Direct Illumination & Monte Carlo Integration

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Announcement

- Homework 1 will be out on Thursday, Apr 13
- Check course website for additional readings

Last Lecture

- Radiometry
  - Physics of light
- BRDFs
  - How materials reflects light

Today's Lecture

- Direct illumination
- Monte Carlo integration I

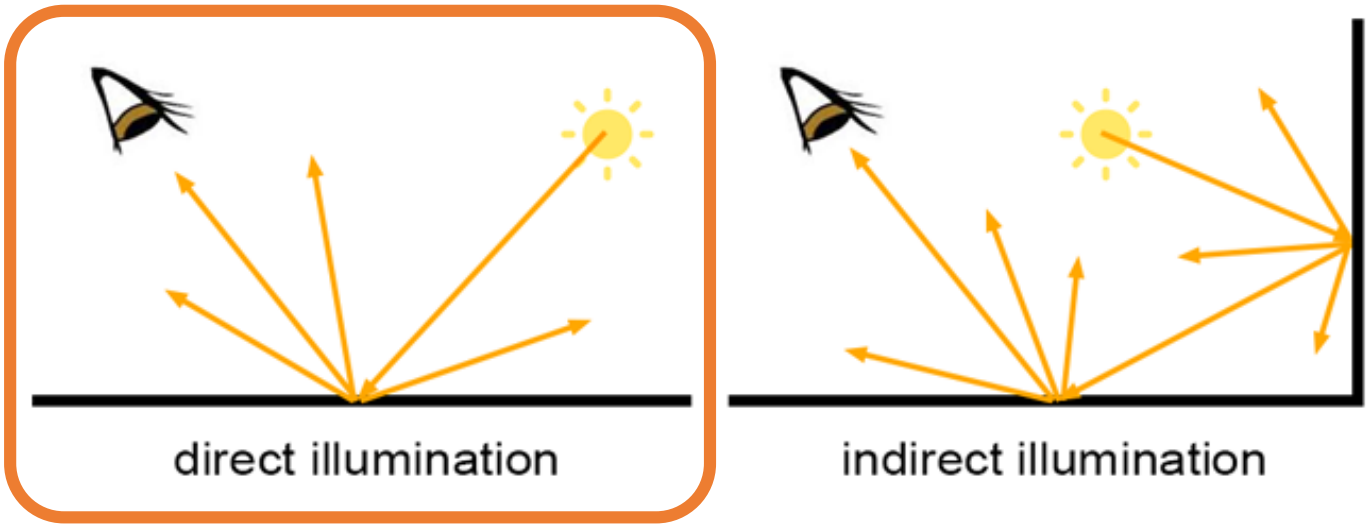
Direct Illumination

CS295 Realistic Image Synthesis

Direct and Indirect Illumination

Today's focus

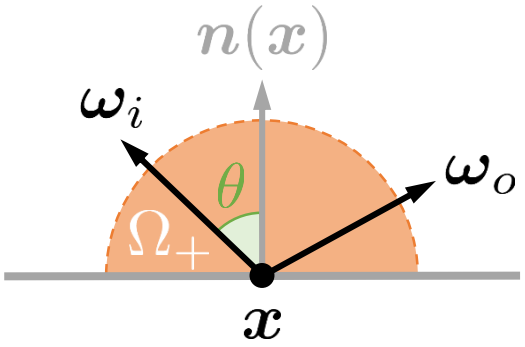
[scratchpixel.com]



Recap: BRDF

- The **B**idirectional **R**eflectance **D**istribution **F**unction (BRDF)

$$f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) := \frac{\mathrm{d}L_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_o)}{L_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{\omega}_i) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i}$$

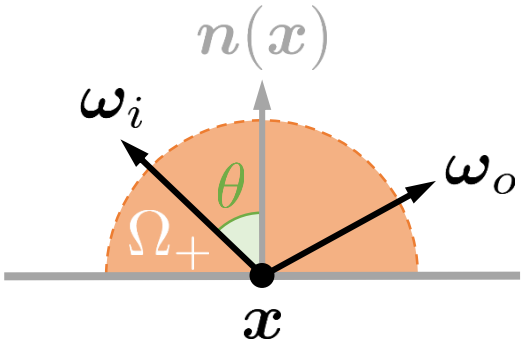


Convention: all vectors point away from ***x***

Reflected Radiance

$$f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) := \frac{\mathrm{d}L_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_o)}{L_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{\omega}_i) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i}$$

$$L_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \int_{\Omega_+} L_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i$$



