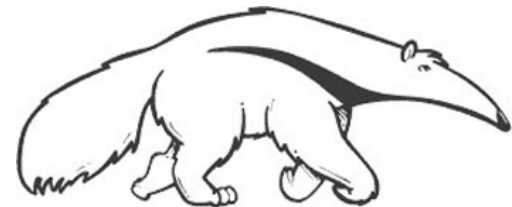


Constraint Satisfaction Problems A: Definition, Search Strategies

Introduction to Artificial Intelligence
Prof. Richard Lathrop



Read Beforehand: R&N 6.1-6.4, except 6.3.3

Constraint Satisfaction Problems

- What is a CSP?
 - Finite set of variables, X_1, X_2, \dots, X_n
 - Nonempty domain of possible values for each: D_1, \dots, D_n
 - Finite set of constraints, C_1, \dots, C_m
 - Each constraint C_i limits the values that variables can take, e.g., $X_1 \neq X_2$
 - Each constraint C_i is a pair: $C_i = (\text{scope}, \text{relation})$
 - Scope = tuple of variables that participate in the constraint
 - Relation = list of allowed combinations of variables
 - May be an explicit list of allowed combinations
 - May be an abstract relation allowing membership testing & listing
- CSP benefits
 - Standard representation pattern
 - Generic goal and successor functions
 - Generic heuristics (no domain-specific expertise required)

Example: Sudoku

- Problem specification

Variables: {A1, A2, A3, ... I7, I8, I9}

Domains: $D_i = \{ 1, 2, 3, \dots, 9 \}$

Constraints:

each row, column “all different”

$\text{alldiff}(A1, A2, A3, \dots, A9), \dots$

each 3x3 block “all different”

$\text{alldiff}(G7, G8, G9, H7, \dots, I9), \dots$

	1	2	3	4	5	6	7	8	9
A			2	4		6			
B	8	6	5	1			2		
C		1				8	6		9
D	9				4		8	6	
E		4	7				1	9	
F		5	8		6				3
G	4		6	9				7	
H			9			4	5	8	1
I				3		2	9		

Task: solve (complete a partial solution)

check “well-posed”: exactly one solution?

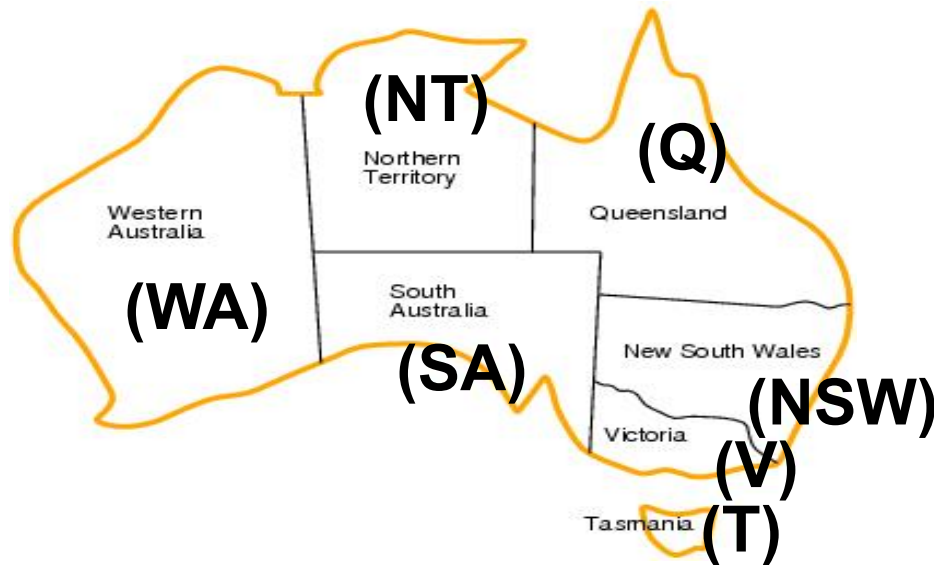
CSPs --- what is a solution?

- A **state** is an **assignment** of values to some variables.
 - **Complete** assignment
 - = every variable has a value.
 - **Partial** assignment
 - = some variables have no values.
 - **Consistent** assignment
 - = assignment does not violate any constraints
- A **solution** is a **complete** and **consistent** assignment.

CSPs with objective functions

- A solution may have to maximize an objective function
 - Preferences, often called “soft” constraints
 - Example: linear objective function
 - => linear programming or integer linear programming
 - Example: “Weighted” CSPs where each variable has a cost
- Examples of CSP applications
 - Scheduling the time of observations on a space telescope
 - Airline flight scheduling
 - Cryptography
 - Job shop scheduling
 - Classroom scheduling
 - Computer vision, image interpretation

