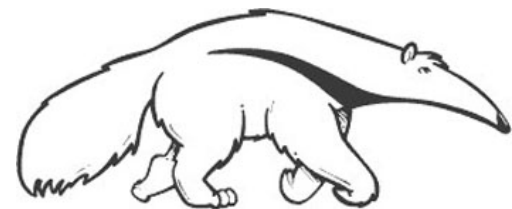


Probability: Reasoning Under Uncertainty

Introduction to Artificial Intelligence
Prof. Richard Lathrop

Read Beforehand: R&N 13



Outline

- Representing uncertainty is useful in knowledge bases
 - Probability provides a coherent framework for uncertainty
- Review of basic concepts in probability
 - Emphasis on conditional probability & conditional independence
- Full joint distributions are intractable to work with
 - Conditional independence assumptions allow much simpler models
- Bayesian networks (next lecture)
 - A useful type of structured probability distribution
 - Exploit structure for parsimony, computational efficiency
- Rational agents cannot violate probability theory

Uncertainty

Let action A_t = leave for airport t minutes before flight
Will A_t get me there on time?

Problems:

1. partial observability (road state, etc.)
2. multi-agent problem (other drivers' plans)
3. noisy sensors (uncertain traffic reports)
4. uncertainty in action outcomes (flat tire, etc.)
5. immense complexity of modeling and predicting traffic

Hence a purely logical approach either

1. risks falsehood: “ A_{25} will get me there on time”, or
2. leads to conclusions that are too weak for decision making:

“ A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact, etc., etc.”

“ A_{1440} should get me there on time but I'd have to stay overnight in the airport.”

Uncertainty in the world

- Uncertainty due to
 - Randomness
 - Overwhelming complexity
 - Lack of knowledge
 - ...
- Probability gives
 - natural way to describe our assumptions
 - rules for how to combine information
- Subjective probability
 - Relate to agent's own state of knowledge: $P(A_{25} | \text{no accidents}) = 0.05$
 - Not assertions about the world; indicate **degrees of belief**
 - Change with new evidence: $P(A_{25} | \text{no accidents, 5am}) = 0.20$

Propositional Logic and Probability

- Their ontological commitments are the same
 - The world is a set of facts that do or do not hold

Ontology is the philosophical study of the nature of being, becoming, existence, or reality; what exists in the world?

- Their epistemological commitments differ
 - **Logic agent** believes true, false, or no opinion
 - **Probabilistic agent** has a numerical degree of belief between 0 (false) and 1 (true)

Epistemology is the philosophical study of the nature and scope of knowledge; how, and in what way, do we know about the world?

Making decisions under uncertainty

- Suppose I believe the following:
 - $P(\text{A25 gets me there on time} \mid \dots) = 0.04$
 - $P(\text{A90 gets me there on time} \mid \dots) = 0.70$
 - $P(\text{A120 gets me there on time} \mid \dots) = 0.95$
 - $P(\text{A1440 gets me there on time} \mid \dots) = 0.9999$
- Which action to choose?
- Depends on my **preferences** for missing flight vs. time spent waiting, etc.
 - **Utility theory** is used to represent and infer preferences
 - **Decision theory** = probability theory + utility theory
- **Expected utility** of action a in state s
 - = $\sum_{\text{outcome in Results}(s,a)} P(\text{outcome}) * \text{Utility}(\text{outcome})$
- A rational agent acts to maximize expected utility

Example: Airport

- Suppose I believe the following:
 - $P(\text{A25 gets me there on time} \mid \dots) = 0.04$
 - $P(\text{A90 gets me there on time} \mid \dots) = 0.70$
 - $P(\text{A120 gets me there on time} \mid \dots) = 0.95$
 - $P(\text{A1440 gets me there on time} \mid \dots) = 0.9999$
 - $\text{Utility}(\text{on time}) = \$1,000$
 - $\text{Utility}(\text{not on time}) = -\$10,000$
- **Expected utility** of action a in state s
 - $$= \sum_{\text{outcome in Results}(s,a)} P(\text{outcome}) * \text{Utility}(\text{outcome})$$
 - $$E(\text{Utility}(\text{A25})) = 0.04 * \$1,000 + 0.96 * (-\$10,000) = -\$9,560$$
 - $$E(\text{Utility}(\text{A90})) = 0.7 * \$1,000 + 0.3 * (-\$10,000) = -\$2,300$$
 - $$E(\text{Utility}(\text{A120})) = 0.95 * \$1,000 + 0.05 * (-\$10,000) = \$450$$
 - $$E(\text{Utility}(\text{A1440})) = 0.9999 * \$1,000 + 0.0001 * (-\$10,000) = \$998.90$$
- Have not yet accounted for disutility of staying overnight at the airport, etc.

Random variables

- **Random Variable:**
 - Basic element of probability assertions
 - Similar to CSP variable, but values reflect probabilities not constraints.
 - Variable: A
 - Domain: $\{a_1, a_2, a_3\}$ <-- events / outcomes
- Types of Random Variables:
 - **Boolean** random variables : $\{true, false\}$
 - e.g., *Cavity* (= do I have a cavity?)
 - **Discrete** random variables : one value from a set of values
 - e.g., *Weather is one of {sunny, rainy, cloudy, snow}*
 - **Continuous** random variables : a value from within constraints
 - e.g., *Current temperature is bounded by (10°, 200°)*
- Domain values must be **exhaustive and mutually exclusive:**
 - One of the values must always be the case (**Exhaustive**)
 - Two of the values cannot both be the case (**Mutually Exclusive**)

Random variables

- **Example: Coin flip**
 - Variable = R, the result of the coin flip
 - Domain = {heads, tails, edge} } <-- must be exhaustive
 - $P(R = \text{heads}) = 0.4999$
 - $P(R = \text{tails}) = 0.4999$ } <-- must be exclusive
 - $P(R = \text{edge}) = 0.0002$

- Shorthand is often used for simplicity:
 - Upper-case letters for variables, lower-case letters for values.
 - E.g., $P(A) \equiv \langle P(A=a_1), P(A=a_2), \dots, P(A=a_n) \rangle$ for all n values in Domain(A)
 - Note: P(A) is a vector giving the probability that A takes on each of its n values in Domain (A)
 - E.g.,

$P(a)$	\equiv	$P(A = a)$
$P(a b)$	\equiv	$P(A = a \mid B = b)$
$P(a, b)$	\equiv	$P(A = a \wedge B = b)$

- Two kinds of probability propositions:
 - **Elementary propositions** are an assignment of a value to a random variable:
 - e.g., *Weather = sunny*; e.g., *Cavity = false* (abbreviated as *¬cavity*)
 - **Complex propositions** are formed from elementary propositions and standard logical connectives :
 - e.g., *Cavity = false \vee Weather = sunny*

Probability

- $P(a)$ is the probability of proposition “a”
 - E.g., $P(\text{it will rain in London tomorrow})$
 - The proposition “a” is actually true or false in the real world
 - $P(a)$ is our degree of belief that proposition “a” is true in the real world
 - $P(a)$ = “prior” or marginal or unconditional probability
 - Assumes no other information is available
- **Axioms of probability:**
 - $0 \leq P(a) \leq 1$
 - $P(\text{NOT}(a)) = 1 - P(a)$
 - $P(\text{true}) = 1$
 - $P(\text{false}) = 0$
 - $P(a \text{ OR } b) = P(a) + P(b) - P(a \text{ AND } b)$
- Any agent that holds degrees of beliefs that contradict these axioms will act sub-optimally in some cases
 - e.g., de Finetti (R&N pp. 489-490) proved that there will be some combination of bets that forces such an unhappy agent to lose money every time.
- **Rational agents cannot violate probability theory.**