Convention: all vectors point away from ***x***

Incident and Exitant Radiance

$$L_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \int_{\Omega_+} L_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i$$

Exitant Incident

- $L_r(\boldsymbol{x}, \boldsymbol{\omega}_o)$: all reflected light *leaving* ***x*** (exitant)
- $L_i(\boldsymbol{x}, \boldsymbol{\omega}_i)$: all light *entering* ***x*** (incident)

- Other exitant quantities
  - $L_e(\boldsymbol{x}, \boldsymbol{\omega}_o)$: all emitted light *leaving* ***x***
    (nonzero for light sources)
  - $L(\boldsymbol{x}, \boldsymbol{\omega}_o)$: all light *leaving* ***x***

$$L(\boldsymbol{x}, \boldsymbol{\omega}_o) = L_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_o) + L_{\mathrm{e}}(\boldsymbol{x}, \boldsymbol{\omega}_o)$$

Incident and Exitant Radiance

- In general:

$$L_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{\omega}) = L(\boldsymbol{x}, -\boldsymbol{\omega}) = L(\boldsymbol{y}, -\boldsymbol{\omega})$$

  - ***y*** is computed by tracing a ray from ***x*** in direction ***ω***

- Recap: invariant of radiance (in vacuum)

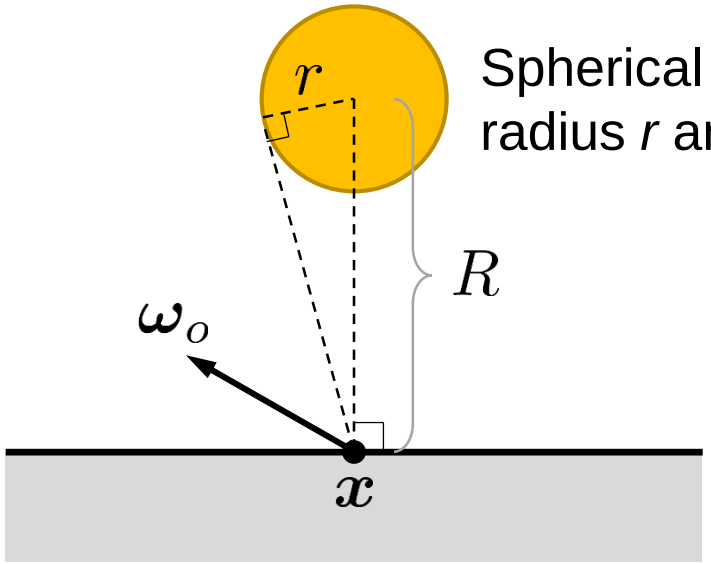
Direct Illumination

$$L_{\mathrm{i}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) = L_{\mathrm{e}}(\boldsymbol{y}, -\boldsymbol{\omega})$$

$$L_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \int_{\Omega_+} L_{\mathrm{i}}(\boldsymbol{x}, \boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i$$

$$\begin{aligned} L_{\mathrm{r}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o) &= \int_{\Omega_+} L_{\mathrm{i}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i \\ &= \int_{\Omega_+} L_{\mathrm{e}}(\boldsymbol{y}, -\boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i \end{aligned}$$

Example: Uniform Spherical Light



Spherical light source with radius r and $L_e(\boldsymbol{x},\boldsymbol{\omega}) \equiv L_0$