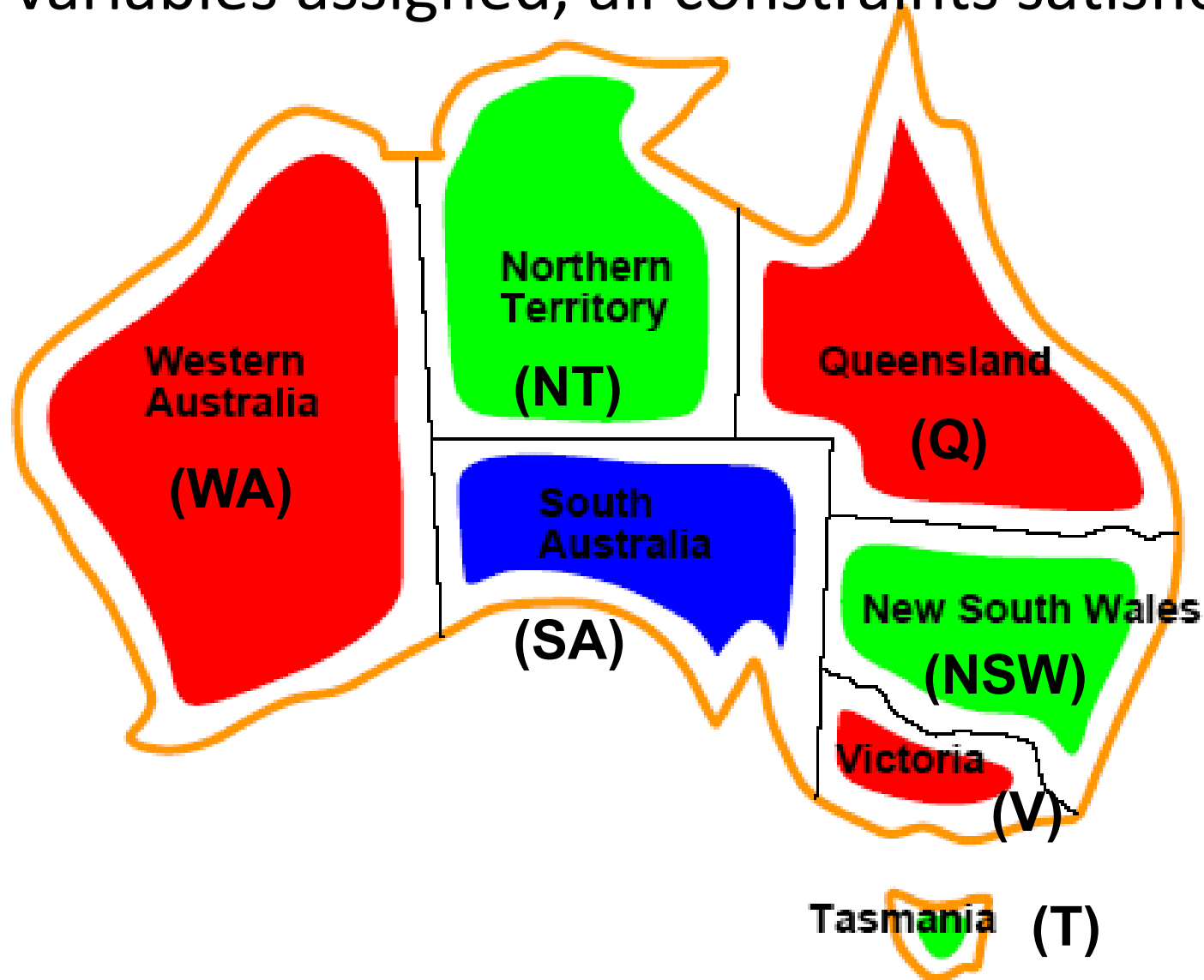
CSP example: map coloring



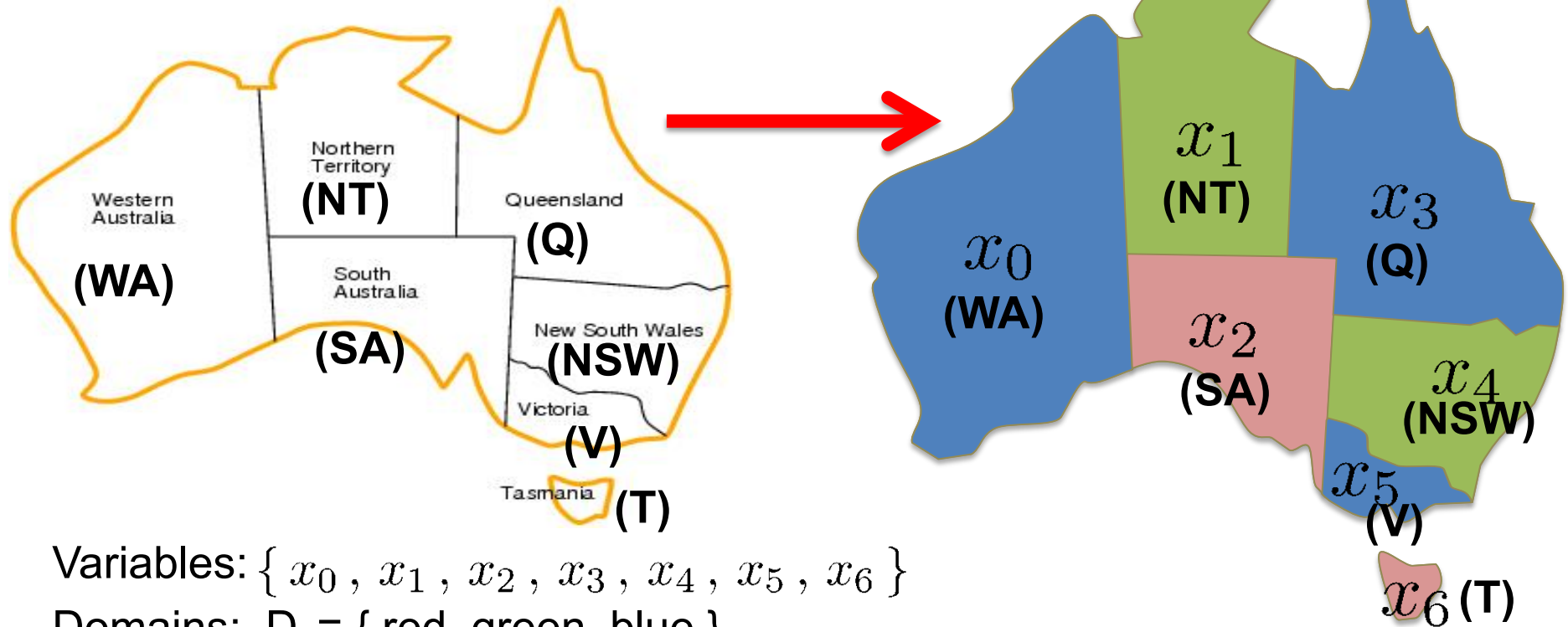
- **Variables:** WA, NT, Q, NSW, V, SA, T
- **Domains:** $D_i = \{red, green, blue\}$
- **Constraints:** Adjacent regions must have different colors, e.g., $WA \neq NT$.

Example: Map coloring solution

All variables assigned, all constraints satisfied.



Example: Map Coloring



Variables: $\{ x_0, x_1, x_2, x_3, x_4, x_5, x_6 \}$

Domains: $D_i = \{ \text{red, green, blue} \}$

Constraints: bordering regions must have different colors:

$$x_0 \neq x_1, x_0 \neq x_2, x_1 \neq x_2, \dots$$

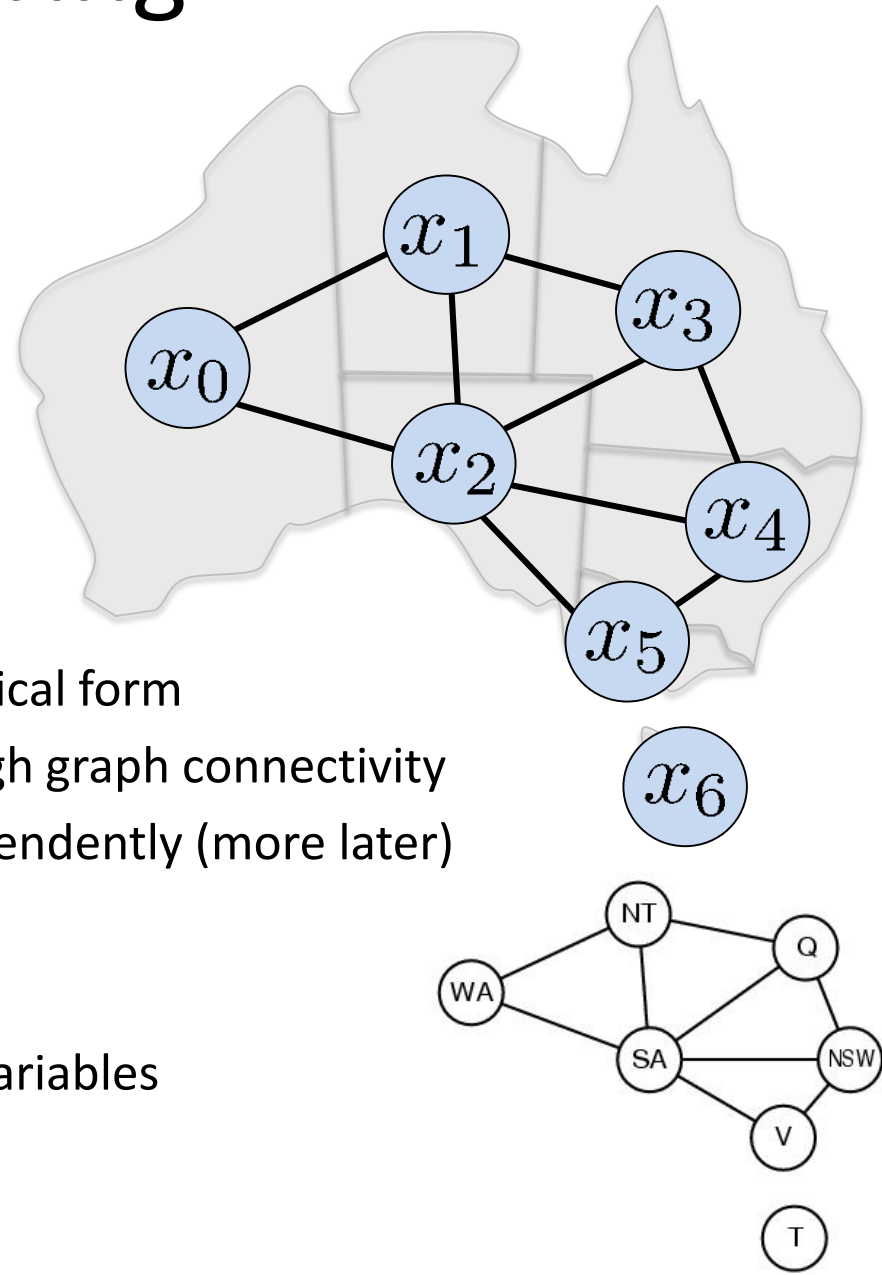
A **solution** is any setting of the variables that satisfies all the constraints, e.g.,

$$x_0 = \text{blue}, x_1 = \text{green}, x_2 = \text{red}, x_3 = \text{blue},$$

$$x_4 = \text{green}, x_5 = \text{blue}, x_6 = \text{red}$$

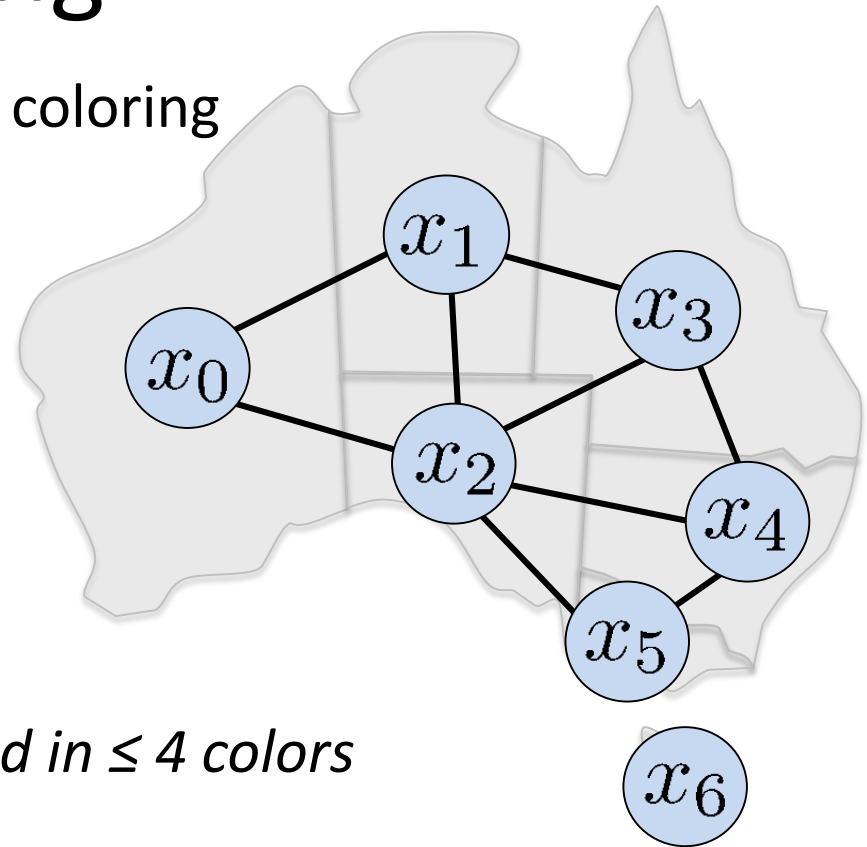
Example: Map Coloring

- Constraint graph
 - Vertices: variables
 - Edges: constraints (connect involved variables)
- Graphical model
 - Abstracts the problem to a canonical form
 - Can reason about problem through graph connectivity
 - Ex: Tasmania can be solved independently (more later)
- Binary CSP
 - Constraints involve at most two variables
 - Sometimes called “pairwise”



Aside: Graph coloring

- More general problem than map coloring
- Planar graph:
graph in 2D plane with no edge crossings
- Guthrie's conjecture (1852)
Every planar graph can be colored in ≤ 4 colors
- Proved (using a computer) in 1977 ([Appel & Haken 1977](#))



Varieties of CSPs

- Discrete variables
 - Finite domains, size $d \Rightarrow O(d^n)$ complete assignments
 - Ex: Boolean CSPs: Boolean satisfiability (NP-complete)
 - Infinite domains (integers, strings, etc.)
 - Ex: Job scheduling, variables are start/end days for each job
 - Need a constraint language, e.g., $\text{StartJob}_1 + 5 < \text{StartJob}_3$
 - Infinitely many solutions
 - Linear constraints: solvable
 - Nonlinear: no general algorithm
- Continuous variables
 - Ex: Building an airline schedule or class schedule
 - Linear constraints: solvable in polynomial time by LP methods

Varieties of constraints

- **Unary** constraints involve a single variable,
 - e.g., $SA \neq \text{green}$
- **Binary** constraints involve pairs of variables,
 - e.g., $SA \neq WA$
- **Higher-order** constraints involve 3 or more variables,
 - Ex: jobs A,B,C cannot all be run at the same time
 - Can always be expressed using multiple binary constraints
- **Preference** (soft constraints)
 - Ex: “red is better than green” can often be represented by a cost for each variable assignment
 - Combines optimization with CSPs

Simplify...

