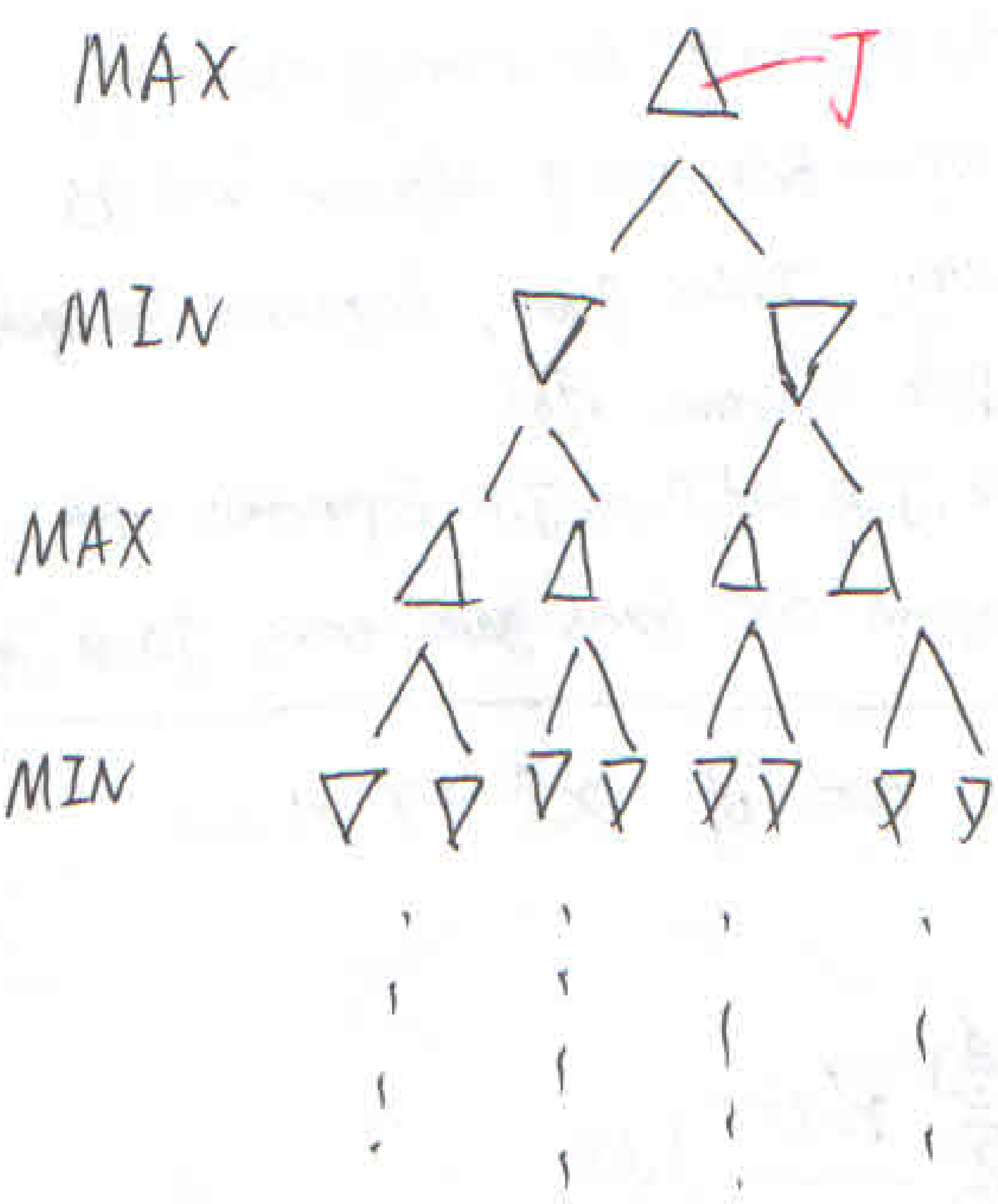
Average Case Analysis

game tree



suppose we have tree using $\alpha-\beta$ prune algorithm. The average terminal nodes (total nodes that are not pruned) we examined is $N_{n,d}$. (n is the branch factor, d is the max depth) of tree.

Denote J is one of maximum node.

