Radiative Transfer & Volume Path Tracing

CS295, Spring 2017

Shuang Zhao

Computer Science Department

University of California, Irvine

Last Lecture

- Refraction & BSDFs
  - How light interacts with refractive interfaces (e.g., glass)

Today's Lecture

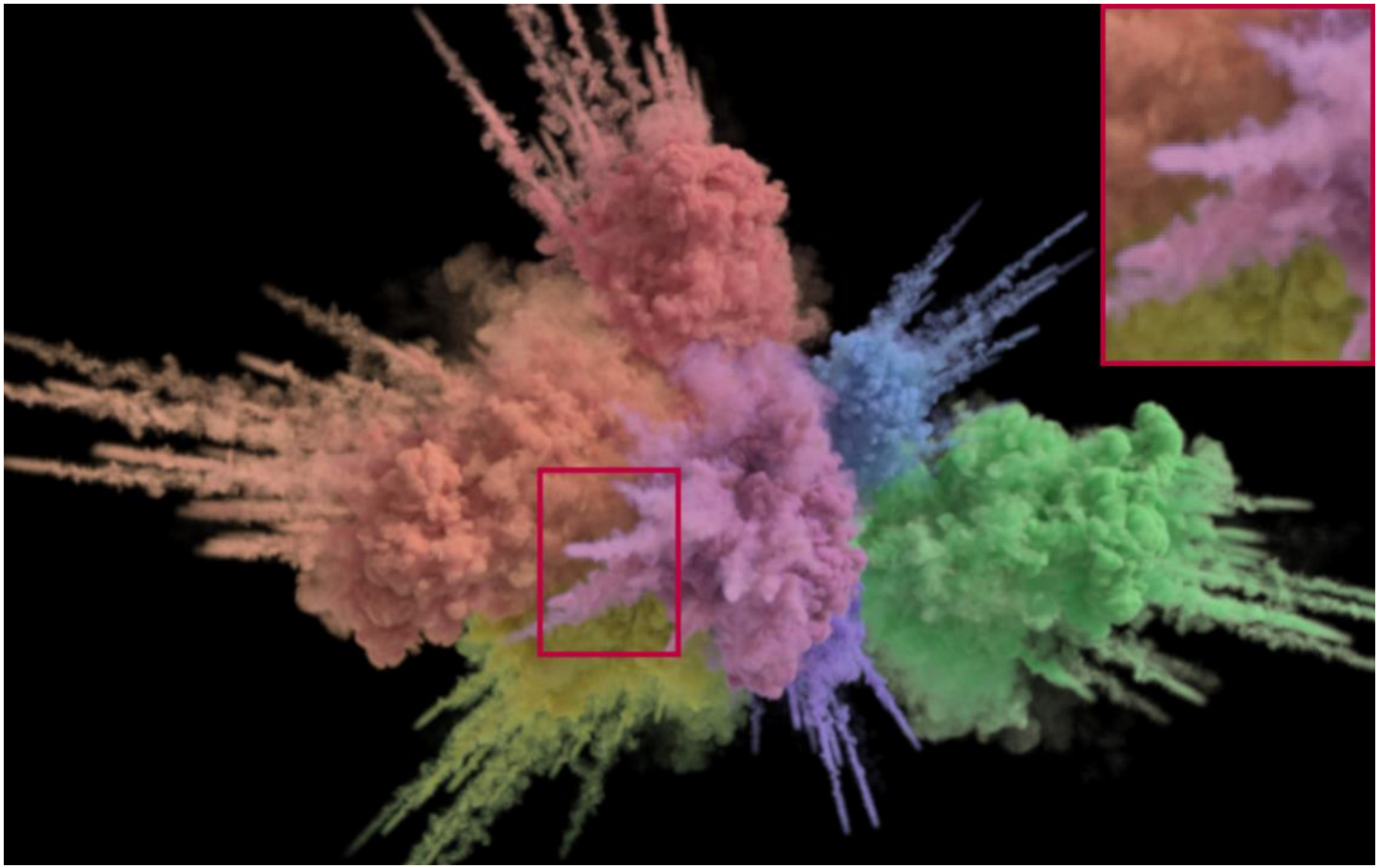
- Radiative transfer
  - The mathematical model to simulate light scattering in participating media (e.g., smoke) and translucent materials (e.g., marble and skin)

- Volume path tracing (VPT)
  - A Monte Carlo solution to the radiative transfer problem
  - Similar to the normal PT from previous lectures

Radiative Transfer

CS295: Realistic Image Synthesis

Participating Media



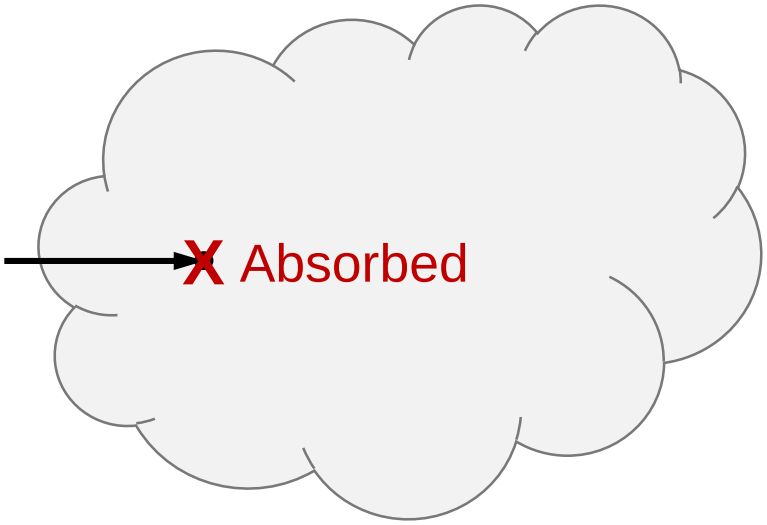
[Kutz et al. 2017]

Translucent Materials

[Gkioulekas et al. 2013]