- We restrict attention to:
 - **Discrete & finite domains**
 - Variables have a discrete, finite set of values
 - **No objective function**
 - Any complete & consistent solution is OK
 - **Solution**
 - Find a complete & consistent assignment
- Example: Sudoku puzzles

Binary CSPs

CSPs only need binary constraints!

- Unary constraints

- Just delete values from the variable's domain

- Higher order (3 or more variables): reduce to binary

- Simple example: 3 variables X, Y, Z
- Domains $D_x = \{1, 2, 3\}$, $D_y = \{1, 2, 3\}$, $D_z = \{1, 2, 3\}$
- Constraint $C[X, Y, Z] = \{X + Y = Z\} = \{(1, 1, 2), (1, 2, 3), (2, 1, 3)\}$
(Plus other variables & constraints elsewhere in the CSP)
- Create a new variable W , taking values as triples (3-tuples)
- Domain of W is $D_w = \{(1, 1, 2), (1, 2, 3), (2, 1, 3)\}$
 - D_w is exactly the tuples that satisfy the higher-order constraint
- Create three new constraints:
 - $C[X, W] = \{ [1, (1, 1, 2)], [1, (1, 2, 3)], [2, (2, 1, 3)] \}$
 - $C[Y, W] = \{ [1, (1, 1, 2)], [2, (1, 2, 3)], [1, (2, 1, 3)] \}$
 - $C[Z, W] = \{ [2, (1, 1, 2)], [3, (1, 2, 3)], [3, (2, 1, 3)] \}$

Other constraints elsewhere involving X, Y, Z are unaffected

Example: Cryptarithmic problems

- Find numeric substitutions that make an equation hold:

$$\begin{array}{r} \\ \\ + \\ \hline = \end{array}$$

For example:

$$O = 4$$

$$R = 8$$

$$W = 3$$

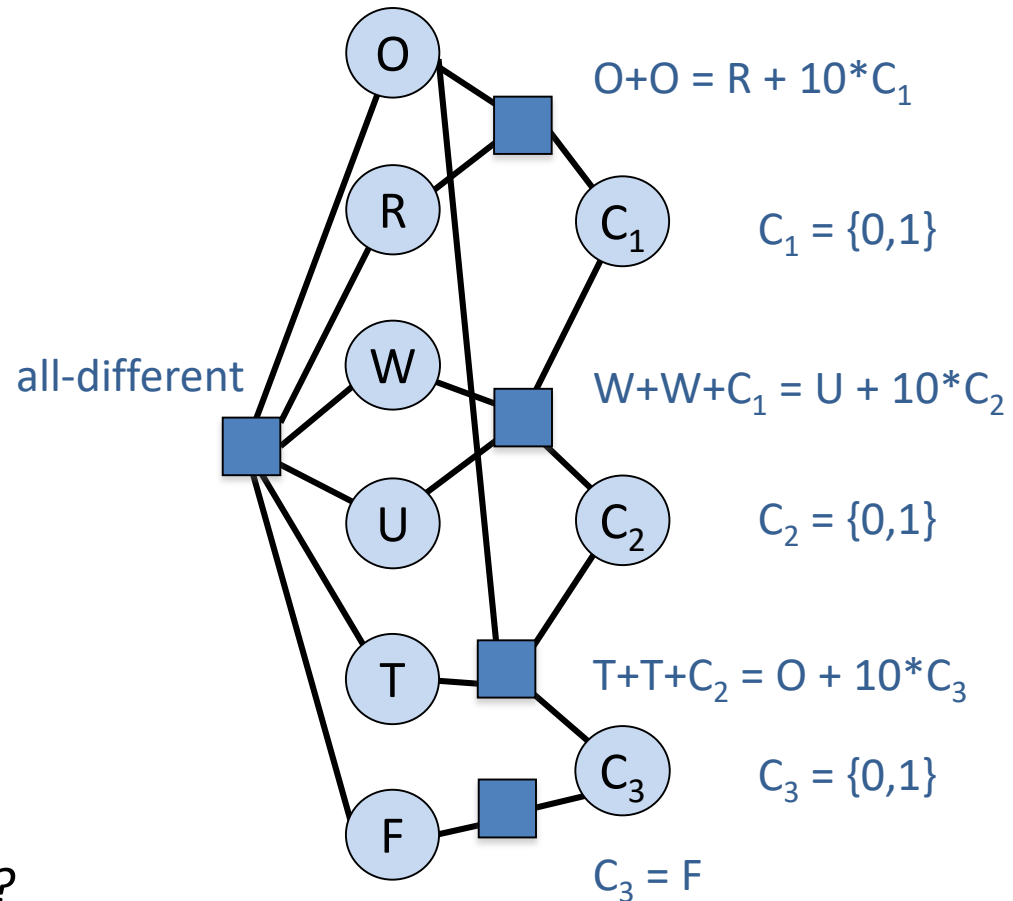
$$U = 6$$

$$T = 7$$

$$F = 1$$

$$\begin{array}{r} \\ \\ + \\ \hline = 1 \ 4 \ 6 \ 8 \end{array}$$

Non-pairwise CSP:



Note: not unique – how many solutions?

Example: Cryptarithmic problems

- Try it yourself at home:

$$\begin{array}{r} \\ \\ + \\ \hline = \end{array}$$

(a frequent request from college students to parents)

Random binary CSPs

- A random binary CSP is defined by a four-tuple (n, d, p_1, p_2)
 - n = the number of variables.
 - d = the domain size of each variable.
 - p_1 = probability a constraint exists between two variables.
 - p_2 = probability a pair of values in the domains of two variables connected by a constraint is incompatible.
 - Note that R&N lists compatible pairs of values instead.
 - Equivalent formulations; just take the set complement.
- (n, d, p_1, p_2) generate random binary constraints
- The so-called “model B” of Random CSP (n, d, n_1, n_2)
 - $n_1 = p_1 n(n-1)/2$ pairs of variables are randomly and uniformly selected and binary constraints are posted between them.
 - For each constraint, $n_2 = p_2 d^2$ randomly and uniformly selected pairs of values are picked as incompatible.
- The random CSP as an optimization problem (minCSP).
 - Goal is to minimize the total sum of values for all variables.

CSP as a standard search problem

- A CSP can easily be expressed as a standard search problem.
- Incremental formulation
 - *Initial State*: the empty assignment $\{\}$
 - *Actions*: Assign a value to an unassigned variable provided that it does not violate a constraint
 - *Goal test*: the current assignment is complete
(by construction it is consistent)
 - *Path cost*: constant cost for every step (not really relevant)

BUT: solution is at depth n (# of variables)

For BFS: branching factor at top level is nd

next level: $(n-1)d$

...

Total: $n! d^n$ leaves! But there are only d^n complete assignments!

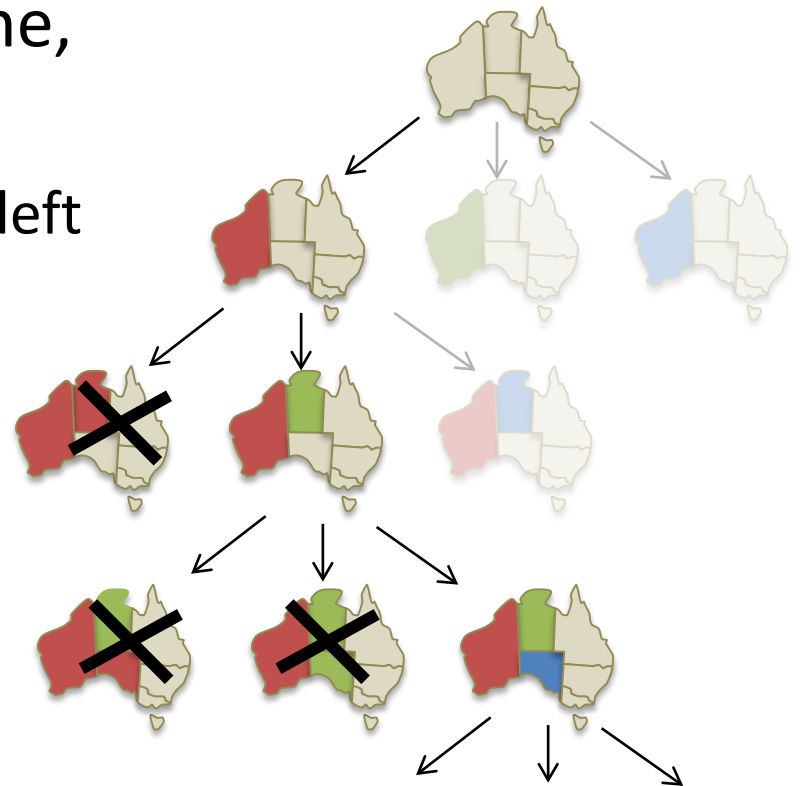
- Aside: can also use complete-state formulation
 - Local search techniques (Chapter 4) tend to work well

Commutativity

- CSPs are commutative.
 - Order of any given set of actions has no effect on the outcome.
 - Example: choose colors for Australian territories, one at a time.
 - **[WA=red then NT=green]** same as **[NT=green then WA=red]**
- All CSP search algorithms can generate successors by considering assignments for only a single variable at each node in the search tree
 - ⇒ there are d^n irredundant leaves
- (Figure out later to which variable to assign which value.)