Interpretations of probability

- **Relative Frequency:** *Usually taught in school*
 - $P(a)$ represents the frequency that event a will happen in repeated trials.
 - Requires event a to have happened enough times for data to be collected.
- **Degree of Belief:** *A more general view of probability*
 - $P(a)$ represents an agent's degree of belief that event a is true.
 - Can predict probabilities of events that occur rarely or have not yet occurred.
 - Does not require new or different rules, just a different interpretation.
- Examples:
 - a = "life exists on another planet"
 - What is $P(a)$? We all will assign different probabilities
 - a = "California will secede from the US"
 - What is $P(a)$?
 - a = "over 50% of the students in this class will get A's"
 - What is $P(a)$?

Concepts of probability

- Unconditional Probability

- $P(\mathbf{a})$, the probability of “a” being true, or $P(\mathbf{a}=\mathbf{True})$
- Does not depend on anything else to be true (**unconditional**)
- Represents the probability prior to further information that may adjust it (**prior**)
- Also sometimes “**marginal**” probability (vs. joint probability)

- Conditional Probability

- $P(\mathbf{a}|\mathbf{b})$, the probability of “a” being true, given that “b” is true
- Relies on “b” = true (**conditional**)
- Represents the prior probability adjusted based upon new information “b” (**posterior**)
- Can be generalized to more than 2 random variables:
 - e.g. $P(\mathbf{a}|\mathbf{b}, \mathbf{c}, \mathbf{d})$

We often use comma to abbreviate AND.

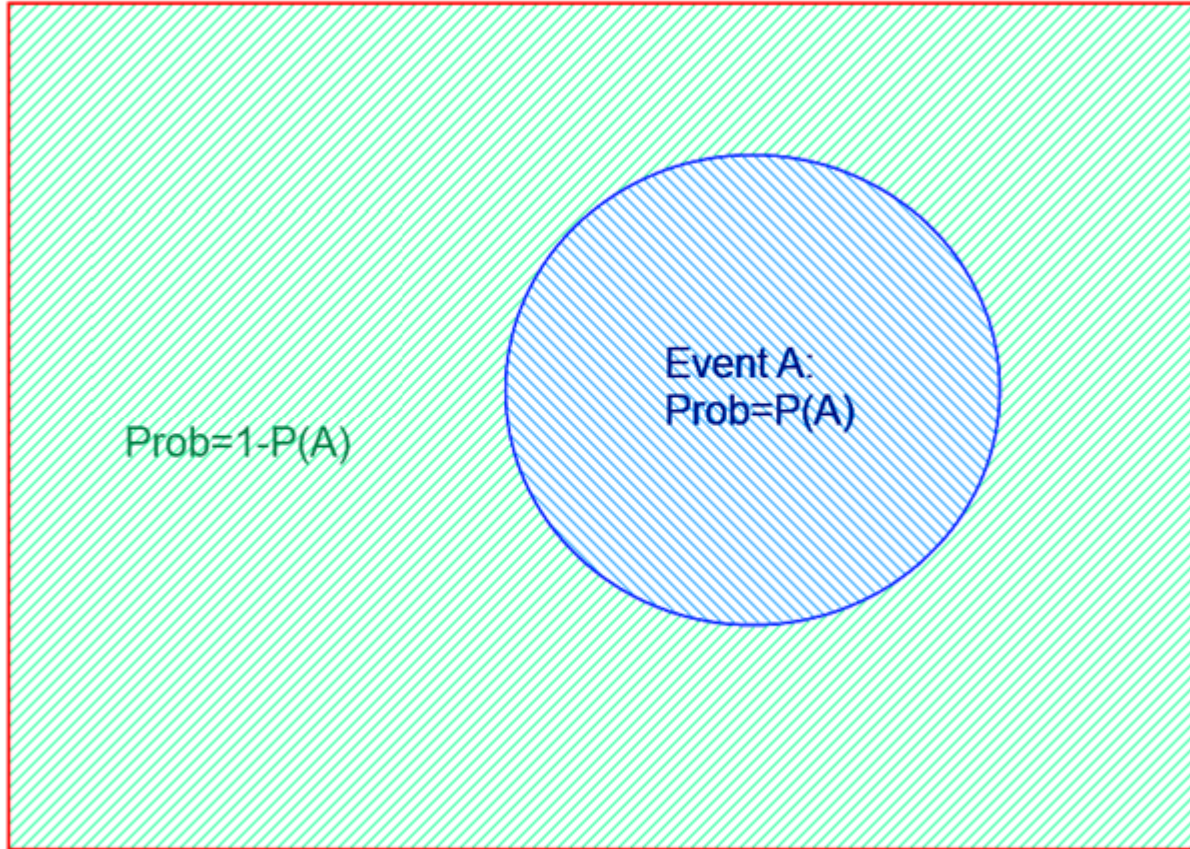
- Joint Probability

- $P(\mathbf{a}, \mathbf{b}) = P(\mathbf{a} \wedge \mathbf{b})$, the probability of “a” and “b” both being true
- Can be generalized to more than 2 random variables:
 - e.g. $P(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$