Subsurface Scattering

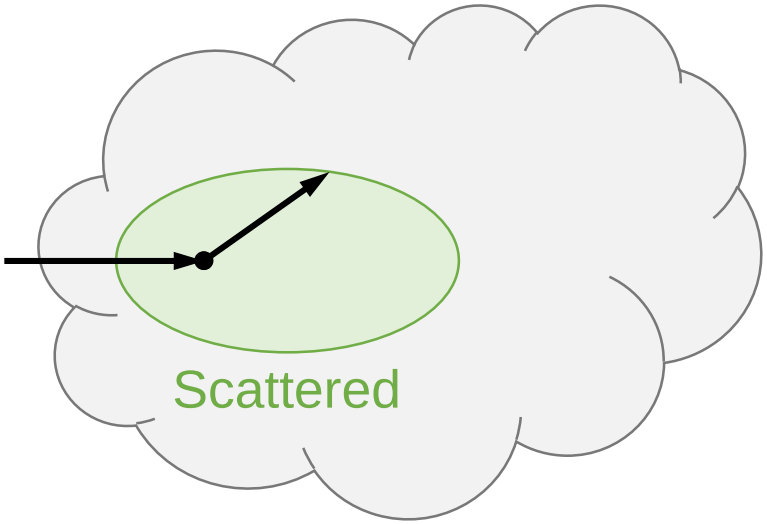
- Light enters a material and scatters around before eventually leaving or absorbed



Participating medium

Subsurface Scattering

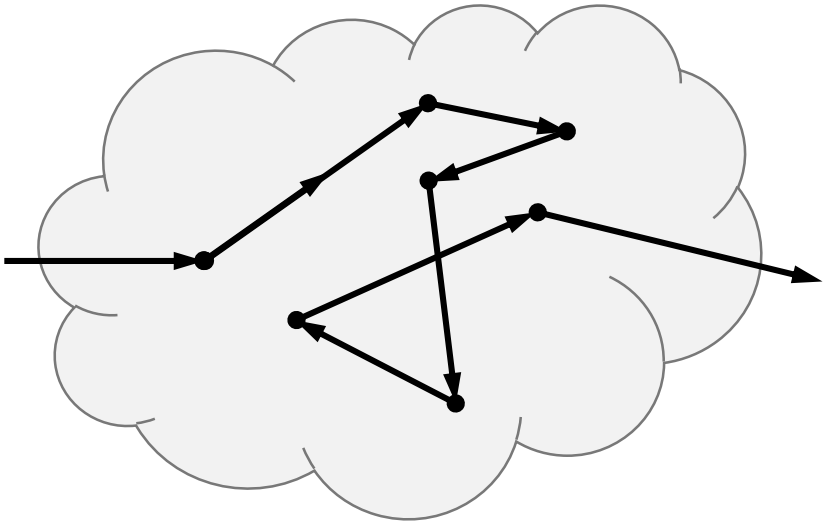
- Light enters a material and scatters around before eventually leaving or absorbed



Participating medium

Subsurface Scattering

- Light enters a material and scatters around before eventually leaving or absorbed

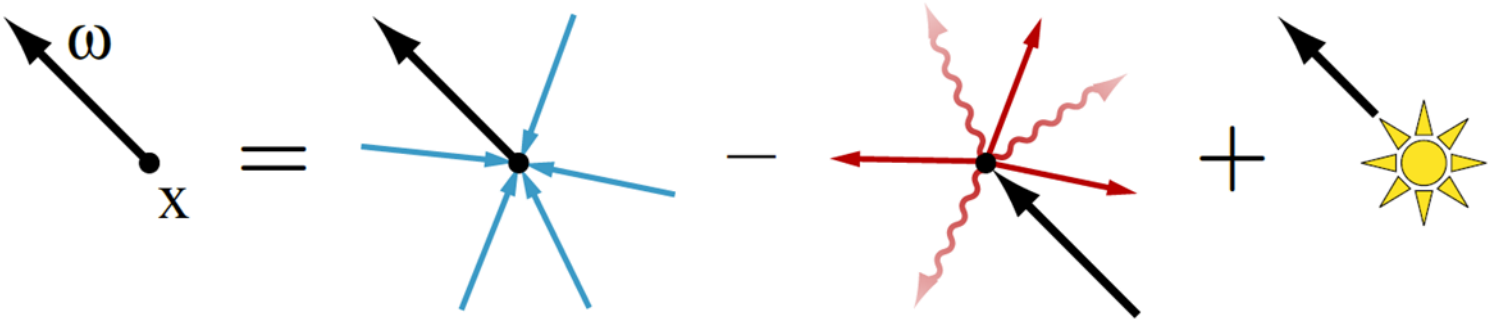


Participating medium

Radiative Transfer

- A mathematical model describing how light interacts with participating media
  - Originated in physics
  - Now used in many areas
    - Astrophysics (light transport in space)
    - Biomedicine (light transport in human tissue)
    - Graphics
    - Nuclear science & engineering (neutron transport)
    - Remote sensing
    - …

Radiative Transfer Equation (RTE)



Differential radiance | In-scattering | Out-scattering & absorption | Emission

$$(\boldsymbol{\omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\omega}) = \sigma_s(\boldsymbol{x})\int_{\mathbb{S}^2} f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{x},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i - \sigma_t(\boldsymbol{x})L(\boldsymbol{x},\boldsymbol{\omega}) + Q(\boldsymbol{x},\boldsymbol{\omega})$$

In-scattering | Out-scattering & absorption | Emission

Radiative Transfer Equation (RTE)

$$(\boldsymbol{\omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\omega}) = \sigma_s(\boldsymbol{x})\int_{\mathbb{S}^2} f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{x},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i - \sigma_t(\boldsymbol{x})L(\boldsymbol{x},\boldsymbol{\omega}) + Q(\boldsymbol{x},\boldsymbol{\omega})$$

In-scattering | Out-scattering & absorption | Emission

- The RTE is a first-order integro-differential equation
- For a participating medium in a volume $\mathcal{V}\subseteq\mathbb{R}^3$ with boundary $\partial\mathcal{V}$, the RTE governs the radiance values inside this volume (i.e., for all $\boldsymbol{x}\in\mathcal{V}^0 := \mathcal{V}\setminus\partial\mathcal{V}$)
- The boundary condition is the radiance field on the boundary $\partial\mathcal{V}$ (i.e., $L(\boldsymbol{x}, \boldsymbol{\omega})$ for all $\boldsymbol{x}\in\partial\mathcal{V}$)