Backtracking search

- Similar to depth-first search
 - At each level, pick a single variable to expand
 - Iterate over the domain values of that variable
- Generate children one at a time,
 - One child per value
 - Backtrack when no legal values left
- Uninformed algorithm
 - Poor general performance



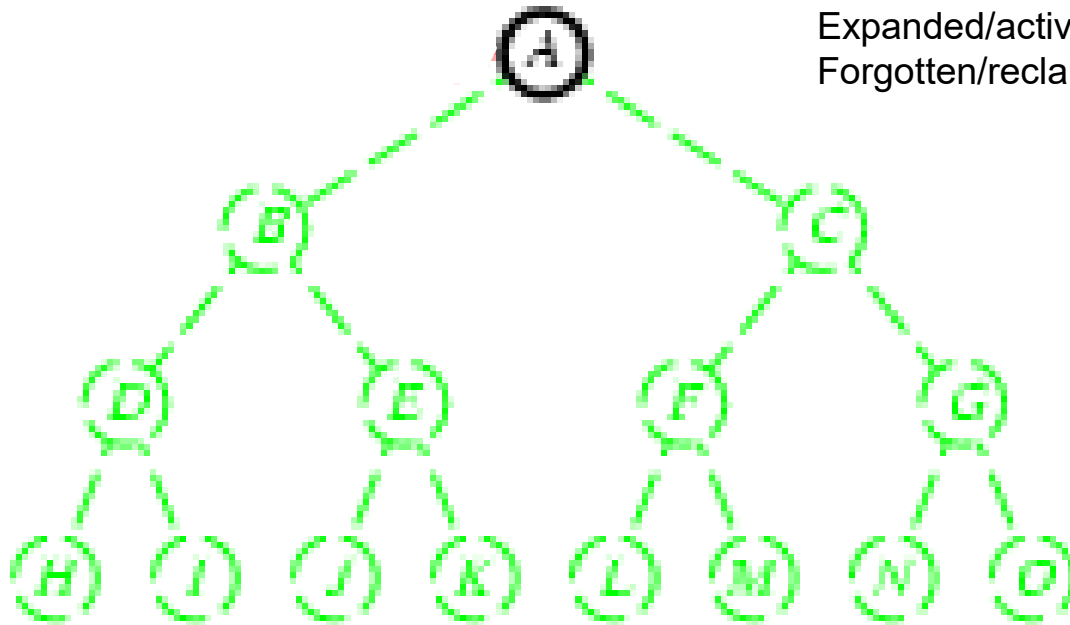
Backtracking search

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    if assignment is complete then return assignment  
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)  
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do  
        if value is consistent with assignment according to CONSTRAINTS[csp] then  
            add {var=value} to assignment  
            result ← RECURSIVE-BACKTRACKING(assignment, csp)  
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            remove {var=value} from assignment  