Diffuse surface
$f_r(\boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \equiv k_d/\pi$

$$L_{\mathrm{r}}^{\mathrm{direct}}(\boldsymbol{x},\boldsymbol{\omega}_o) = \int_{\Omega_+} L_{\mathrm{e}}(\boldsymbol{y},-\boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x},\boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \,\mathrm{d}\boldsymbol{\omega}_i$$

$$= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\arcsin\frac{r}{R}} L_0 \frac{k_d}{\pi} \cos\theta \sin\theta \,\mathrm{d}\theta$$

$$= 2\,L_0\,k_d \int_0^{\arcsin\frac{r}{R}} \cos\theta \sin\theta \,\mathrm{d}\theta = \frac{L_0\,k_d\,r^2}{R^2}$$

Recall:
$\mathrm{d}\boldsymbol{\omega}_i = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$

Example: Uniform Spherical Light

$$L_{\mathrm{r}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \frac{L_0 \, k_d \, r^2}{R^2}$$

- For a spherical light source with fixed radiant power Φ_0 (i.e., $4\pi^2 r^2 \, L_0 \equiv \Phi_0$):

$$L_{\mathrm{r}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \frac{\Phi_0 \, k_d}{4\pi^2 R^2}$$

- Further, when *r* goes to zero (point source):

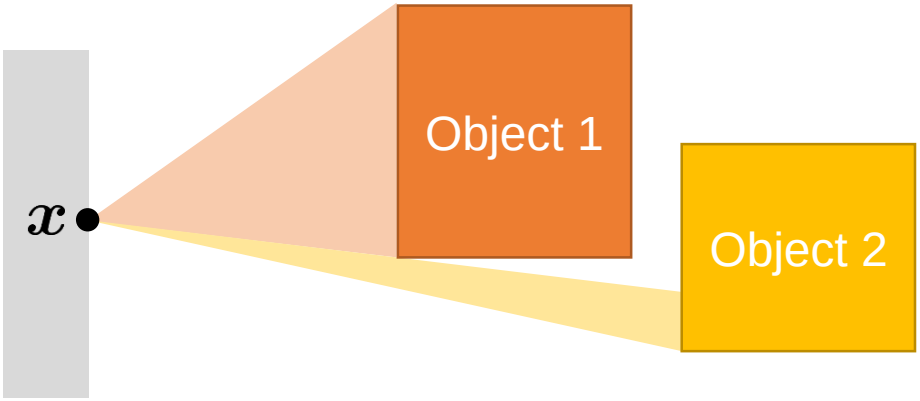
Recall: $I_0 = \frac{\Phi_0}{4\pi}$

$$L_{\mathrm{r}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \frac{\Phi_0 \, k_d}{4\pi^2 R^2} = \frac{I_0}{R^2} \frac{k_d}{\pi}$$

Computing Direct Illumination

$$L_{\mathrm{r}}^{\mathrm{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o) = \int_{\Omega_+} L_{\mathrm{e}}(\boldsymbol{y}, -\boldsymbol{\omega}_i) f_{\mathrm{r}}(\boldsymbol{x}, \boldsymbol{\omega}_i \leftrightarrow \boldsymbol{\omega}_o) \cos\theta \, \mathrm{d}\boldsymbol{\omega}_i$$

- Challenges
  - $\boldsymbol{y}$ changes discontinuously with $\boldsymbol{\omega}_i$ due to occlusion



  - f_r can be complicated (e.g., microfacet BRDFs)

Monte Carlo Integration I

CS295 Realistic Image Synthesis

Monte Carlo Integration

- A powerful framework for computing **integrals**

$$\int_{\Omega_{\boldsymbol{x}}} f(\boldsymbol{x})\,\mathrm{d}\mu(\boldsymbol{x}) = ?$$

- Numerical
- Nondeterministic (i.e., using randomness)
- Scalable to *high-dimensional* problems

Recap: Random Variables

- (Discrete) random variable X

- Possible outcomes: x_1, x_2, …, x_n
  - with probabilities p_1, p_2, …, p_n such that

$$\sum_{i=1}^{n} p_i = 1$$

- E.g., “fair” coin
  - Outcomes: x_1 = “head”, x_2 = “tail”
  - Probabilities: $p_1 = p_2 = ½$