Radiative Transfer Equation (RTE)

$$(\boldsymbol{\omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\omega}) = \sigma_s(\boldsymbol{x})\int_{\mathbb{S}^2} f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{x},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i - \sigma_t(\boldsymbol{x})L(\boldsymbol{x},\boldsymbol{\omega}) + Q(\boldsymbol{x},\boldsymbol{\omega})$$

In-scattering | Out-scattering & absorption | Emission

- Differential radiance

$$(\boldsymbol{\omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\omega}) = \frac{\mathrm{d}L(\boldsymbol{x}+\tau\boldsymbol{\omega},\boldsymbol{\omega})}{\mathrm{d}\tau}\bigg|_{\tau=0} = \lim_{\tau\to 0}\frac{L(\boldsymbol{x}+\tau\boldsymbol{\omega},\boldsymbol{\omega})-L(\boldsymbol{x},\boldsymbol{\omega})}{\tau}$$

- Scattering coefficient: $\sigma_s(\boldsymbol{x})\in\mathbb{R}_{\geq 0}$,
  Phase function: $f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega})$, a probability density over $\mathbb{S}^2$ given $\boldsymbol{x}$ and $\boldsymbol{\omega}_i$
- Extinction coefficient: $\sigma_t(\boldsymbol{x})\in\mathbb{R}_{\geq\sigma_s(\boldsymbol{x})}$
- Source term: $Q(\boldsymbol{x},\boldsymbol{\omega})\in\mathbb{R}_{\geq 0}$

Radiative Transfer Equation (RTE)

$$(\boldsymbol{\omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\omega}) = \sigma_s(\boldsymbol{x})\int_{\mathbb{S}^2} f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{x},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i - \sigma_t(\boldsymbol{x})L(\boldsymbol{x},\boldsymbol{\omega}) + Q(\boldsymbol{x},\boldsymbol{\omega})$$

In-scattering | Out-scattering & absorption | Emission

- σ_t controls how frequently light scatters and is also known as the *optical density*

- The ratio between σ_s and σ_t controls the fraction of radiant energy *not* being absorbed at each scattering and is also known as the *single-scattering albedo*

Radiative Transfer Equation (RTE)

$$(\boldsymbol{\omega}\cdot\nabla)L(\boldsymbol{x},\boldsymbol{\omega}) = \sigma_s(\boldsymbol{x})\int_{\mathbb{S}^2} f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{x},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i - \sigma_t(\boldsymbol{x})L(\boldsymbol{x},\boldsymbol{\omega}) + Q(\boldsymbol{x},\boldsymbol{\omega})$$

In-scattering | Out-scattering & absorption | Emission

- The phase function f_p is usually parameterized as a function on the angle between $\boldsymbol{\omega}_i$ and $\boldsymbol{\omega}$. Namely,

$$f_p(\boldsymbol{x},\boldsymbol{\omega}_i\to\boldsymbol{\omega}) = f_p(\boldsymbol{x},\langle\boldsymbol{\omega}_i,\boldsymbol{\omega}\rangle)$$

- Example: the Henyey-Greenstein (HG) phase function with parameter $-1 < g < 1$:

$$f_p^{\mathrm{HG}}(\cos\theta;\ g) = \frac{1}{4\pi}\frac{1-g^2}{(1+g^2-2g\cos\theta)^{1.5}}$$

The Integral Form of the RTE

$$(\boldsymbol{\omega} \cdot \nabla) L(\boldsymbol{x}, \boldsymbol{\omega}) = \sigma_s(\boldsymbol{x}) \int_{\mathbb{S}^2} f_p(\boldsymbol{x}, \boldsymbol{\omega}_i \to \boldsymbol{\omega}) L(\boldsymbol{x}, \boldsymbol{\omega}_i) \, \mathrm{d}\boldsymbol{\omega}_i - \sigma_t(\boldsymbol{x}) L(\boldsymbol{x}, \boldsymbol{\omega}) + Q(\boldsymbol{x}, \boldsymbol{\omega})$$

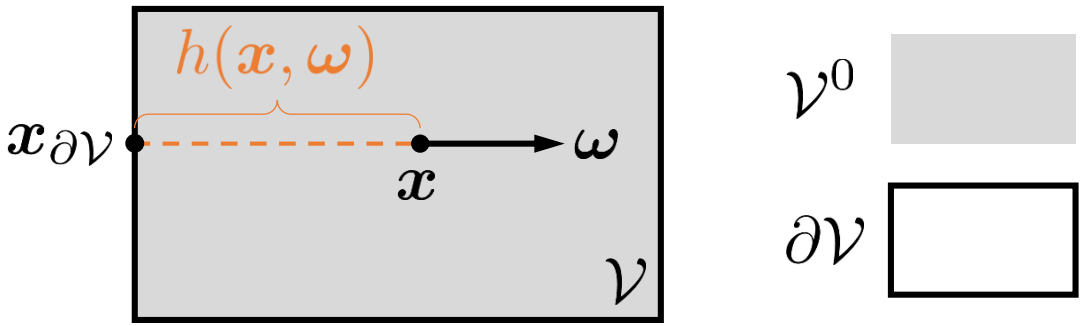
Integro-differential equation

$$L(\boldsymbol{x}, \boldsymbol{\omega}) = \int \ldots$$

Integral equation

- It is desirable to rewrite the RTE as an integral equation
  - which can then be solved numerically using Monte Carlo methods