    return failure
```

Backtracking search

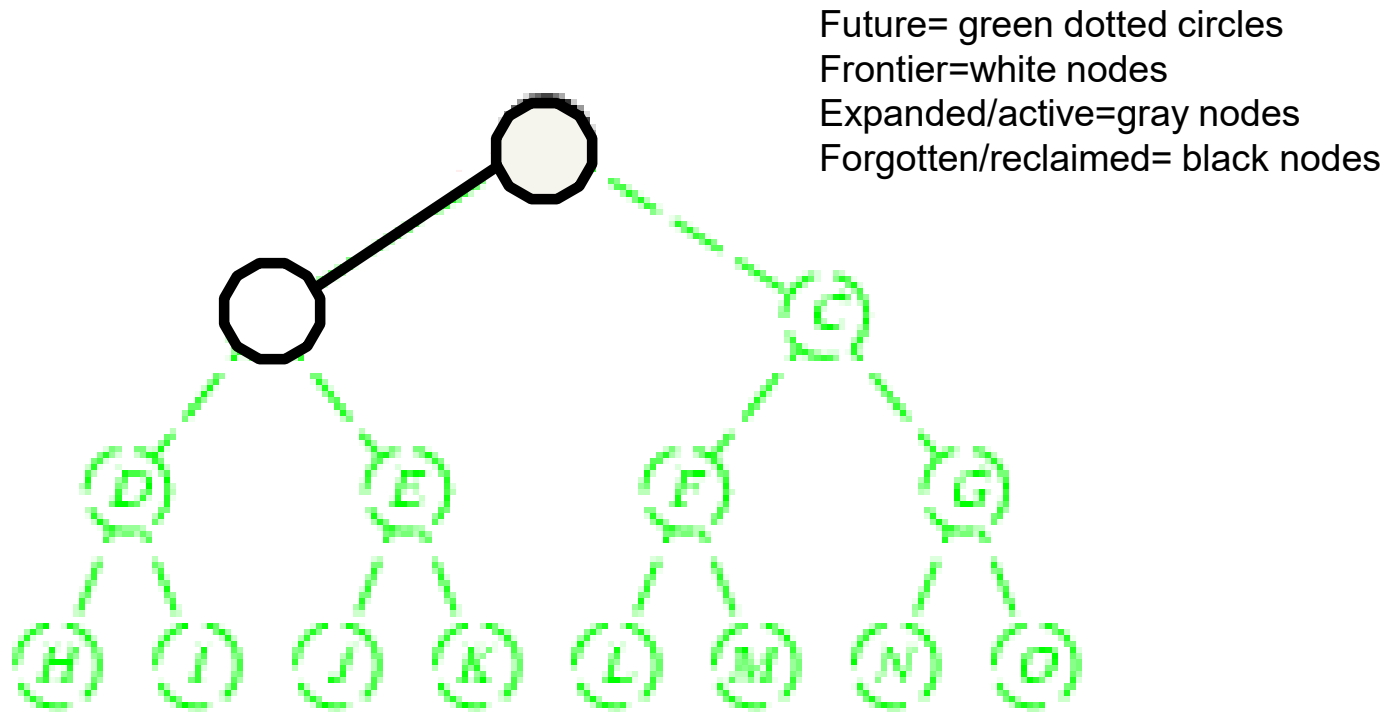
- Expand *deepest* unexpanded node
- Generate **only one** child at a time.
- *Goal-Test* when inserted.
 - For CSP, Goal-test at bottom



Future= green dotted circles
Frontier=white nodes
Expanded/active=gray nodes
Forgotten/reclaimed= black nodes

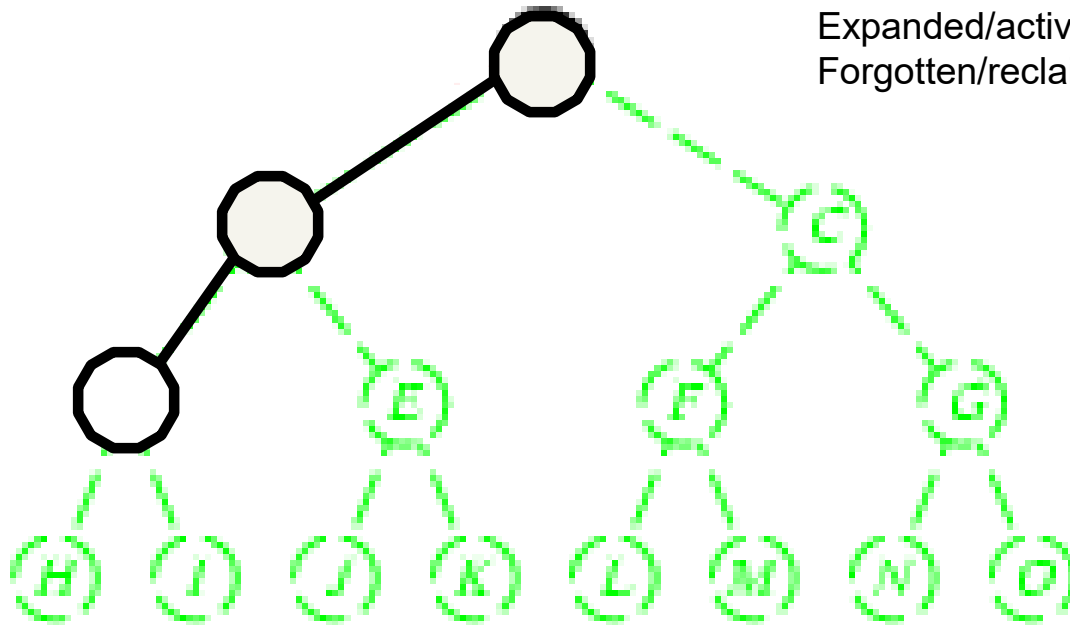
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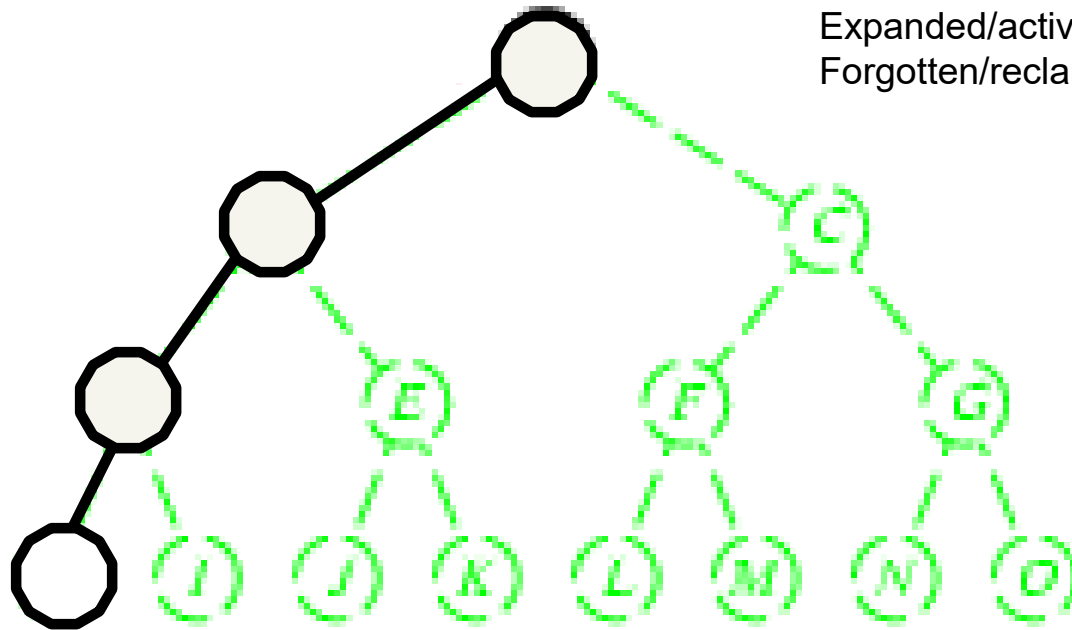
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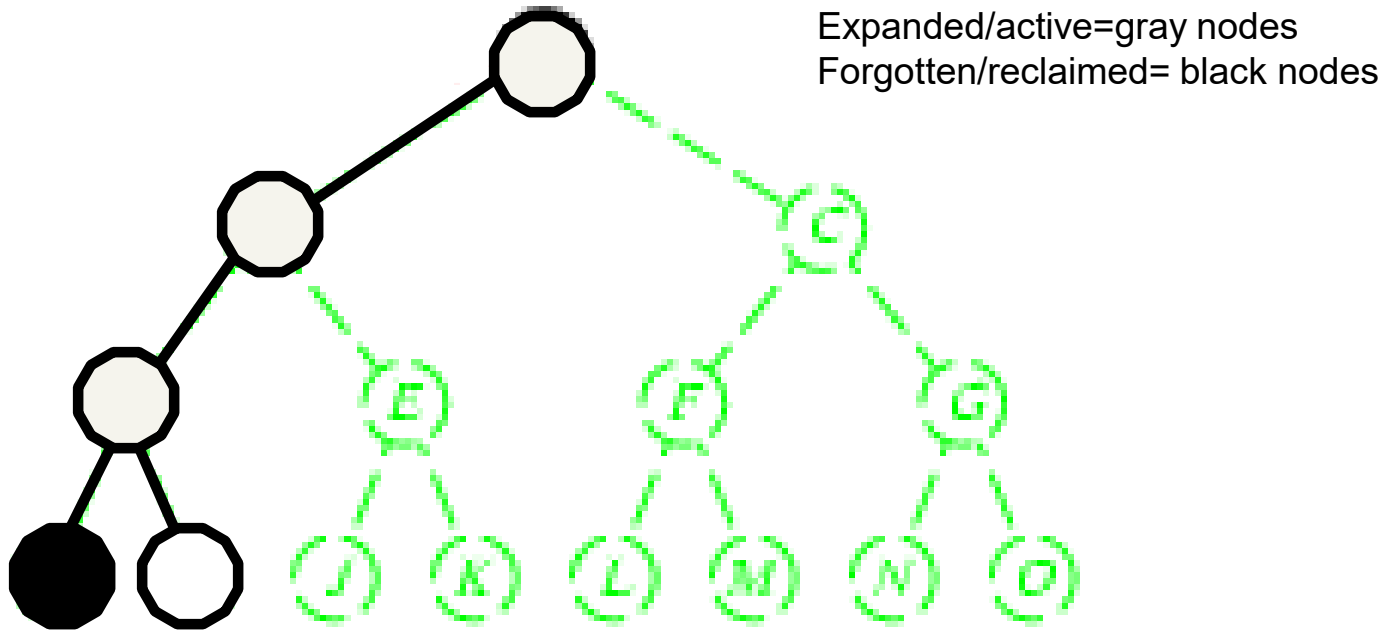
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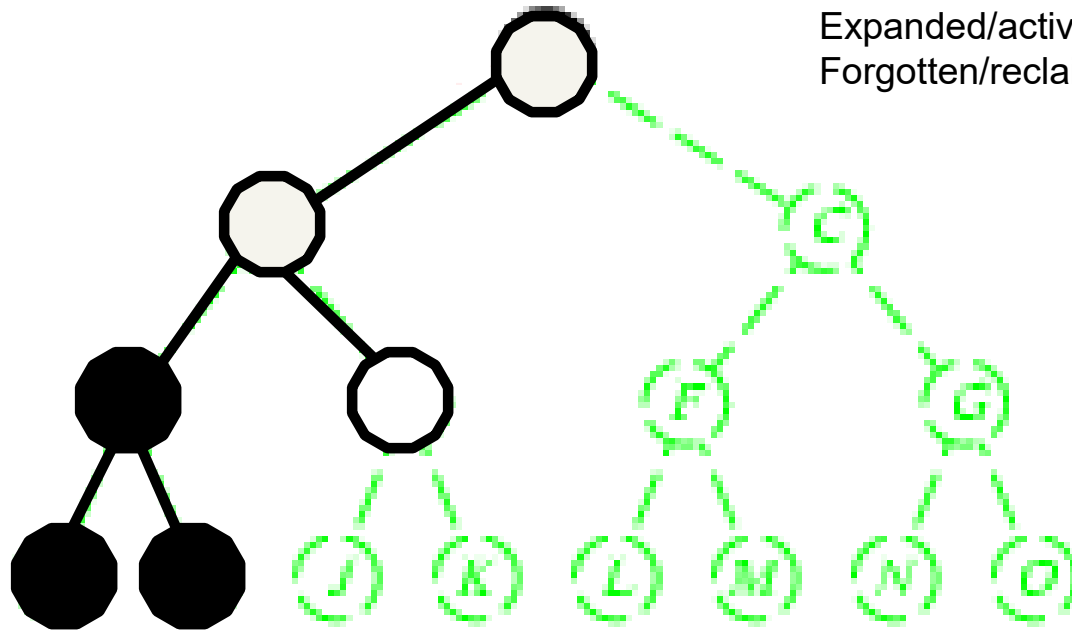
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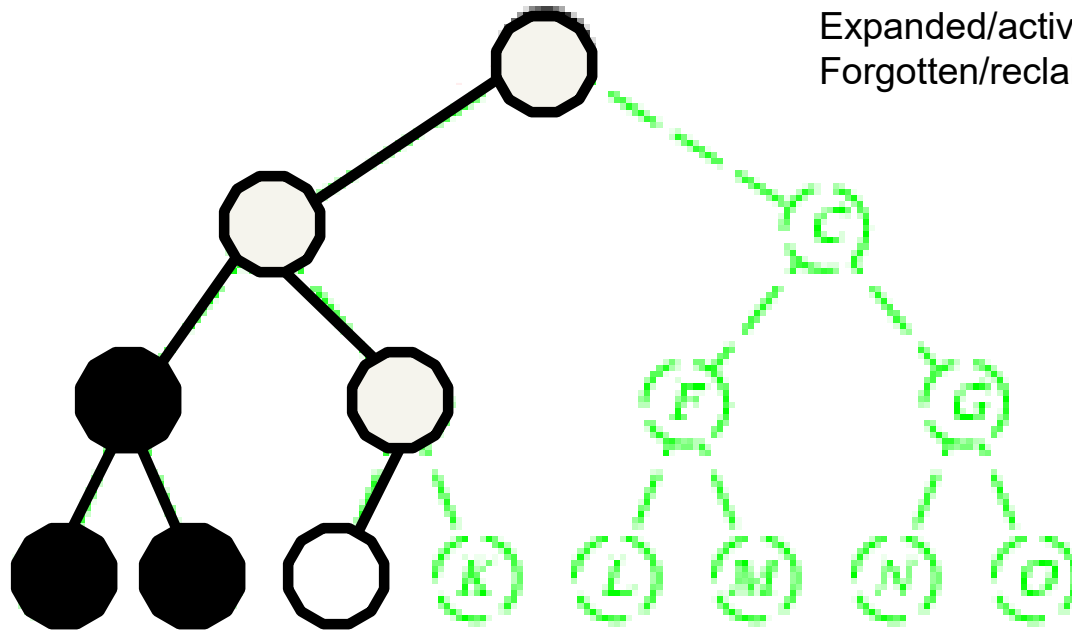
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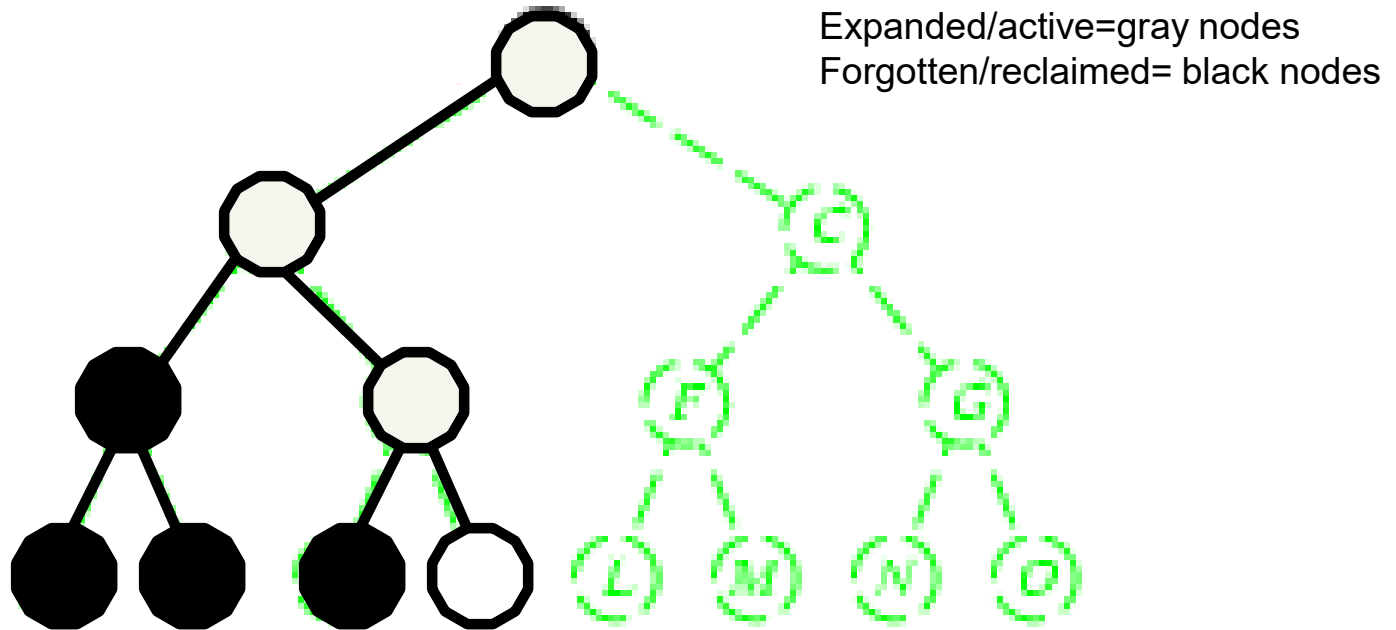