Probability Space

$$P(A) + P(\neg A) = 1$$

Entire Sample Space: $P(S)=1$

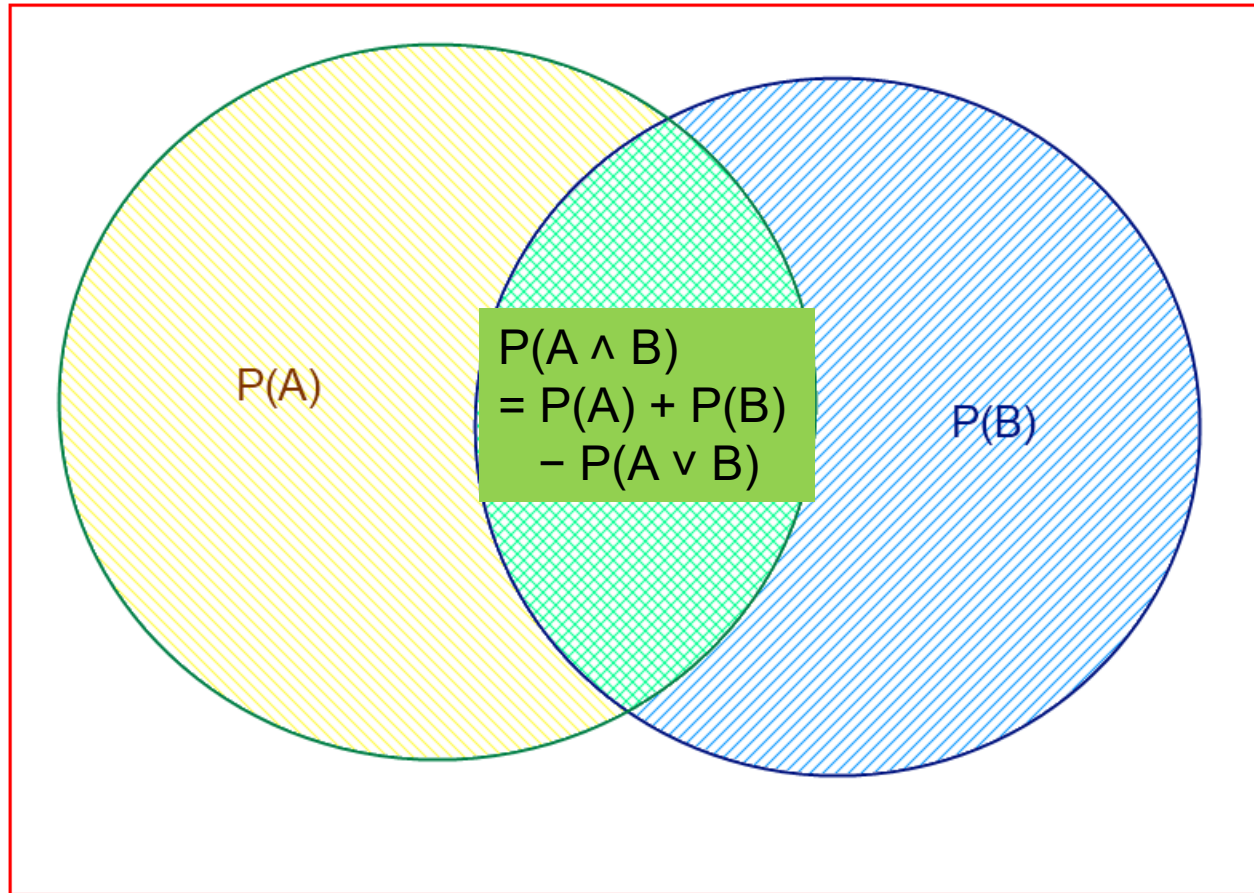


Area = Probability of Event

AND Probability

$$P(A, B) = P(A \wedge B) = P(A) + P(B) - P(A \vee B)$$

Entire Sample Space: $P(S)=1$

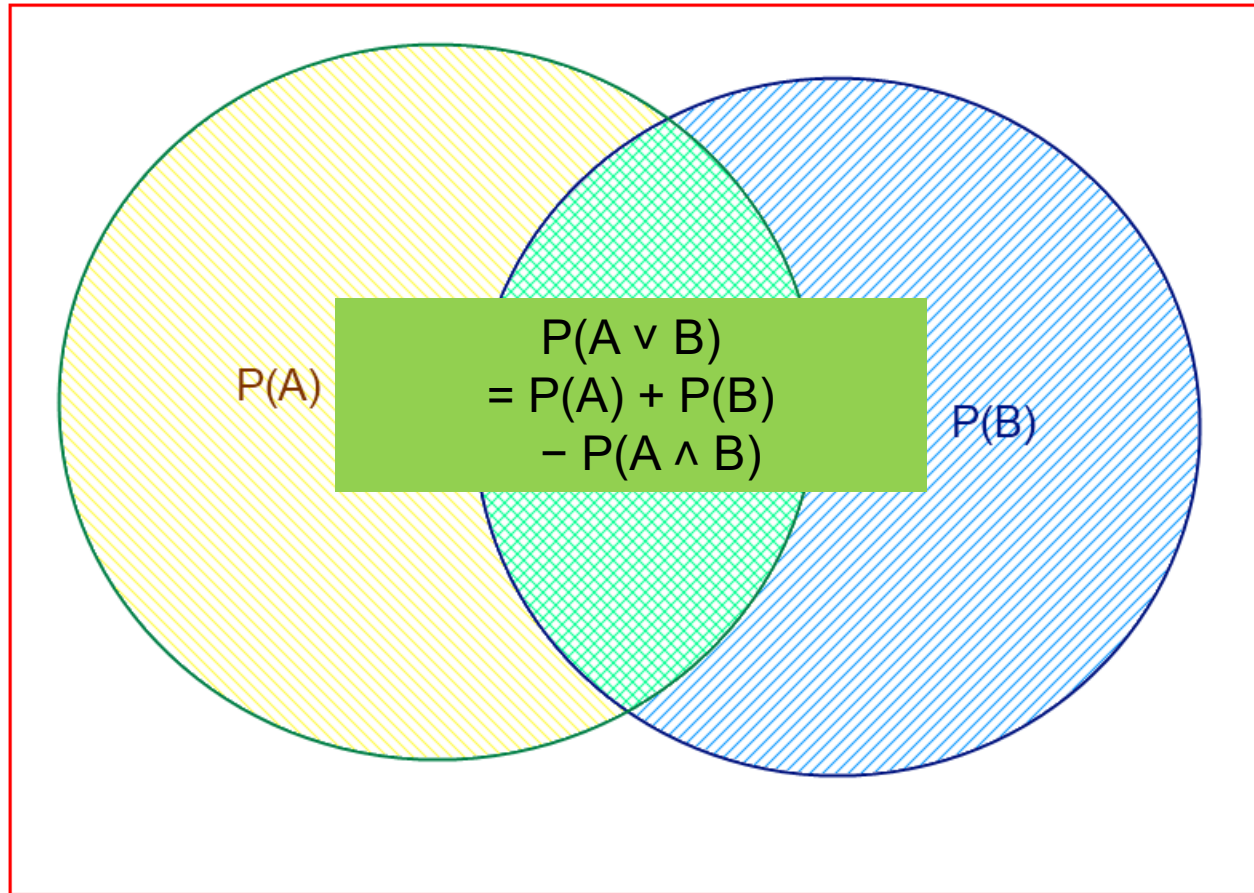


Area = Probability of Event

OR Probability

$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$

Entire Sample Space: $P(S)=1$

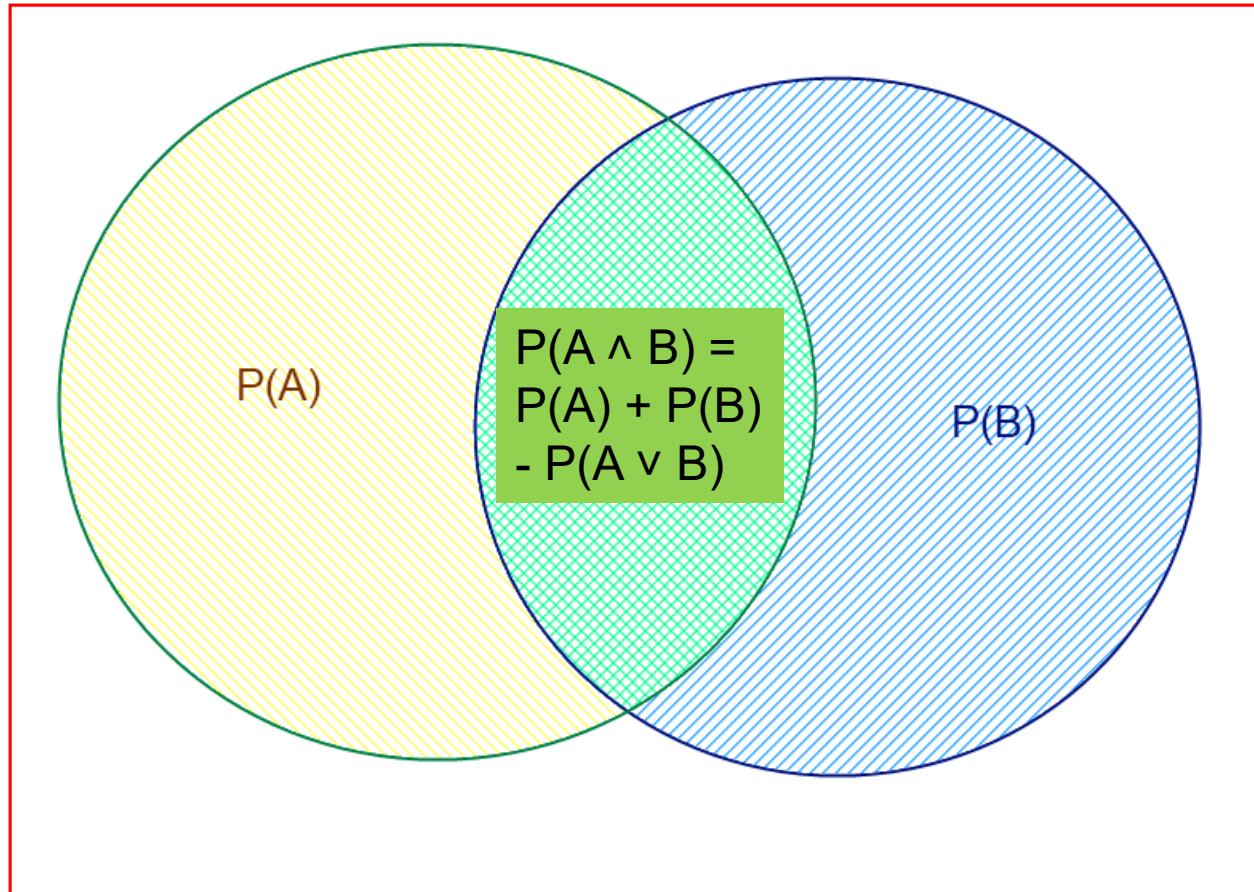


Area = Probability of Event

Conditional Probability

$$P(A | B) = P(A, B) / P(B) = P(A \wedge B) / P(B)$$

Entire Sample Space: $P(S)=1$

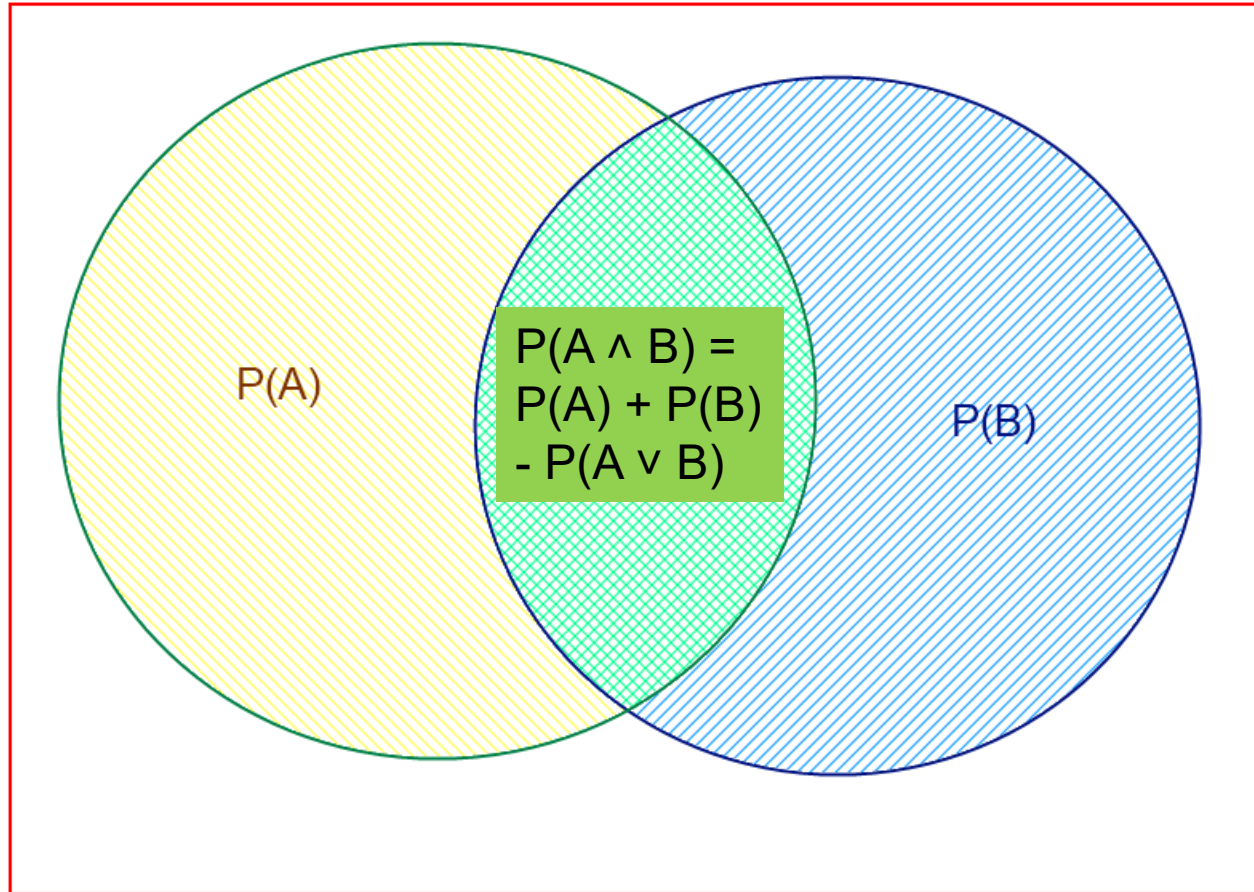


Area = Probability of Event

Product Rule

$$P(A, B) = P(A|B) P(B)$$

Entire Sample Space: $P(S)=1$



Area = Probability of Event

Using the Product Rule

- **Applies to any number of variables:**

- $P(a, b, c) = P(a, b | c) P(c) = P(a | b, c) P(b, c)$
- $P(a, b, c | d, e) = P(a | b, c, d, e) P(b, c | d, e)$

- **Factoring:** (AKA **Chain Rule** for probabilities)

- By the product rule, we can always write:

$$P(a, b, c, \dots y, z) = P(a | b, c, \dots y, z) P(b, c, \dots y, z)$$

- Repeating this idea, we can completely factor $P(a, b, \dots, z)$:

$$\begin{aligned} P(a, b, c, \dots y, z) \\ = P(a | b, c, \dots y, z) P(b | c, \dots y, z) P(c | \dots y, z) \dots P(y | z) P(z) \end{aligned}$$

- These relationships hold for any ordering of the variables

We often use comma to abbreviate AND.