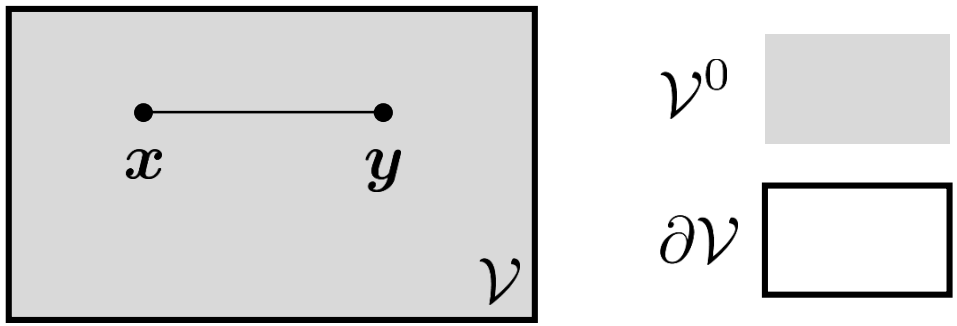
Integral Form of the RTE



- For any $\boldsymbol{x} \in \mathcal{V}^0,\ \boldsymbol{\omega} \in \mathbb{S}^2$, let $h(\boldsymbol{x}, \boldsymbol{\omega})$ denotes the minimal distance for the ray ($\boldsymbol{x}$, -$\boldsymbol{\omega}$) to hit the boundary $\partial\mathcal{V}$. In other words, $h(\boldsymbol{x}, \boldsymbol{\omega}) := \inf\{\tau \in \mathbb{R}_{\geq 0} : \boldsymbol{x} - \tau\boldsymbol{\omega} \in \partial\mathcal{V}\}$
- When ($\boldsymbol{x}$, -$\boldsymbol{\omega}$) never hits the boundary, $h(\boldsymbol{x}, \boldsymbol{\omega}) = +\infty$
  - This can happen when the volume is infinite
- For any $\boldsymbol{x} \in \mathcal{V}^0,\ \boldsymbol{\omega} \in \mathbb{S}^2$ with $h(\boldsymbol{x}, \boldsymbol{\omega}) < +\infty$, let

$$\boldsymbol{x}_{\partial\mathcal{V}} := \boldsymbol{x} - h(\boldsymbol{x}, \boldsymbol{\omega})\boldsymbol{\omega}$$

Integral Form of the RTE



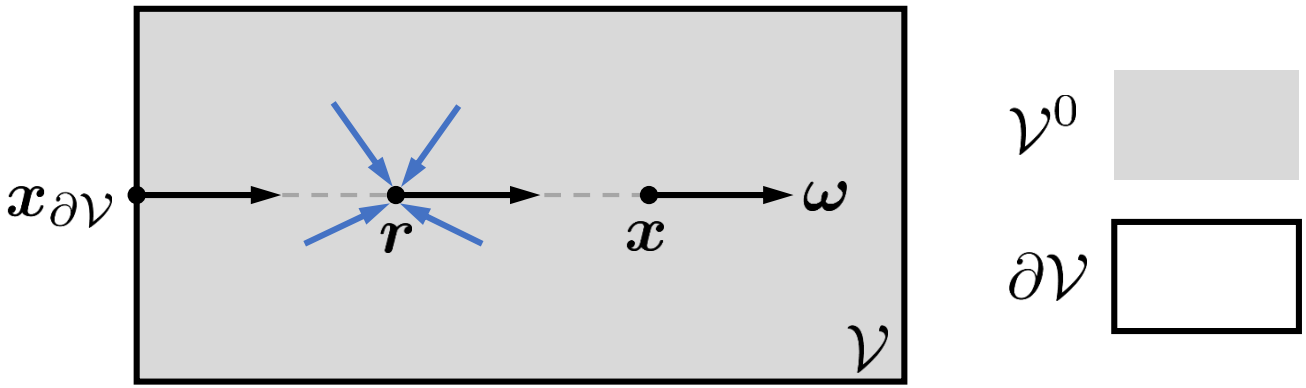
- For any $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{V}$, the attenuation between ***x*** and ***y*** is

$$T(\boldsymbol{x} \leftrightarrow \boldsymbol{y}) := \exp\left(-\int_{(\boldsymbol{x},\boldsymbol{y})} \sigma_t(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r}\right)$$

  - A line integral between ***x*** and ***y***
  - $0 \leq T(\boldsymbol{x} \leftrightarrow \boldsymbol{y}) \leq 1$ for all ***x*** and ***y***
  - For homogeneous media with $\sigma_t(\boldsymbol{x}) \equiv \sigma_t$,

$$T(\boldsymbol{x} \leftrightarrow \boldsymbol{y}) = \exp(-\|\boldsymbol{x} - \boldsymbol{y}\|\sigma_t)$$

Integral Form of the RTE



Attenuation In-scattering Emission

$$L(\boldsymbol{x},\boldsymbol{\omega}) = \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r}\leftrightarrow\boldsymbol{x})\left[\sigma_s(\boldsymbol{r})\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + Q(\boldsymbol{r},\boldsymbol{\omega})\right]\mathrm{d}\tau$$

$$+\,T(\boldsymbol{x}_{\partial\mathcal{V}}\leftrightarrow\boldsymbol{x})\,L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega})$$

where $\boldsymbol{r} := \boldsymbol{x} - \tau\boldsymbol{\omega}$

Attenuation Boundary cond.

(The second term vanishes when $h(\boldsymbol{x},\boldsymbol{\omega}) = +\infty$)

Kernel Form of the RTE

$$
\begin{aligned}
L(\boldsymbol{x},\boldsymbol{\omega}) &= \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r}\leftrightarrow\boldsymbol{x})\left[\sigma_s(\boldsymbol{r})\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + Q(\boldsymbol{r},\boldsymbol{\omega})\right]\mathrm{d}\tau \\
&\quad + T(\boldsymbol{x}_{\partial\mathcal{V}}\leftrightarrow\boldsymbol{x})\,L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) \\
&= \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})}\int_{\mathbb{S}^2} \underbrace{T(\boldsymbol{r}\leftrightarrow\boldsymbol{x})\,\sigma_s(\boldsymbol{r})\,f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})}_{=:\ K((\boldsymbol{r},\boldsymbol{\omega}_i)\to(\boldsymbol{x},\boldsymbol{\omega}))}\,L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i\,\mathrm{d}\tau \\
&\quad + \underbrace{\int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r}\leftrightarrow\boldsymbol{x})\,Q(\boldsymbol{r},\boldsymbol{\omega})\,\mathrm{d}\tau + T(\boldsymbol{x}_{\partial\mathcal{V}}\leftrightarrow\boldsymbol{x})\,L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega})}_{=:\ S(\boldsymbol{x},\boldsymbol{\omega})} \\
&= \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})}\int_{\mathbb{S}^2} K((\boldsymbol{r},\boldsymbol{\omega}_i)\to(\boldsymbol{x},\boldsymbol{\omega}))\,L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i\,\mathrm{d}\tau + S(\boldsymbol{x},\boldsymbol{\omega}),
\end{aligned}
$$