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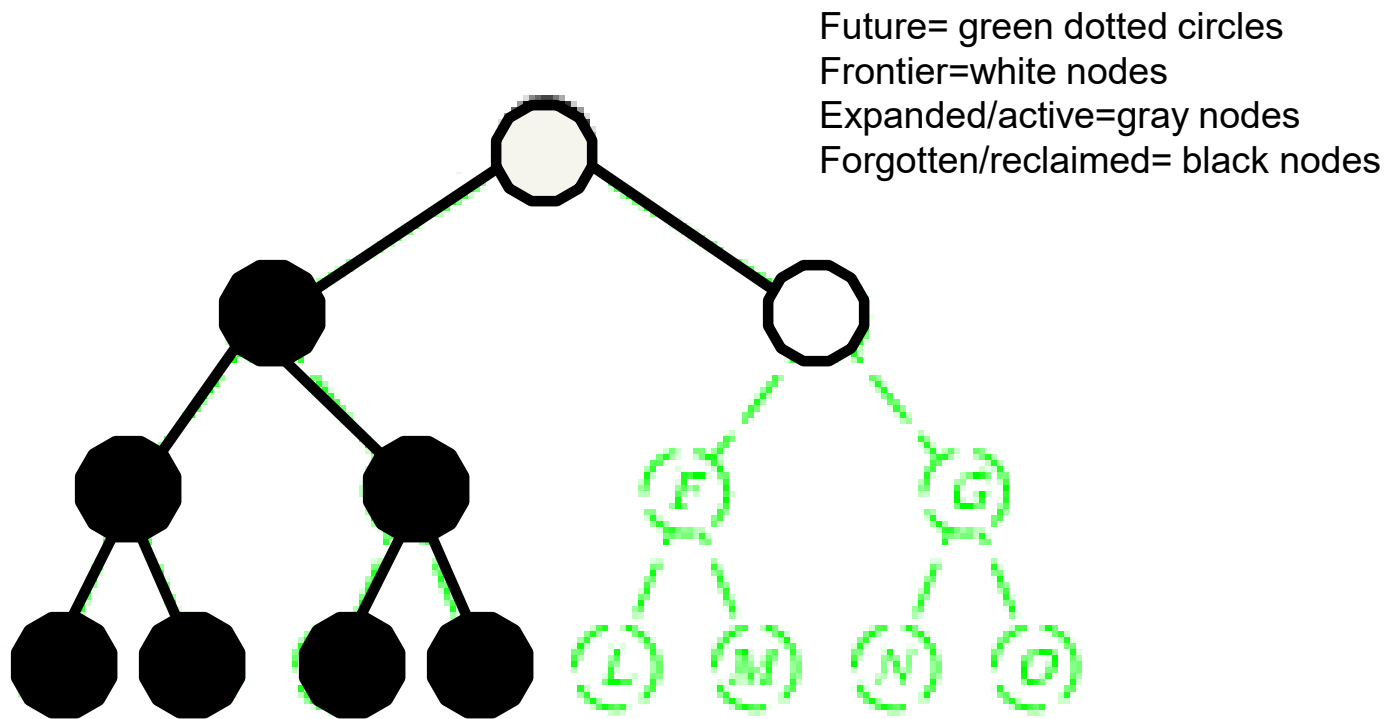
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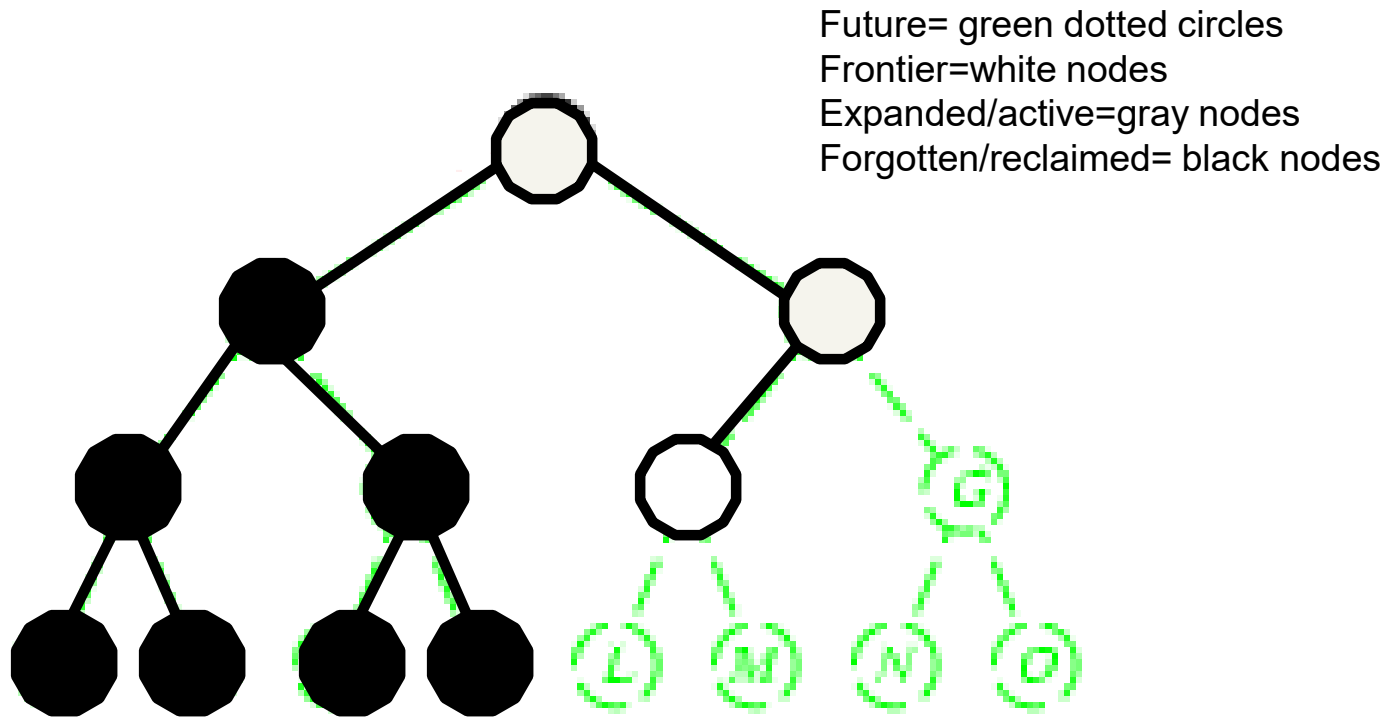
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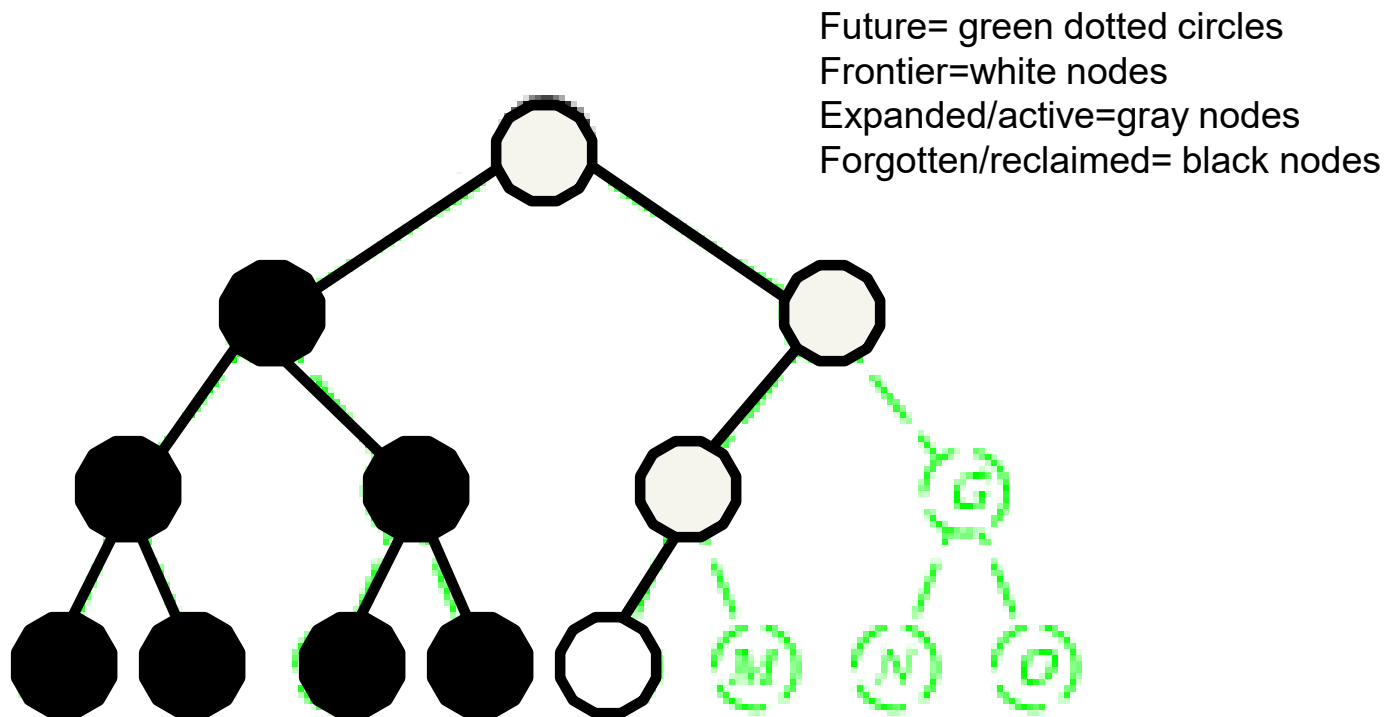
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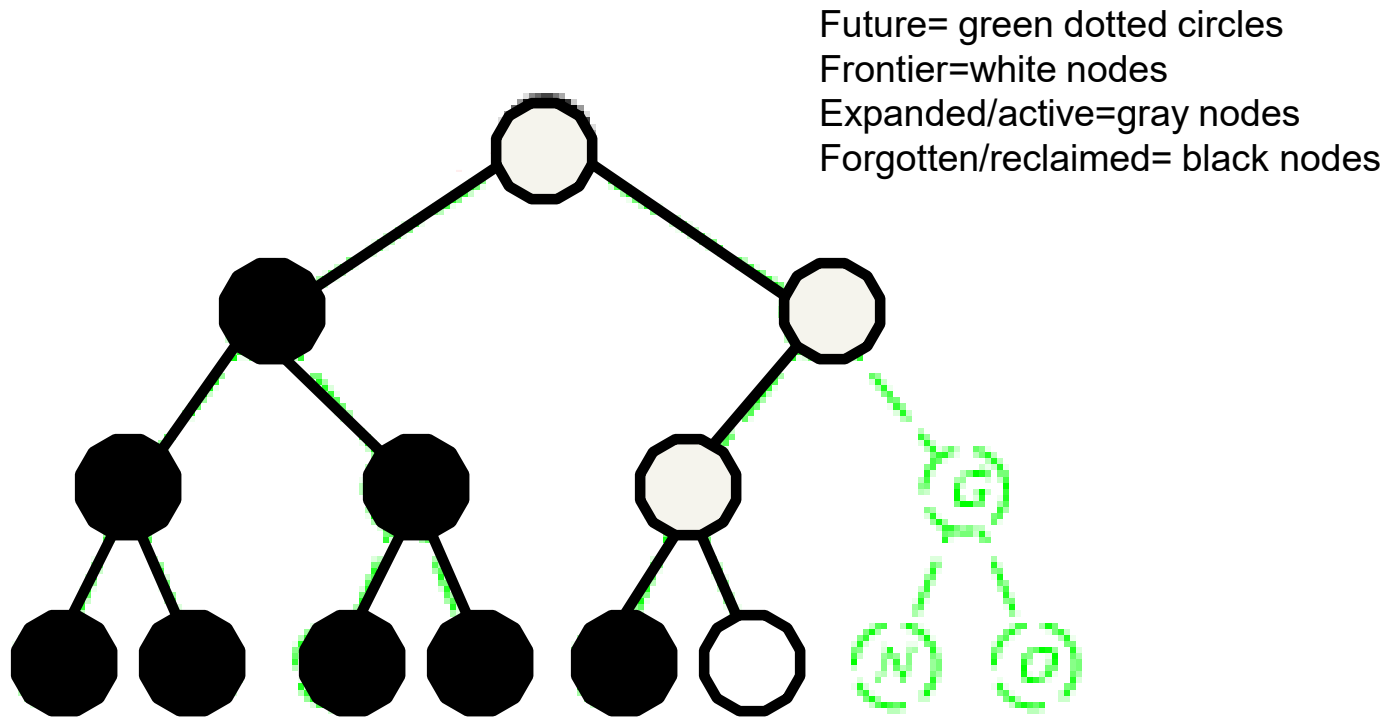
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Improving Backtracking $O(\exp(n))$

- Make our search more “informed” (e.g. heuristics)
 - General purpose methods can give large speed gains
 - CSPs are a generic formulation; hence heuristics are more “generic” as well
- Before search:
 - **Reduce the search space**
 - Arc-consistency, path-consistency, i-consistency
 - Variable ordering (fixed)
- During search:
 - **Look-ahead schemes:**
 - Detecting failure early; reduce the search space if possible