Recap: Random Variables

- (Continuous) random variable *X*

- Possible outcomes: [*a*, *b*]
  - with probability density *p*(*x*) such that

$$\int_a^b p(x)\mathrm{d}x = 1$$

Recap: Expected Value & Variance

- Expected value: $\mathbb{E}[X] := \int_a^b x\, p(x)\, \mathrm{d}x$

$$\mathbb{E}[g(X)] := \int_a^b g(x)\, p(x)\, \mathrm{d}x$$

- Variance: $\mathrm{Var}[X] := \mathbb{E}\left[(X - \mathbb{E}[X])^2\right]$

$$= \int_a^b (x - \mathbb{E}[X])^2 p(x)\, \mathrm{d}x$$

$$= \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$$

$$\mathrm{Var}[g(X)] = \mathbb{E}[g(X)^2] - (\mathbb{E}[g(X)])^2$$

Recap: Expected Value & Variance

Let X and Y be two random variables, then:

Today's focus

$$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$
$$\mathbb{E}[aX] = a\,\mathbb{E}[X] \quad (a \text{ constant})$$
$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X]\,\mathbb{E}[Y] \quad (\text{if } X \text{ and } Y \text{ are independent})$$

$$\mathrm{Var}[aX] = a^2\,\mathrm{Var}[X] \quad (a \text{ constant})$$
$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] \quad (\text{if } X \text{ and } Y \text{ are independent})$$

Recap: Strong Law of Large Numbers

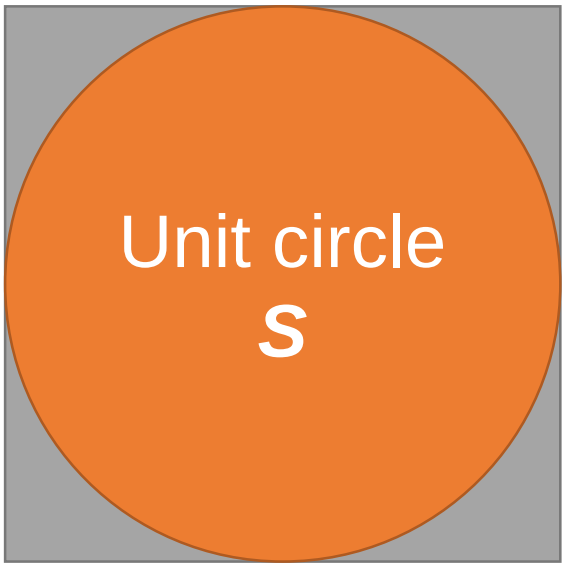
Let x_1, x_2, …, x_n be n independent observations (aka. **samples**) of X

$$\bar{x} := \frac{1}{n}\sum_{i=1}^{n} x_i \xrightarrow{n\to\infty} \mathbb{E}[X]$$

Sample mean

"Actual" mean

Example: Evaluating π



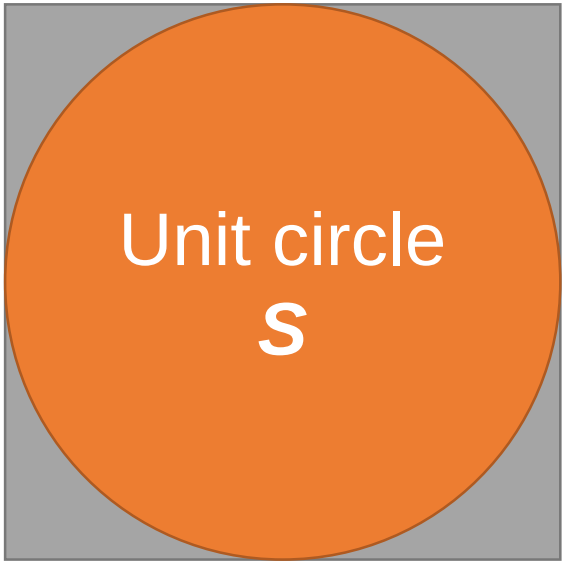
Circle area = π

Square area = 4

- Let $\boldsymbol{X}$ be a point uniformly distributed in the square
- Probability for $\boldsymbol{X}$ inside $\boldsymbol{S}$: $\pi/4$
- Let $Y := \begin{cases} 4 & \boldsymbol{X} \in S \\ 0 & \boldsymbol{X} \notin S \end{cases}$
- Then

$$\mathbb{E}[Y] = 4 \cdot \mathbb{P}[Y = 4] + 0 \cdot \mathbb{P}[Y = 0]$$
$$= 4 \cdot \frac{\pi}{4} = \pi$$