Kernel function

Source function

where $\boldsymbol{r} := \boldsymbol{x} - \tau\boldsymbol{\omega}$

Operator Form of the RTE

- Phase space: $\Gamma := \mathcal{V}^0 \times \mathbb{S}^2$
- For any real-valued function *g* on *Γ*, define operator ***K*** as

$$(\boldsymbol{K}g)(\boldsymbol{x},\boldsymbol{\omega}) := \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} \int_{\mathbb{S}^2} K((\boldsymbol{r},\boldsymbol{\omega}_i) \to (\boldsymbol{x},\boldsymbol{\omega}))\, g(\boldsymbol{r},\boldsymbol{\omega}_i)\, \mathrm{d}\boldsymbol{\omega}_i\, \mathrm{d}\tau,$$

where $\boldsymbol{r} := \boldsymbol{x} - \tau\boldsymbol{\omega}$

- Then, the RTE becomes

$$L = \boldsymbol{K}L + S$$

  - Similar to the RE!
  - Yield Neumann series $L = (\boldsymbol{I} - \boldsymbol{K})^{-1}S = \sum_{i=0}^{\infty} \boldsymbol{K}^i S$

Volume Path Tracing

CS295: Realistic Image Synthesis

Solving the RTE

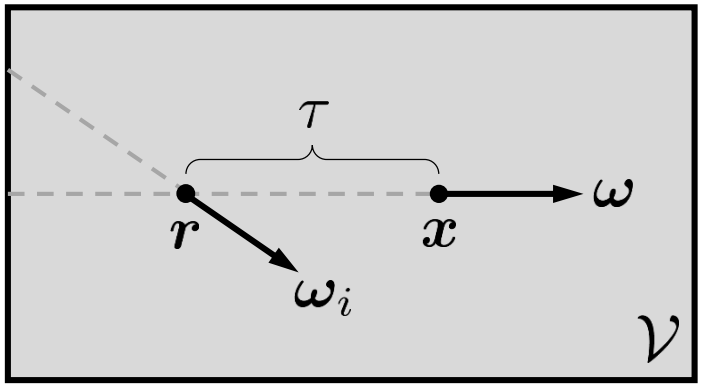
- Given the similarity between the RTE and the RE, Monte Carlo solutions to the RE can be adapted to solve the RTE
  - Volume path tracing
  - Volume adjoint particle tracing
  - Volume bidirectional path tracing
  - …

Volume Path Tracing

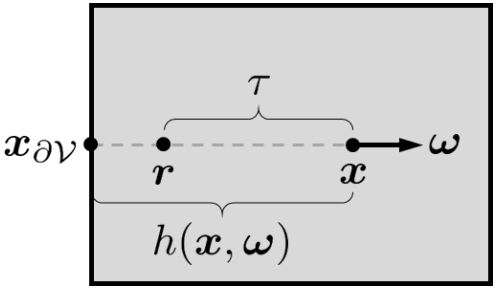
$$L(\boldsymbol{x},\boldsymbol{\omega}) = \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r} \leftrightarrow \boldsymbol{x}) \left[\sigma_s(\boldsymbol{r}) \int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega}) L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + Q(\boldsymbol{r},\boldsymbol{\omega}) \right] \mathrm{d}\tau$$

$$+ \underbrace{T(\boldsymbol{x}_{\partial\mathcal{V}} \leftrightarrow \boldsymbol{x})\, L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega})}_{\text{Known}} \qquad \text{where } \boldsymbol{r} := \boldsymbol{x} - \tau\boldsymbol{\omega}$$

- Basic idea
  - Draw τ from $p(\tau)$
  - Draw $\boldsymbol{\omega}_i$ from $p(\boldsymbol{\omega}_i)$
  - Evaluate $L(\boldsymbol{r}, \boldsymbol{\omega}_i)$ recursively



Free Distance Sampling



- τ is called the "free distance" and is sampled from

$$p(\tau) = \lambda(\tau) \exp\left(-\int_0^\tau \lambda(t)\,\mathrm{d}t\right),$$

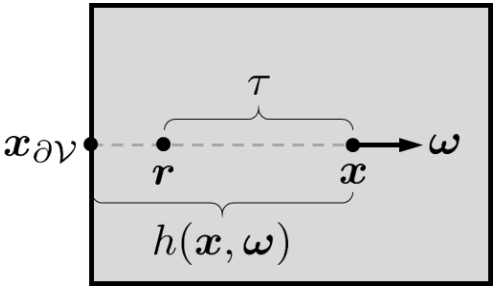
where

$$\lambda(\tau) := \begin{cases} \sigma_t(\boldsymbol{x} - \tau\boldsymbol{\omega}) & \tau < h(\boldsymbol{x}, \boldsymbol{\omega}), \\ \lambda_0 & \tau \geq h(\boldsymbol{x}, \boldsymbol{\omega}), \end{cases}$$

with λ_0 being an arbitrary positive number

- *p* gives an exponential distribution with varying parameters

Free Distance Sampling



$$p(\tau) = \lambda(\tau) \exp\left(-\int_0^{\tau} \lambda(t)\,\mathrm{d}t\right)$$

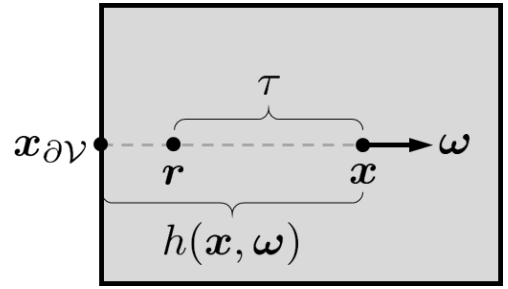
- For all $\tau \geq 0$, it holds that

$$\int_0^{\tau} p(t)\,\mathrm{d}t = 1 - \exp\left(-\int_0^{\tau} \lambda(t)\,\mathrm{d}t\right),$$

$$\int_{\tau}^{+\infty} p(t)\,\mathrm{d}t = \exp\left(-\int_0^{\tau} \lambda(t)\,\mathrm{d}t\right).$$

$$T(\boldsymbol{x}_{\partial\mathcal{V}} \leftrightarrow \boldsymbol{x}) := \exp\left(-\int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} \underbrace{\sigma_t(\boldsymbol{x} - t\boldsymbol{\omega})}_{=\lambda(t)}\,\mathrm{d}t\right) = \int_{h(\boldsymbol{x},\boldsymbol{\omega})}^{+\infty} p(t)\,\mathrm{d}t$$

