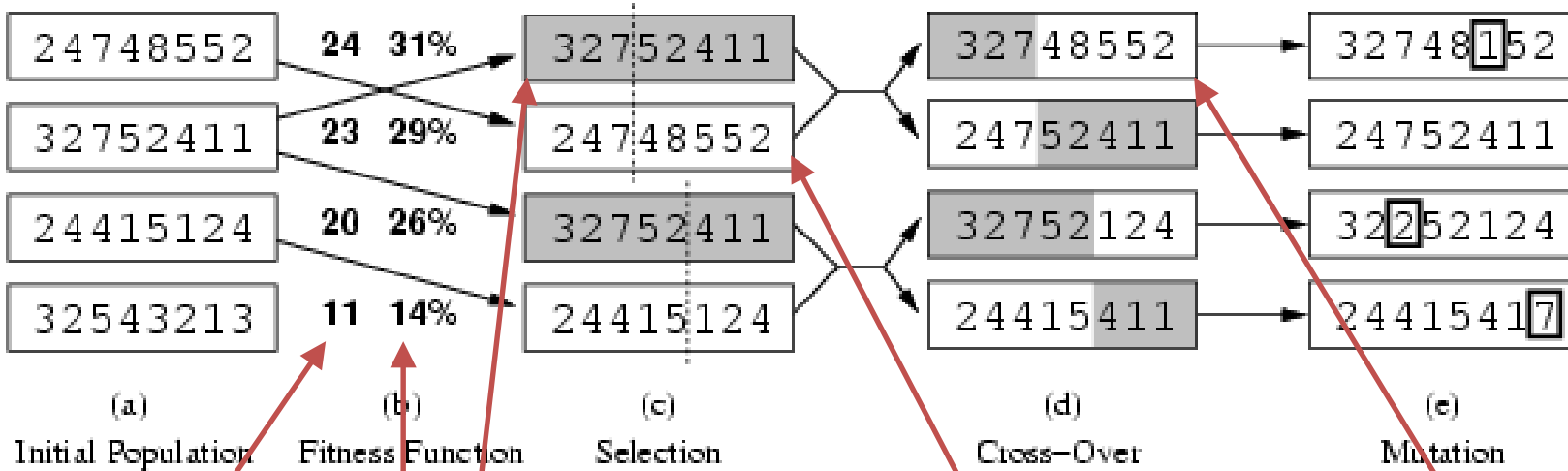
fitness = #non-attacking queens

probability of being in next generation = fitness/(Σ_i fitness_i)
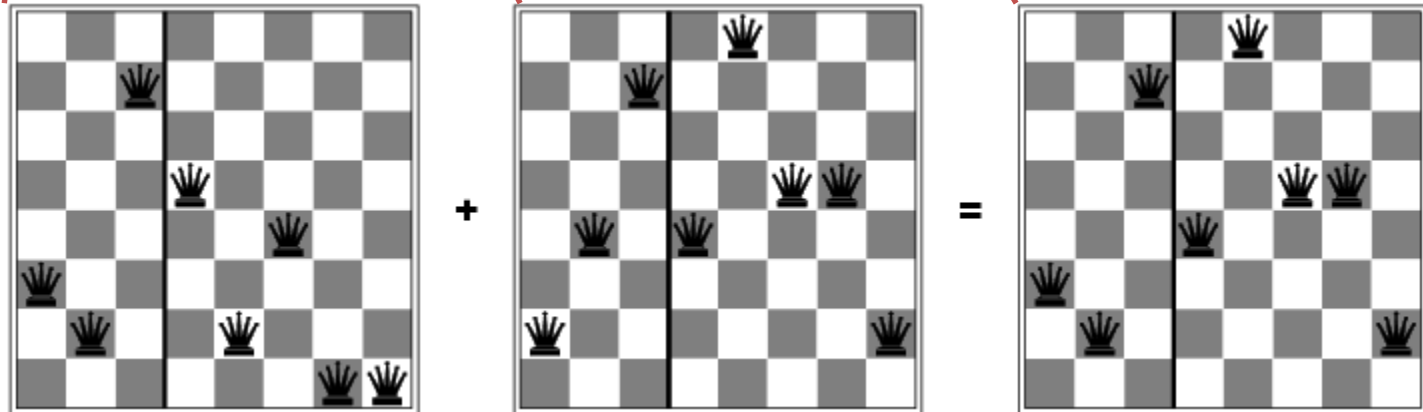
How to convert a fitness value into a probability of being in the next generation.

- Fitness function: #non-attacking queen pairs
  - min = 0, max = 8 × 7/2 = 28
- Σ_i fitness_i = 24+23+20+11 = 78
- P(pick child_1 for next gen.) = fitness_1/(Σ_i fitness_i) = 24/78 = 31%
- P(pick child_2 for next gen.) = fitness_2/(Σ_i fitness_i) = 23/78 = 29%; etc

Linear Programming

- Maximize: $z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$
- Primary constraints: $x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_n \geq 0$
- Arbitrary additional linear constraints:

$a_{i1} x_1 + a_{i2} x_2 + \ldots + a_{in} x_n \leq a_i,\ (a_i \geq 0)$
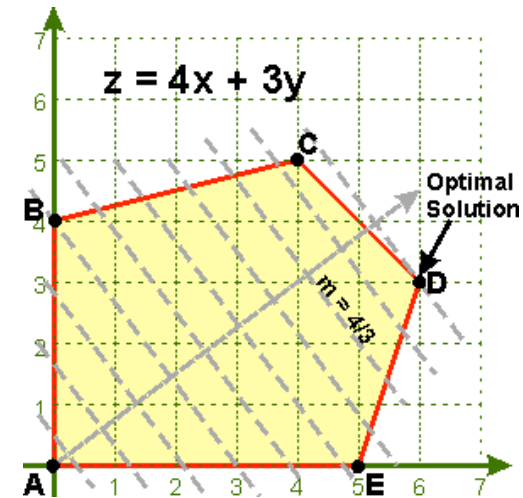
$a_{j1} x_1 + a_{j2} x_2 + \ldots + a_{jn} x_n \geq a_j \geq 0$

$b_{k1} x_1 + b_{k2} x_2 + \ldots + b_{kn} x_n = b_k \geq 0$

- <u>Restricted</u> class of linear problems.
  - <u>Efficient for very large problems(!!)</u> in this class.

Linear Programming
Efficient <u>Optimal</u> Solution
For a <u>Restricted</u> Class of Problems

- Very efficient "off-the-shelf" solvers are available for LPs.

- They quickly solve large problems with thousands of variables.

Summary

- Local search maintains a complete solution
  - Maintains a complete solution, seeks consistent (or at least good)
  - vs: Path search maintains a consistent solution; seeks complete
  - Goal of both: consistent & complete solution

- Types:
  - hill climbing, gradient ascent
  - simulated annealing, other Monte Carlo methods
  - Population methods: beam search; genetic / evolutionary algorithms
  - Wrappers: random restart; tabu search

- Local search often works well on very large problems
  - Abandons optimality
  - Always has some answer available (best found so far)
  - Often requires a very long time to achieve a good result