 - Which variable should be assigned next?
 - Which value should we explore first?
 - **Look-back schemes:**
 - Backjumping
 - Constraint recording
 - Dependency-directed backtracking

Look-ahead: Variable and value orderings

- Intuition:
 - Apply propagation at each node in the search tree (reduce future branching)
 - Choose a **variable** that will detect failures early (low branching factor)
 - Choose **value** least likely to yield a dead-end (find solution early if possible)
- Forward-checking
 - (check each unassigned variable separately)
- Maintaining arc-consistency (MAC)
 - (apply full arc-consistency)

Backtracking search (Figure 6.5)

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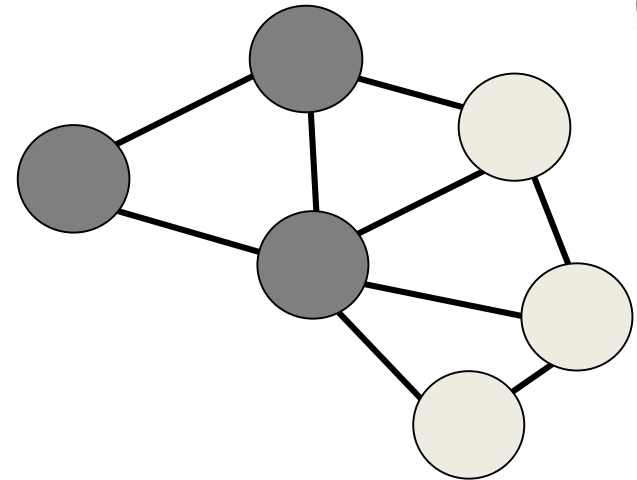
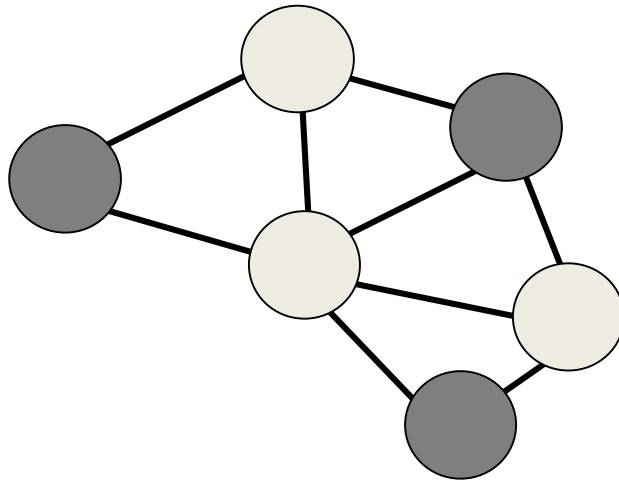
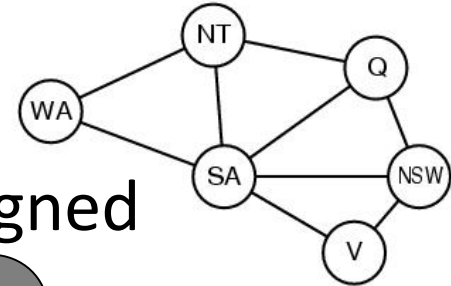
```
            remove {var=value} from assignment
```