Free Distance Sampling



$$L(\boldsymbol{x},\boldsymbol{\omega}) = \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r}\leftrightarrow\boldsymbol{x})\left[\sigma_s(\boldsymbol{r})\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + Q(\boldsymbol{r},\boldsymbol{\omega})\right]\mathrm{d}\tau$$

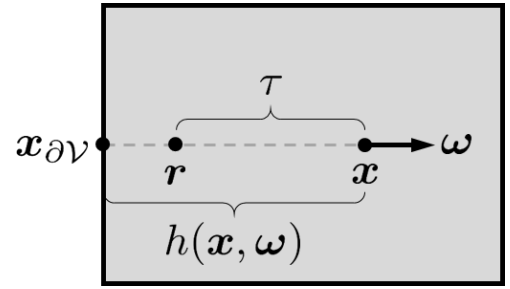
$$+ L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega})\underbrace{\int_{h(\boldsymbol{x},\boldsymbol{\omega})}^{+\infty} p(t)\,\mathrm{d}t}_{=T(\boldsymbol{x}_{\partial\mathcal{V}}\leftrightarrow\boldsymbol{x})}$$

$$= \int_0^{+\infty} f(\tau)\,\mathrm{d}\tau$$

where

$$f(\tau) := \begin{cases} T(\boldsymbol{r}\leftrightarrow\boldsymbol{x})\left[\sigma_s(\boldsymbol{r})\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + Q(\boldsymbol{r},\boldsymbol{\omega})\right] & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ p(\tau)\,L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

Free Distance Sampling



- By applying Monte Carlo integration, we have

$$\begin{aligned}\langle L(\boldsymbol{x},\boldsymbol{\omega})\rangle &= \frac{f(\tau)}{p(\tau)} \\ &= \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})}\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + \frac{Q(\boldsymbol{r},\boldsymbol{\omega})}{\sigma_t(\boldsymbol{r})} & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}\end{aligned}$$

- Pseudocode:
  - Draw τ from *p*
  - If $\tau < h(\boldsymbol{x},\boldsymbol{\omega})$, return $\frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})}\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i\to\boldsymbol{\omega})L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + \frac{Q(\boldsymbol{r},\boldsymbol{\omega})}{\sigma_t(\boldsymbol{r})}$
  - Otherwise, return $L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega})$

Direction Sampling

- One extra integral remains:

$$\langle L(\boldsymbol{x},\boldsymbol{\omega})\rangle = \frac{f(\tau)}{p(\tau)}$$
$$= \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} \int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega}) L(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + \frac{Q(\boldsymbol{r},\boldsymbol{\omega})}{\sigma_t(\boldsymbol{r})} & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

- $\boldsymbol{\omega}_i$ can be sampled based on $f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega})$
  - In practice, $p(\boldsymbol{\omega}_i) = f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega})$ is usually a valid probability density on $\boldsymbol{\omega}_i$, yielding

$$\langle L(\boldsymbol{x},\boldsymbol{\omega})\rangle = \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} L(\boldsymbol{r},\boldsymbol{\omega}_i) + \frac{Q(\boldsymbol{r},\boldsymbol{\omega})}{\sigma_t(\boldsymbol{r})} & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

Volume Path Tracing

$$\langle L(\boldsymbol{x},\boldsymbol{\omega})\rangle = \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} L(\boldsymbol{r},\boldsymbol{\omega}_i) + \frac{Q(\boldsymbol{r},\boldsymbol{\omega})}{\sigma_t(\boldsymbol{r})} & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

```
radiance(x, ω):
    compute h = h(x, ω) # using ray tracing
    draw τ
    if τ < h:
        r = x - τ*ω
        draw ωi
        return σs(r)/σt(r)*radiance(r, ωi) + Q(r, ω)/σt(r)
    else:
        return boundaryRadiance(x - h*ω, ω)
```

How to implement this?

Free Distance Sampling Methods

$$p(\tau) = \lambda(\tau)\exp\left(-\int_0^\tau \lambda(t)\,\mathrm{d}t\right) \text{ where } \lambda(\tau) := \begin{cases} \sigma_t(\boldsymbol{x} - \tau\boldsymbol{\omega}) & \tau < h(\boldsymbol{x}, \boldsymbol{\omega}), \\ \lambda_0 & \tau \geq h(\boldsymbol{x}, \boldsymbol{\omega}). \end{cases}$$

- How to draw samples from this distribution?

- Homogeneous media
  - $\sigma_t(\boldsymbol{x}) \equiv \sigma_t,\ \forall \boldsymbol{x} \in \mathcal{V}^0$
  - Let $\lambda_0 = \sigma_t$, then $\lambda(\tau) \equiv \sigma_t,\ \forall \tau \geq 0$ and
    $$p(\tau) = \sigma_t \exp(-\sigma_t \tau)$$
  - In this case, τ can be drawn using the inversion method: $\tau = -\log \xi / \sigma_t$ where $\xi \sim U(0,1)$

Free Distance Sampling Methods

$$p(\tau) = \lambda(\tau)\exp\left(-\int_0^\tau \lambda(t)\,\mathrm{d}t\right) \text{ where } \lambda(\tau) := \begin{cases} \sigma_t(\boldsymbol{x} - \tau\boldsymbol{\omega}) & \tau < h(\boldsymbol{x}, \boldsymbol{\omega}), \\ \lambda_0 & \tau \geq h(\boldsymbol{x}, \boldsymbol{\omega}). \end{cases}$$

- Heterogeneous media
  - $\sigma_t(\boldsymbol{x})$ varies with ***x***, causing $\lambda(\tau)$ to vary with τ
  - *p* does not have a close-form expression in general

- Common sampling methods
  - Ray marching
  - Delta tracking

Ray Marching

$$\mathrm{CDF}(\tau) := \int_0^\tau p(t)\,\mathrm{d}t = 1 - \exp\left(-\int_0^\tau \lambda(t)\,\mathrm{d}t\right)$$