Examples of complete Factoring

Using the Product Rule

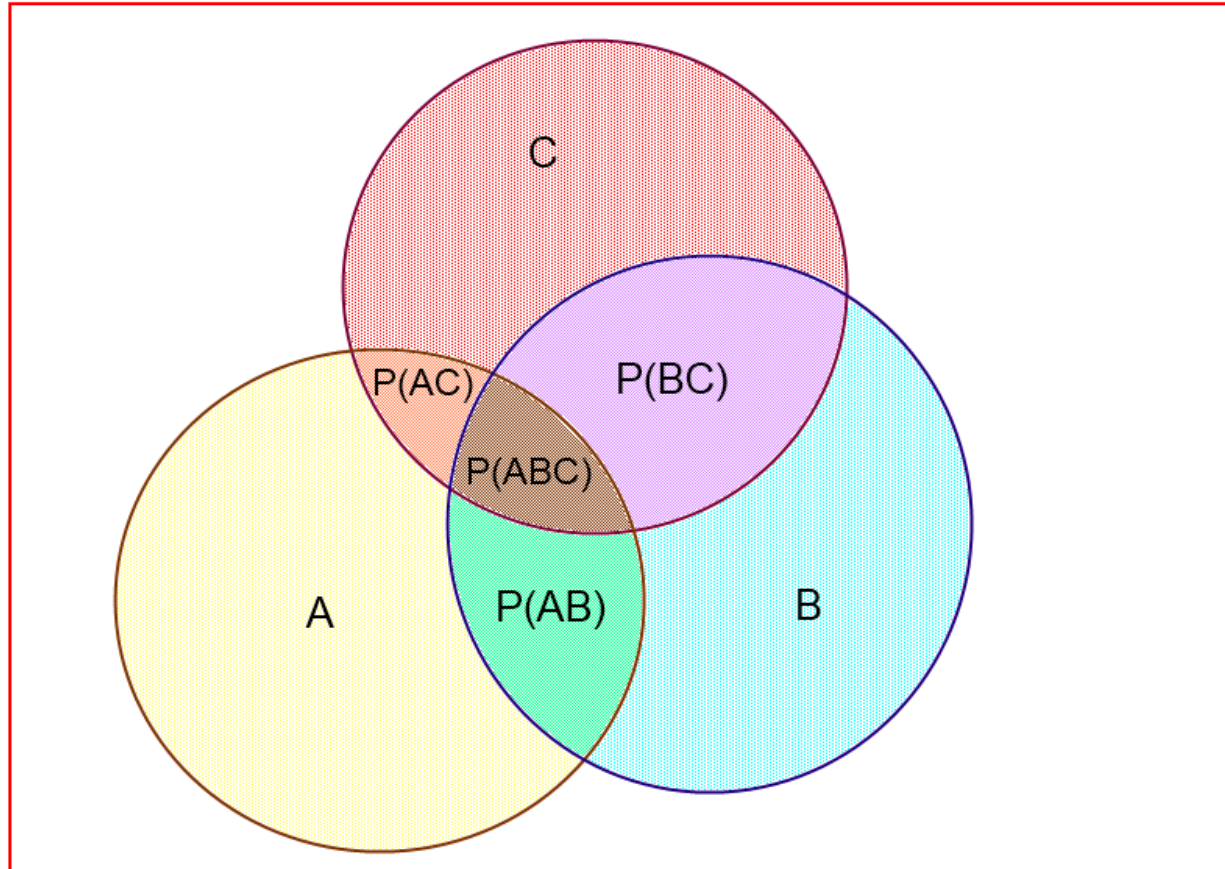
(can use any variable ordering)

- $P(a, b) = P(a|b)P(b)$
- $P(a, b, c) = P(a|b, c)P(b, c)$
 $= P(a|b, c)P(b|c)P(c)$ \leq complete factoring
- $P(a, b, c, d) = P(a|b, c, d)P(b, c, d)$
 $= P(a|b, c, d)P(b|c, d)P(c, d)$
 $= P(a|b, c, d)P(b|c, d)P(c|d)P(d)$ \leq complete factoring
- $P(a, b, c, d, e) = P(a|b, c, d, e)P(b, c, d, e)$
 $= P(a|b, c, d, e)P(b|c, d, e)P(c, d, e)$
 $= P(a|b, c, d, e)P(b|c, d, e)P(c|d, e)P(d, e)$
 $= P(a|b, c, d, e)P(b|c, d, e)P(c|d, e)P(d|e)P(e)$ \leq complete

Sum Rule

$$P(A) = \sum_{B,C} P(A,B,C) = \sum_{b \in B, c \in C} P(A,b,c)$$

Entire Sample Space: $P(S)=1$



Area = Probability of Event

Using the Sum Rule

- We can marginalize variables out of any joint distribution by simply summing over that variable:

- $P(b) = \sum_a \sum_c \sum_d P(a, b, c, d)$

- $P(a, d) = \sum_b \sum_c P(a, b, c, d)$

We often use comma to abbreviate AND.

- **For Example:** Determine probability of catching a fish

- Given a set of probabilities $P(\text{CatchFish}, \text{Day}, \text{Lake})$

- Where:

- $\text{CatchFish} = \{true, false\}$

- $\text{Day} = \{mon, tues, wed, thurs, fri, sat, sun\}$

- $\text{Lake} = \{blue\ lake, ralph\ lake, crystal\ lake\}$

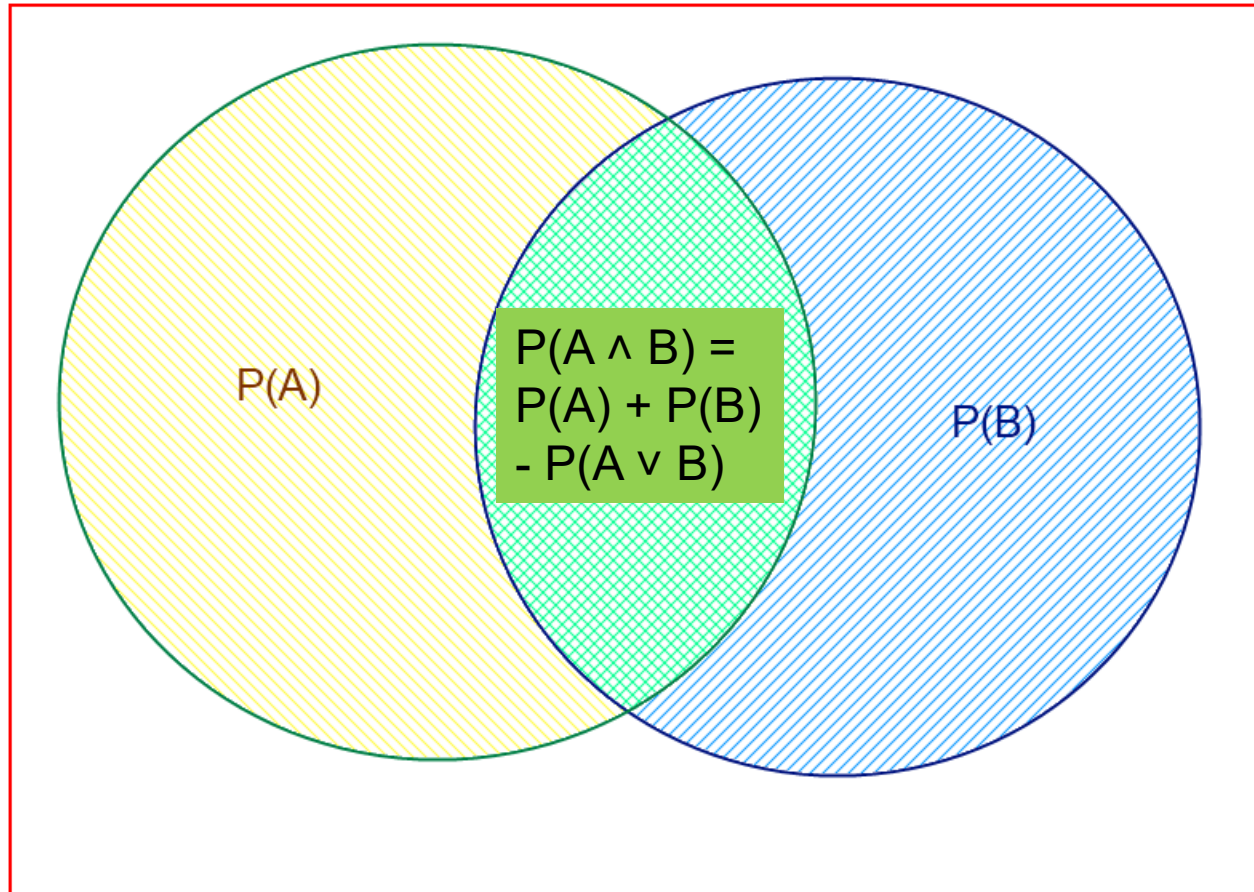
- Need to find $P(\text{CatchFish} = \text{True})$:

- $P(\text{CatchFish} = true) = \sum_{day} \sum_{lake} P(\text{CatchFish} = true, day, lake)$

Bayes' Rule

$$P(B|A) = P(A|B) P(B) / P(A)$$

Entire Sample Space: $P(S)=1$



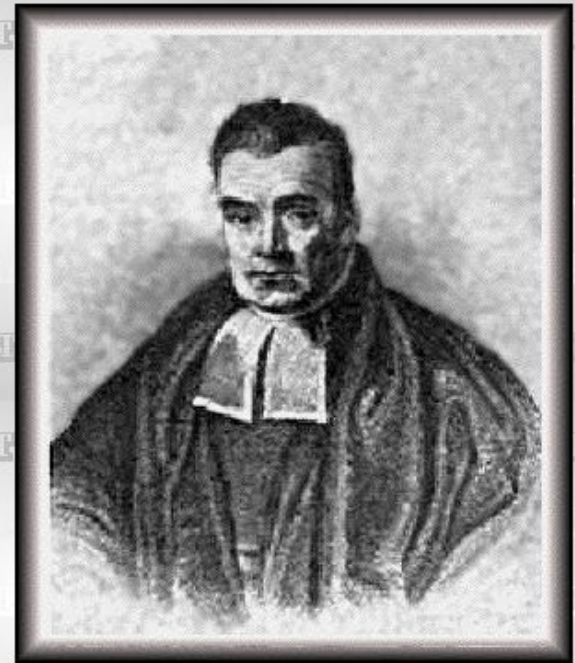
Area = Probability of Event

Derivation of Bayes' Rule

- **Start from Product Rule:**
 - $P(a, b) = P(a|b) P(b) = P(b|a) P(a)$
- **Isolate Equality on Right Side:**
 - $P(a|b) P(b) = P(b|a) P(a)$
- **Divide through by P(b):**
 - $P(a|b) = P(b|a) P(a) / P(b)$ <-- Bayes' Rule
- **“Bayes' rule underlies most modern approaches to uncertain reasoning in AI systems.” — R&N p. 9**

Who's Bayes?

- Reverend Thomas Bayes (c. 1701 – 1761) was an English minister and mathematician. **His ideas have created much controversy and debate among statisticians....**
- The paper that describes Bayes' Theorem (or Bayes' Rule) was discovered in his office after his death. Allegedly, he was trying to prove the existence of God by mathematics; though this is not certain and other motives also are alleged. His paper was sent to the Royal Society with a note, "Some of your members may be interested in this." It was published by, and read to, the Royal Society. **Nowadays, it has given rise to an immense body of statistical and probabilistic work.**



Thomas Bayes

Portrait purportedly of Bayes used in a 1936 book, but it is doubtful the portrait is actually of him. No earlier claimed portrait survives.

Summary of probability rules

- **Product Rule:** (aka **Chain Rule**)
 - $P(\mathbf{a}, \mathbf{b}) = P(\mathbf{a} | \mathbf{b}) P(\mathbf{b}) = P(\mathbf{b} | \mathbf{a}) P(\mathbf{a})$ Probability of “a” and “b” occurring is the same as probability of “a” occurring given “b” is true, times the probability of “b” occurring.
 - e.g., $P(\text{rain}, \text{cloudy}) = P(\text{rain} | \text{cloudy}) * P(\text{cloudy})$
 - $P(\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots, \mathbf{y}, \mathbf{z}) = P(\mathbf{a} | \mathbf{b}, \mathbf{c}, \dots, \mathbf{y}, \mathbf{z}) P(\mathbf{b} | \mathbf{c}, \dots, \mathbf{y}, \mathbf{z}) \dots P(\mathbf{y} | \mathbf{z}) P(\mathbf{z})$
- **Sum Rule:** (aka **Law of Total Probability**)
 - $P(\mathbf{a}) = \sum_b P(\mathbf{a}, \mathbf{b}) = \sum_b P(\mathbf{a} | \mathbf{b}) P(\mathbf{b})$, where B is any random variable
 - Probability of “a” occurring is the same as the sum of all joint probabilities including the event, provided the joint probabilities represent all possible events.
 - Can be used to “marginalize” out other variables from probabilities, resulting in prior probabilities also being called marginal probabilities.
 - e.g., $P(\text{rain}) = \sum_{\text{Windspeed}} P(\text{rain}, \text{Windspeed})$
where $\text{Windspeed} = \{0\text{-}10\text{mph}, 10\text{-}20\text{mph}, 20\text{-}30\text{mph}, \text{etc.}\}$
- **Bayes’ Rule:**
 - $P(\mathbf{b} | \mathbf{a}) = P(\mathbf{a} | \mathbf{b}) P(\mathbf{b}) / P(\mathbf{a})$
 - Acquired from rearranging the product rule.
 - Allows conversion between conditionals, from $P(\mathbf{b} | \mathbf{a})$ to $P(\mathbf{a} | \mathbf{b})$.
 - e.g., b = disease, a = symptoms
More natural to encode knowledge as $P(\mathbf{a} | \mathbf{b})$ than as $P(\mathbf{b} | \mathbf{a})$.

Full Joint Distribution

- We can fully specify a probability space by a **full joint distribution**:

- A full joint distribution contains a probability for every possible combination of variable values. This requires:

$\prod_{\text{vars}} (n_{\text{var}})$ probabilities

where n_{var} is the number of values in the domain of variable **var**

- E.g. $P(A, B, C)$, where A,B,C have 4 values each;
Full joint distribution specified by 4^3 values = 64 values

T	D	C	P(T,D,C)
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108

- For n variables each with m values, requires m^n probabilities
 - E.g., a realistic problem of 100 Boolean variables requires $> 10^{30}$ probabilities (intractable)
- Using a full joint distribution, we can use the product rule, sum rule, and Bayes' rule to create any combination of joint, marginal, and conditional probabilities.