```
    return failure
```

Dependence on variable ordering

- Example: coloring

- Dark nodes assigned, light nodes unassigned



(1) Assign WA, Q, V first:

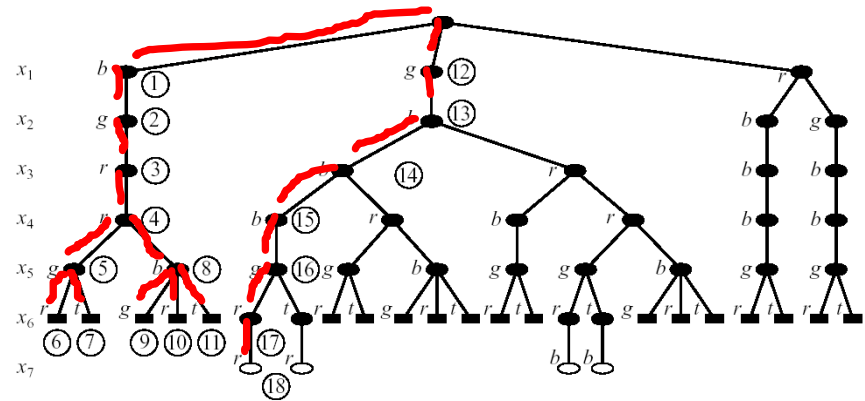
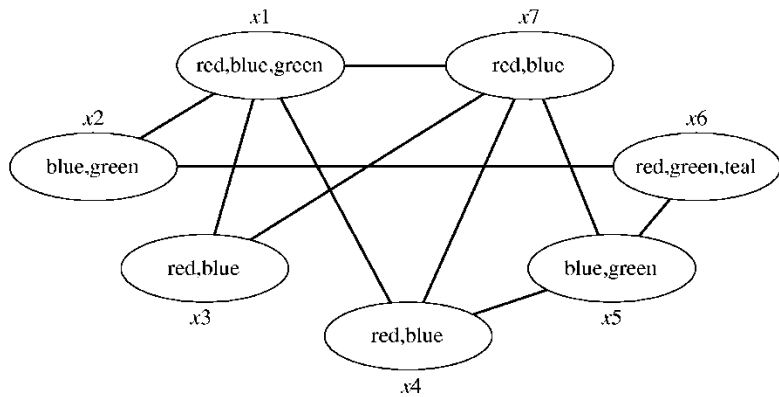
- $27 = 3^3$ ways to color assigned nodes consistently
- none inconsistent (yet)
- only 3 lead to solutions...

(2) Assign WA, SA, NT first:

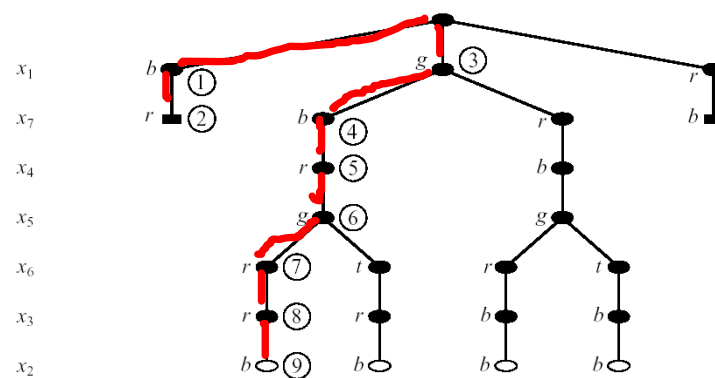
- $6 = 3!$ ways to color assigned nodes consistently
- all lead to solutions
- no backtracking

Dependence on variable ordering

- Another graph coloring example:



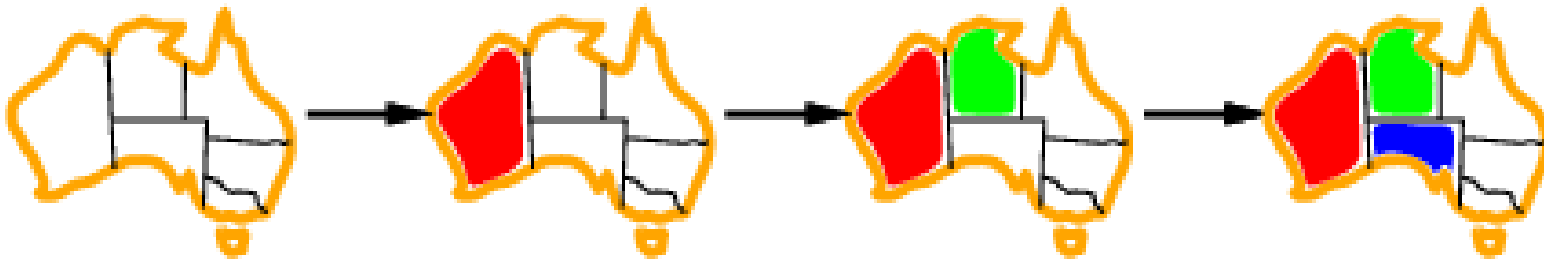
(a)



(b)

Minimum remaining values (MRV)

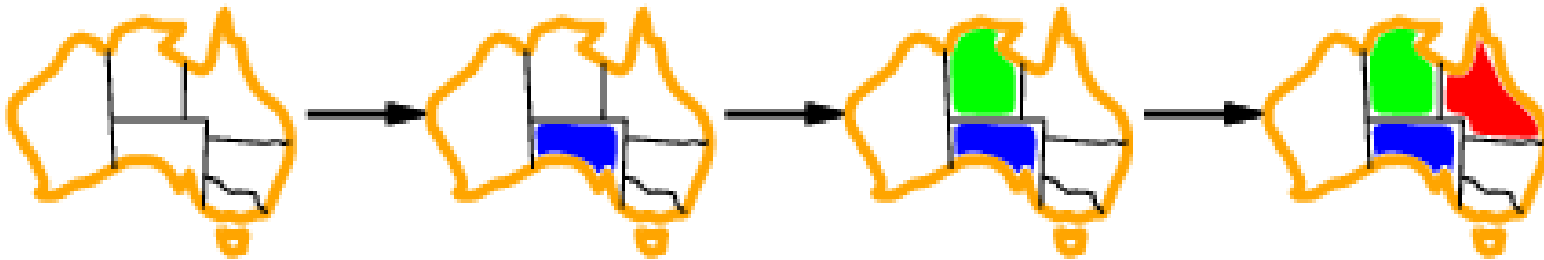
- A heuristic for selecting the next variable
 - a.k.a. **most constrained variable (MCV)** heuristic



- choose the variable with the fewest legal values
- will immediately detect failure if X has no legal values
- (Related to forward checking, later)

Degree heuristic

- Another heuristic for selecting the next variable
 - a.k.a. **most constraining variable** heuristic



- Select variable involved in the most constraints on other unassigned variables
- Useful as a tie-breaker among most constrained variables

What about the order to try values?

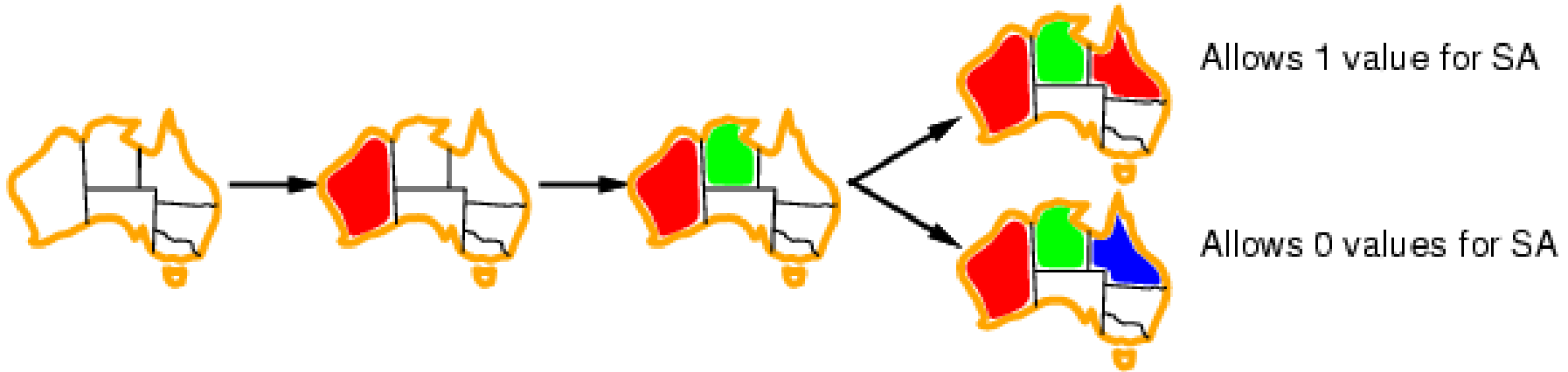
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```

Least Constraining Value

- Heuristic for selecting what value to try next
- Given a variable, choose the least constraining value:
 - the one that rules out the fewest values in the remaining variables



- Makes it more likely to find a solution early

Variable and value orderings

- Minimum remaining values for variable ordering
- Least constraining value for value ordering
 - Why do we want these? Is there a contradiction?
- **Intuition:**
 - Choose a **variable** that will detect failures early (low branching factor)
 - Choose **value** least likely to yield a dead-end (find solution early if possible)
- MRV for variable selection reduces current branching factor
 - Low branching factor throughout tree = fast search
 - Hopefully, when we get to variables with currently many values, forward checking or arc consistency will have reduced their domains & they'll have low branching too
- LCV for value selection increases the chance of success
 - If we're going to fail at this node, we'll have to examine every value anyway
 - If we're going to succeed, the earlier we do, the sooner we can stop searching

Summary

- CSPs
 - special kind of problem: states defined by values of a fixed set of variables, goal test defined by constraints on variable values
- Backtracking = depth-first search with one variable assigned per node
- Heuristics
 - Variable ordering and value selection heuristics help significantly
- Variable ordering (selection) heuristics
 - Choose variable with Minimum Remaining Values (MRV)
 - Degree Heuristic – break ties after applying MRV
- Value ordering (selection) heuristic
 - Choose Least Constraining Value