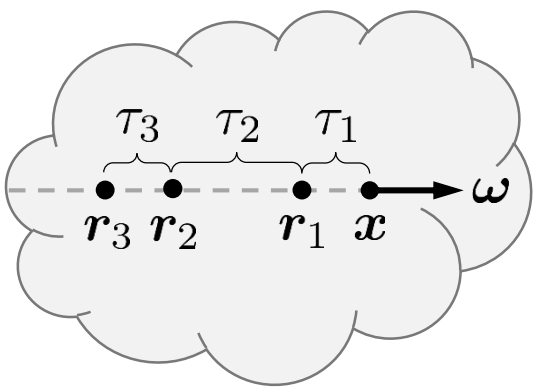
- One can apply the inversion method by
  1. Drawing ξ from U(0, 1)
  2. Finding τ satisfying $\mathrm{CDF}(\tau) = \xi$
     - This is usually achieved numerically by iteratively increasing τ with some fixed step size $\Delta\tau$ until $\mathrm{CDF}(\tau)$ reaches ξ
     - The step size $\Delta\tau$ is generally picked according to the underlying representation of $\sigma_t(\boldsymbol{x})$ (e.g., voxel size)

Ray Marching

- Pros
  - For each sample τ , $p(\tau)$ can be obtained easily
- Cons
  - Biased (for any finite step size $\Delta\tau$)
  - Resolution dependent
    - $\Delta\tau$ needs to be picked based on the resolution of the density (σ_t) field
    - Slow for high-resolution density fields

Delta Tracking

- Also known as Woodcock tracking
- Basic idea
  - Consider the medium to have homogeneous density $\sigma_t^{\max} := \max\{\sigma_t(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathcal{V}^0\}$, and use it to draw free distances
  - To compensate the fact that “phantom” densities have been introduced, the sampling process continues with probability $1 - \sigma_t(\boldsymbol{r}_i)/\sigma_t^{\max}$ at each $\boldsymbol{r}_i$



$$\tau_i = -\frac{\log \xi_i}{\sigma_t^{\max}}$$

Delta Tracking

- Pseudocode:

```
deltaTracking(x, ω, σt^max)
    compute h using ray tracing
    τ = 0
    while τ < h:
        τ += -log(rand())/σt^max
        r = x - τ*ω
        if rand() < σt(r)/σt^max:
            break
    return τ
```

Delta Tracking

- Pros
  - Unbiased
  - Resolution independent

- Cons
  - For each sample τ , $p(\tau)$ is not immediately available
  - Slow for density fields with widely varying σ_t values (i.e., $\sigma_t^{max} >> \sigma_t(\boldsymbol{x})$ for many $\boldsymbol{x}$)

Volume Path Tracing (VPT)

$$\langle L(\boldsymbol{x},\boldsymbol{\omega})\rangle = \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} L(\boldsymbol{r},\boldsymbol{\omega}_i) + \frac{Q(\boldsymbol{r},\boldsymbol{\omega})}{\sigma_t(\boldsymbol{r})} & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ L(\boldsymbol{x}_{\partial\mathcal{V}},\boldsymbol{\omega}) & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

```
radiance(x, ω):
    compute h = h(x, ω)
    draw τ
    if τ < h:
        r = x - τ*ω
        draw ωi
        return σs(r)/σt(r)*radiance(r, ωi) + Q(r, ω)/σt(r)
    else:
        return boundaryRadiance(x - h*ω, ω)
```

This basic version can be improved using techniques we have seen earlier:

- *Russian roulette*
- *Next-event estimation*
- *Multiple importance sampling*

VPT with Next-Event Estimation

- The RTE $L = \boldsymbol{K}L + S$ implies that $\boldsymbol{K}L = \boldsymbol{K}(\boldsymbol{K}L + S)$. Namely,

$$(\boldsymbol{K}L)(\boldsymbol{x}, \boldsymbol{\omega}) = \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r} \leftrightarrow \boldsymbol{x})\, \sigma_s(\boldsymbol{r}) \int_{\mathbb{S}^2} f_p(\boldsymbol{r}, \boldsymbol{\omega}_i \to \boldsymbol{\omega})[(\boldsymbol{K}L)(\boldsymbol{r}, \boldsymbol{\omega}_i) + S(\boldsymbol{r}, \boldsymbol{\omega}_i)]\, \mathrm{d}\boldsymbol{\omega}_i\, \mathrm{d}\tau$$

where $\boldsymbol{r} := \boldsymbol{x} - \tau\boldsymbol{\omega}$

- By drawing τ from the aforementioned exponential distribution, we have

$$\langle (\boldsymbol{K}L)(\boldsymbol{x}, \boldsymbol{\omega}) \rangle = \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} \int_{\mathbb{S}^2} f_p(\boldsymbol{r}, \boldsymbol{\omega}_i \to \boldsymbol{\omega})[(\boldsymbol{K}L)(\boldsymbol{r}, \boldsymbol{\omega}_i) + S(\boldsymbol{r}, \boldsymbol{\omega}_i)]\, \mathrm{d}\boldsymbol{\omega}_i & \tau < h(\boldsymbol{x}, \boldsymbol{\omega}), \\ 0 & \tau \geq h(\boldsymbol{x}, \boldsymbol{\omega}) \end{cases}$$

VPT with Next-Event Estimation

$$\langle (\boldsymbol{K}L)(\boldsymbol{x},\boldsymbol{\omega})\rangle = \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} \boxed{\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega})[(\boldsymbol{K}L)(\boldsymbol{r},\boldsymbol{\omega}_i) + S(\boldsymbol{r},\boldsymbol{\omega}_i)]\,\mathrm{d}\boldsymbol{\omega}_i} & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ 0 & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

- The remaining integral is then split into two:

$$\int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega})(\boldsymbol{K}L)(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i + \int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega})S(\boldsymbol{r},\boldsymbol{\omega}_i)\,\mathrm{d}\boldsymbol{\omega}_i$$

Estimate recursively by drawing $\boldsymbol{\omega}_i$ based on f_p
"indirect illumination"

Estimate directly by area sampling or MIS
"direct illumination"