$\begin{cases} A(J) \text{ is the maximum value of J's descendant at MIN stage,} \\ B(J) \text{ is the minimum value of J's descendant at MAX stage} \end{cases}$

if we have a path from leaves to J, and J is not pruned, $A(J) < B(J)$

Therefore, we have $N_{n,d} = \sum_{\substack{\text{All node we have} \\ \text{visited in tree}}} P[A(J) < B(J)]$

Suppose terminal nodes are chosen independently in common distribution function $f_0(x) = P[V_0 \le x]$

$f_k(x) = 1 - [1 - f_{k-1}(x)]^n$ ① (at k level of tree), V_k – value of node at MIN in K level

Thus, we have $P[A(J) < B(J)] = \int_{-\infty}^{\infty} F_{A(J)}(x) F'_{B(J)}(x)\, dx$ (A(J), B(J) independent)

$$\Rightarrow \quad N_{n,d} = \int_{-\infty}^{\infty} \Big[\sum_{\text{All nodes}} F_{A(J)}(x) F'_{B(J)}(x)\, dx \Big] + \underline{n^{\lceil d/2 \rceil} + n^{\lfloor d/2 \rfloor} - 1}$$

this is lower bound proven by Slagle. J.R and Dixon J.K

$r_d(t) = \dfrac{1 - [f_{d-1}(t)]^n}{1 - f_{d-1}(t)}$ ②

$R_d(t) = r_1(t) \times .. \times r_{\lceil d/2 \rceil}(t)$ ③

$s_d(t) = \dfrac{f_d(t)}{[f_{d-1}(t)]^n}$ ④

$S_d(t) = s_1(t) \times .. \times s_{\lfloor d/2 \rfloor}(t)$ ⑤

$$= \int_0^1 R'_d(t) S_d(t)\, dt + n^{\lfloor d/2 \rfloor}$$

$$= \int_a^b \frac{d R_d[\phi(x)]}{dx} S_d[\phi(x)]\, dx + n^{\lfloor d/2 \rfloor}$$

where $\phi(x) = g[\phi(ax)]$ ⑥, $g(\phi) = 1 - (1 - \phi^n)^n$ ⑦

According to ①~⑦.

$f_d(x) = \phi(x/a^d)$, where $r(x) = \dfrac{1-[\phi(x)]^n}{1-\phi(x)}$ ⑨

$r_d(x) = r(x/a^{d-1})$ ⑧ $\Leftarrow$

$S_d(x) = s(x/a^{d-1})$ $s(x) = \dfrac{1-[1-\phi(x)]^n}{\phi(x)^n}$

Ref:

[1] The solution for the branching factor of the alpha-beta pruning algorithm and its optimility, Juden Pearl, Programming Techniques and Data Structure, 1981

[2] Slagle, J, R and Dixon, J.K. Experiments with some programs that search game trees. JACM, 1969

Equation ⑥ is Poincare equation, $\phi(0) = \xi_n$, ξ_n is the root of $x^n + x - 1 = 0$

and $a = \left[\dfrac{\xi_n}{n(1-\xi_n)}\right]^2 < 1$

Therefore, $N_{n,d} = n^{\lfloor d/2 \rfloor} + \int_{-\infty}^{\infty} \prod_{i=0}^{\lfloor \frac{d}{2} \rfloor} p\left(\frac{x}{a^i}\right) \left(\sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \frac{r_i'(x)}{r_i(x)} \right) dx$

where $p(x) = r(x) \cdot s(x) = P[\phi(x)]$,

$P[\phi] = \dfrac{1-\phi^n}{1-\phi} \cdot \dfrac{1-(1-\phi^n)^n}{\phi^n}$, because $\dfrac{r_i'(x)}{r_i(x)} \le n\phi'\left(\dfrac{x}{a^{i-1}}\right) \cdot \dfrac{1}{a^{i-1}}$ (according to ⑧ and ⑨)

Therefore, $N_{n,d} \le n^{\lfloor d/2 \rfloor} + n \int_{-\infty}^{\infty} \prod_{i=0}^{\lfloor \frac{d}{2} \rfloor} p\left(\frac{x}{a^i}\right) \cdot \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \left(\phi'\left(\frac{x}{a^{i-1}}\right) \cdot \frac{1}{a^{i-1}} \right) dx$

$\exists \alpha > 0,\ p(x) \le p(0) - \alpha x\ \forall x \le 0$

$A(n) = e^{\frac{-\alpha x_0}{p(0)(1+\alpha)}}$ $\Rightarrow$

$\le n^{\lfloor d/2 \rfloor} + n A(n) [p(0)]^{\frac{d}{2}} \cdot \int_{-\infty}^{\infty} \sum_{i=0}^{\frac{d}{2}} \left[\phi'\left(\frac{x}{a^i}\right)\left(\frac{1}{a^i}\right)\right] dx$

$= n^{d/2} + n A(n) [p(0)]^{\frac{d}{2}} \cdot \frac{d}{2}$

because $p(0) = \left(\dfrac{\xi_n}{1-\xi_n}\right)^2 > n$

Thus, the "real branch factor" in average case

$n' = \lim_{d \to \infty} (N_{n,d})^{1/d} \le \dfrac{\xi_n}{1-\xi_n} = O(n/\log n)$ ← according to Knuth

$\approx O(n / 0.925 \cdot n^{0.747}) \approx O(n^{\frac{3}{4}})$