Marginal Probability

- Can fully specify a probability space by constructing a full joint distribution
- Example: dentist
 - T: have a toothache
 - D: dental probe catches
 - C: have a cavity
- Joint distribution
 - Assigns each event (T=t, D=d, C=c) a probability
 - Probabilities sum to 1.0
- Law of total probability:

T	D	C	P(T,D,C)
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108

$$\begin{aligned}
 p(C = 1) &= \sum_{t,d} P(T = t, D = d, C = 1) \\
 &= 0.008 + 0.072 + 0.012 + 0.108 = 0.20
 \end{aligned}$$

- *Some* value of (T,D) must occur; values are disjoint
- “Marginal probability” of C; “marginalize” or “sum over” T,D
- Early actuaries wrote row & column totals in their probability table margins

The effect of evidence

T	D	C	P(T,D,C)
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
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T	D	C	P(T,D,C)
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1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108

- Example: dentist
 - T: have a toothache
 - D: dental probe catches
 - C: have a cavity
- Recall $p(C=1) = 0.20$
- Suppose we observe $D=0, T=0$?

$$p(C = 1 | D = 0, T = 0) = \frac{p(C = 1, D = 0, T = 0)}{p(D = 0, T = 0)}$$

$$= \frac{0.008}{0.576 + 0.008} = 0.012$$

Called *posterior probabilities* or *conditional probabilities*

- Observe $D=1, T=1$?

$$p(C = 1 | D = 1, T = 1) = \frac{0.108}{0.016 + 0.108} = 0.871$$

The effect of evidence

- Example: dentist
 - T: have a toothache
 - D: dental probe catches
 - C: have a cavity

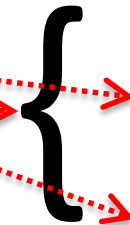
- Combining these rules:

$$p(C = 1 | T = 1) = \frac{p(C = 1, T = 1)}{p(T = 1)}$$

$$= \frac{0.012 + 0.108}{0.064 + 0.012 + 0.016 + 0.108} = 0.60$$

$$p(T = 1) = 0.20$$

T	D	C	P(T,D,C)
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108



Called the *probability of evidence*

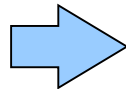
Computing posteriors

- Sometimes it is easiest to normalize last

$$p(C|T = 1) = \frac{1}{p(T = 1)} p(C, T = 1) \propto p(C, T = 1) = \sum_d p(C, d, T = 1)$$

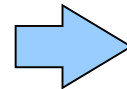
T	D	C	P(T,D,C)
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108

Assign T=1



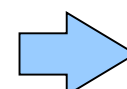
D	C	F(D,C)
0	0	0.064
0	1	0.012
1	0	0.016
1	1	0.108

Sum over D



C	G(C)
0	0.08
1	0.120

Normalize



C	P(C T=1)
0	0.40
1	0.60

- The normalizing constant α is used to abbreviate normalization

$$p(C|T = 1) = \alpha \sum_d p(C, d, T = 1) = \sum_d p(C, d, T = 1) / p(T = 1)$$



Independence

- X, Y independent:
 - $p(X=x, Y=y) = p(X=x) p(Y=y)$ for all x, y
 - Shorthand: $p(X, Y) = P(X) P(Y)$
 - Equivalent: $p(X|Y) = p(X)$ or $p(Y|X) = p(Y)$ (if $p(Y), p(X) > 0$)
 - Intuition: knowing X has no information about Y (or vice versa)

Independent probability distributions:

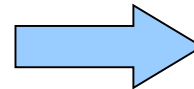
$$P(A, B, C) = P(A) * P(B) * P(C)$$

A	P(A)
0	0.4
1	0.6

B	P(B)
0	0.7
1	0.3

C	P(C)
0	0.1
1	0.9

Joint:



A	B	C	P(A,B,C)
0	0	0	.4 * .7 * .1 = .028
0	0	1	.4 * .7 * .9 = .252
0	1	0	.4 * .3 * .1 = .012
0	1	1	.4 * .3 * .9 = .108
1	0	0	.6 * .7 * .1 = .042
1	0	1	.6 * .7 * .9 = .378
1	1	0	.6 * .3 * .1 = .018
1	1	1	.6 * .3 * .9 = .162

This property can **greatly** reduce representation size!

Note: it is hard to “read” independence from the joint distribution.

We can “test” for it, but to do so requires a number of tests equal to the size of the joint distribution.

We may omit leading zeroes to save space and effort.

Conditional Independence

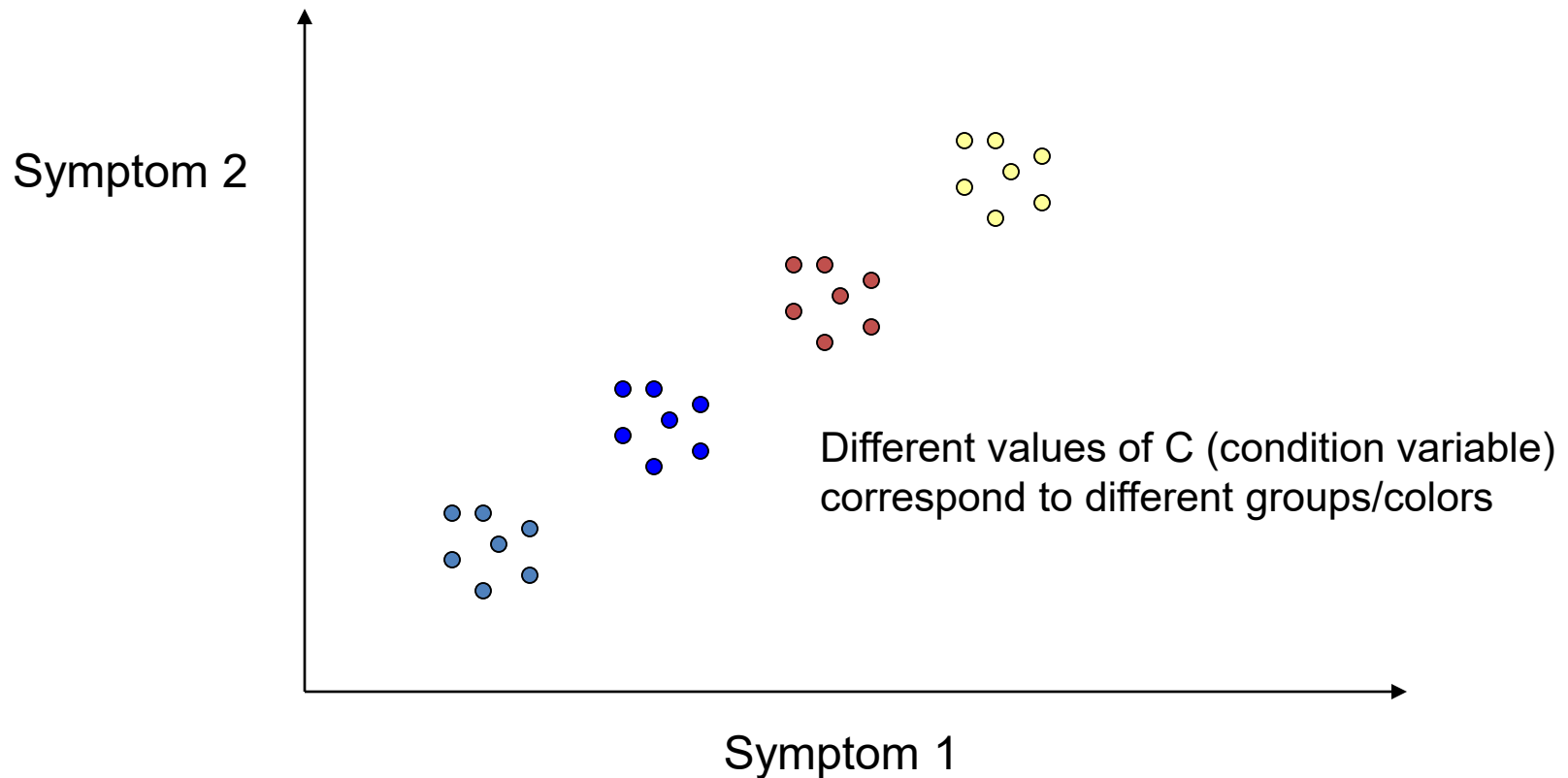
- X, Y independent given Z
 - $p(X=x, Y=y | Z=z) = p(X=x | Z=z) p(Y=y | Z=z)$ for all x, y, z
 - Equivalent: $p(X|Y,Z) = p(X|Z)$ or $p(Y|X,Z) = p(Y|Z)$ (if all > 0)
 - Intuition: X has no additional info about Y beyond Z's