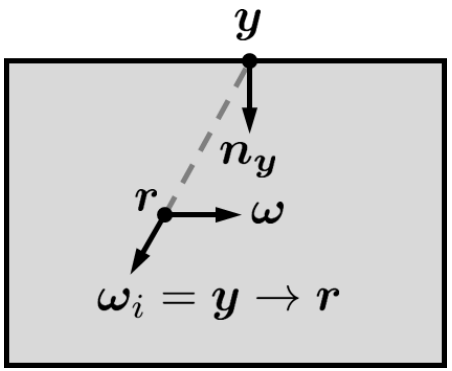
VPT with Next-Event Estimation

$$\langle (\boldsymbol{K}L)(\boldsymbol{x},\boldsymbol{\omega})\rangle = \begin{cases} \frac{\sigma_s(\boldsymbol{r})}{\sigma_t(\boldsymbol{r})} \int_{\mathbb{S}^2} f_p(\boldsymbol{r},\boldsymbol{\omega}_i \to \boldsymbol{\omega})[(\boldsymbol{K}L)(\boldsymbol{r},\boldsymbol{\omega}_i) + S(\boldsymbol{r},\boldsymbol{\omega}_i)]\,\mathrm{d}\boldsymbol{\omega}_i & \tau < h(\boldsymbol{x},\boldsymbol{\omega}), \\ 0 & \tau \geq h(\boldsymbol{x},\boldsymbol{\omega}) \end{cases}$$

- Pseudocode:

```
scatteredRadiance(x, ω):
    compute h = h(x, ω) # using ray tracing
    draw τ
    if τ < h:
        r = x - τ*ω
        rad = directIllumination(r, ω)
        draw ωi
        rad += scatteredRadiance(r, ωi)
        return σs(r)/σt(r)*rad
    else:
        return 0
```

Direct Illumination for VPT



- Recall that

$$S(\boldsymbol{x}, \boldsymbol{\omega}) := \int_0^{h(\boldsymbol{x},\boldsymbol{\omega})} T(\boldsymbol{r} \leftrightarrow \boldsymbol{x})\, Q(\boldsymbol{r}, \boldsymbol{\omega})\, \mathrm{d}\tau + T(\boldsymbol{x}_{\partial\mathcal{V}} \leftrightarrow \boldsymbol{x})\, L(\boldsymbol{x}_{\partial\mathcal{V}}, \boldsymbol{\omega})$$

- For non-emissive materials, *Q* vanishes and

$$S(\boldsymbol{x}, \boldsymbol{\omega}) = T(\boldsymbol{x}_{\partial\mathcal{V}} \leftrightarrow \boldsymbol{x})\, L(\boldsymbol{x}_{\partial\mathcal{V}}, \boldsymbol{\omega})$$

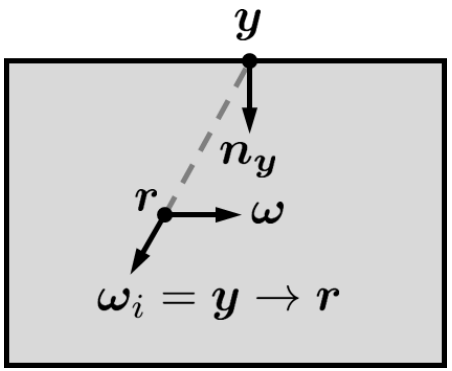
- In this case,

$$\int_{\mathbb{S}^2} f_p(\boldsymbol{r}, \boldsymbol{\omega}_i \to \boldsymbol{\omega})\, S(\boldsymbol{r}, \boldsymbol{\omega}_i)\, \mathrm{d}\boldsymbol{\omega}_i$$

$$= \int_{\partial\mathcal{V}} f_p(\boldsymbol{r}, (\boldsymbol{y} \to \boldsymbol{r}) \to \boldsymbol{\omega})\, T(\boldsymbol{y} \leftrightarrow \boldsymbol{r})\, \underbrace{L(\boldsymbol{y} \to \boldsymbol{r})}_{\text{Boundary radiance}}\, \underbrace{\frac{V(\boldsymbol{y} \leftrightarrow \boldsymbol{r})\, |\langle \boldsymbol{n}_{\boldsymbol{y}}, \boldsymbol{y} \to \boldsymbol{r} \rangle|}{\|\boldsymbol{y} - \boldsymbol{r}\|^2}}_{\text{Change of measure}}\, \mathrm{d}\boldsymbol{y}$$

  - The area integral can be further restricted to the subset of $\partial\mathcal{V}$ where the boundary radiance is non-zero

Direct Illumination for VPT



- Phase function sampling:

$$\int_{\mathbb{S}^2} f_p(\boldsymbol{r}, \boldsymbol{\omega}_i \to \boldsymbol{\omega})\, S(\boldsymbol{r}, \boldsymbol{\omega}_i)\, \mathrm{d}\boldsymbol{\omega}_i \quad \Rightarrow \quad S(\boldsymbol{r}, \boldsymbol{\omega}_i)$$

Draw $\boldsymbol{\omega}_i$ based on f_p

- Area sampling:

$$\int_{\partial\mathcal{V}} f_p(\boldsymbol{r}, (\boldsymbol{y} \to \boldsymbol{r}) \to \boldsymbol{\omega})\, T(\boldsymbol{y} \leftrightarrow \boldsymbol{r})\, L(\boldsymbol{y} \to \boldsymbol{r})\, \frac{V(\boldsymbol{y} \leftrightarrow \boldsymbol{r})\, |\langle \boldsymbol{n}_{\boldsymbol{y}}, \boldsymbol{y} \to \boldsymbol{r} \rangle|}{\|\boldsymbol{y} - \boldsymbol{r}\|^2}\, \mathrm{d}\boldsymbol{y}$$

$$\Downarrow$$

$$\frac{1}{|\partial\mathcal{V}|} f_p(\boldsymbol{r}, (\boldsymbol{y} \to \boldsymbol{r}) \to \boldsymbol{\omega})\, T(\boldsymbol{y} \leftrightarrow \boldsymbol{r})\, L(\boldsymbol{y} \to \boldsymbol{r})\, \frac{V(\boldsymbol{y} \leftrightarrow \boldsymbol{r})\, |\langle \boldsymbol{n}_{\boldsymbol{y}}, \boldsymbol{y} \to \boldsymbol{r} \rangle|}{\|\boldsymbol{y} - \boldsymbol{r}\|^2}$$

Draw $\boldsymbol{y}$ from $\partial\mathcal{V}$

- The two strategies can be combined using MIS

Next Lecture

- Metropolis light transport (MLT)
  - Applying the Metropolis-Hasting algorithm to rendering