- Example

X = height $p(\text{height} | \text{reading}, \text{age}) = p(\text{height} | \text{age})$
Y = reading ability $p(\text{reading} | \text{height}, \text{age}) = p(\text{reading} | \text{age})$
Z = age

Height and reading ability are dependent (not independent), but are conditionally independent given age

Conditional Independence



Symptom 1 and symptom 2 are conditionally independent, given group.

But clearly, symptom 1 and 2 are marginally dependent, unconditionally.

Conditional Independence Example:

- X, Y independent given Z
 - $p(X=x, Y=y | Z=z) = p(X=x | Z=z) p(Y=y | Z=z)$ for all x, y, z
- A box contains two coins: one regular coin, $P(\text{heads}) = .5$, and one fake two-headed coin, $P(\text{heads})=1$. I choose a coin at random and toss it twice. Define the following events.
 - $A =$ First coin toss results in heads
 - $B =$ Second coin toss results in heads
 - $C =$ Coin 1 (regular) has been selected
- $P(A \wedge B) = 5/8 \neq P(A) P(B) = 9/16$, so A and B are not independent
 - Event A makes it more likely I selected the two-headed coin, which makes Event B more likely. Knowing Event A gives information about Event B .
- $P(A \wedge B | C) = 1/4 = P(A | C) P(B | C)$, so A and B are independent given C
 - Given C , knowing Event A gives **no** information about Event B .

Conditional Independence Example:

- X, Y independent given Z
 - $p(X=x, Y=y | Z=z) = p(X=x | Z=z) p(Y=y | Z=z)$ for all x, y, z
- Consider two brothers John and Joseph, both having a genetic disease. These two events are dependent as they are brothers.
- However, given the condition that Joseph is an adopted son of the family makes the events conditionally independent.

Conditional Independence Example:

- X, Y independent given Z
 - $p(X=x, Y=y | Z=z) = p(X=x | Z=z) p(Y=y | Z=z)$ for all x, y, z
- Rain causes both increased umbrella usage and worsened road conditions. These events are not independent because seeing lots of umbrellas makes worsened road conditions more likely.
- However, given the condition that it is raining makes the events conditionally independent. Once you know it is raining, seeing umbrellas tells you nothing more about road conditions.

Conditional Independence

- X, Y independent given Z
 - $p(X=x, Y=y | Z=z) = p(X=x | Z=z) p(Y=y | Z=z)$ for all x, y, z
 - Equivalent: $p(X|Y, Z) = p(X|Z)$ or $p(Y|X, Z) = p(Y|Z)$
 - Intuition: X has no additional info about Y beyond Z's

T	D	C	P(T,D,C)
0	0	0	0.576
0	0	1	0.008
0	1	0	0.144
0	1	1	0.072
1	0	0	0.064
1	0	1	0.012
1	1	0	0.016
1	1	1	0.108

- Example: Dentist Conditionally independent distributions:

– $P(T,D|C) = P(T|C) * P(D|C)$

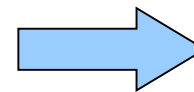
T	C	P(T C)
0	0	.9
0	1	.4
1	0	.1
1	1	.6

Again, hard to “read” from the joint probabilities; only from the conditional probabilities.

Like independence, can **greatly** reduce representation size!

We may omit leading zeroes to save space and effort.

D	C	P(D C)
0	0	.8
0	1	.1
1	0	.2
1	1	.9



Joint:

Conditional probabilities:

T	D	C	P(T,D C)
0	0	0	.9 * .8 = .72
0	0	1	.4 * .1 = .04
0	1	0	.9 * .2 = .18
0	1	1	.4 * .9 = .36
1	0	0	.1 * .8 = .08
1	0	1	.6 * .1 = .06
1	1	0	.1 * .2 = .02
1	1	1	.6 * .9 = .54

Conditional Independence

- Formal Definition:

- 2 random variables A and B are **conditionally independent** given C iff:

$$P(a, b | c) = P(a | c) P(b | c), \quad \text{for all values } a, b, c$$

- Informal Definition:

- 2 random variables A and B are **conditionally independent** given C iff:

$$P(a | b, c) = P(a | c) \quad \text{OR} \quad P(b | a, c) = P(b | c), \quad \text{for all values } a, b, c$$

- $P(a | b, c) = P(a | c)$ tells us that learning about b, given that we already know c, provides no change in our probability for a, and thus b contains no information about a beyond what c provides.

- Naïve Bayes Model:

- Often a single variable can directly influence a number of other variables, all of which are conditionally independent, given the single variable.
- E.g., k different symptom variables X_1, X_2, \dots, X_k , and $C = \text{disease}$, reducing to:

$$P(C | X_1, X_2, \dots, X_k) = \alpha P(C) \prod P(X_i | C)$$



The normalizing constant α is used to abbreviate normalization

Full Joint vs Conditional Independence

- Example : 4 Binary Random Variable (A,B,C,D)
 - Full Joint Probability Table
 - 1 Table with 16 rows
 - Conditional Independence
 - $P(A,B,C,D) = P(A) P(B|A) P(C|A, B) P(D|A, B, C)$ (no saving yet..)
 - if... $P(C|A, B) = P(C|A)$, $P(D|A, B, C) = P(D|A)$ [Naïve Bayes Model]
 - $P(A,B,C,D) = P(A) P(B|A) P(C|A) P(D|A)$
 - 4 Tables with $2 + 4 + 4 + 4 = 14$ rows
- If we had N Binary Random Variables
 - Full Joint Probability Table
 - 1 Table with 2^N Rows; $N = 100$, $2^{100} \approx 10^{30}$
 - Naïve Bayes Model (Conditional Independence)
 - N tables with at most 4 rows!

Conclusions...

- Representing uncertainty is useful in knowledge bases.
- Probability provides a framework for managing uncertainty.
- Using a full joint distribution and probability rules, we can derive any probability relationship in a probability space.
- Number of required probabilities can be reduced through independence and conditional independence relationships
- Probabilities allow us to make better decisions by using decision theory and expected utilities.
- **Rational agents cannot violate probability theory.**