Example: Evaluating π



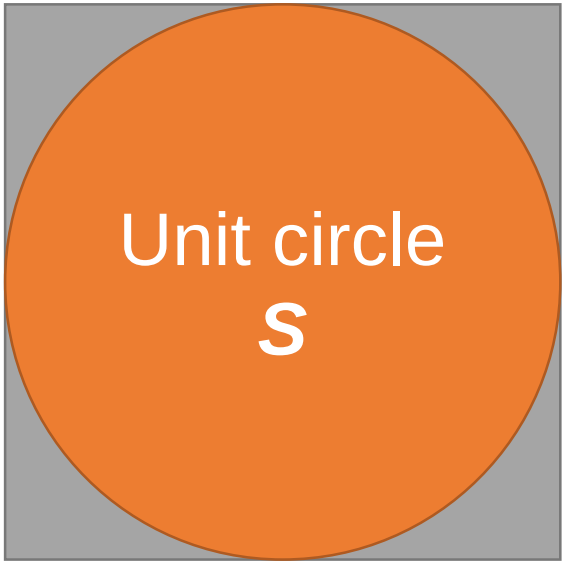
Circle area = π

Square area = 4

$$Y := \begin{cases} 4 & \boldsymbol{X} \in S \\ 0 & \boldsymbol{X} \notin S \end{cases}$$

- (Simple) solution for computing π:
  - Generating n independent samples of Y based on n i.i.d. $\boldsymbol{X}$ samples
  - Computing their mean

Example: Evaluating π



Circle area = π

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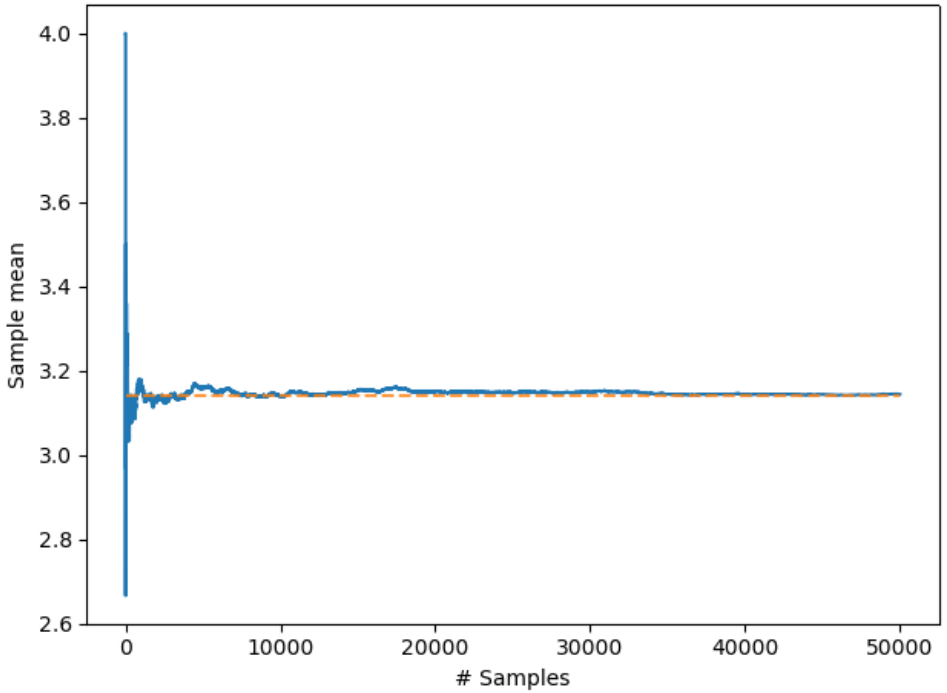
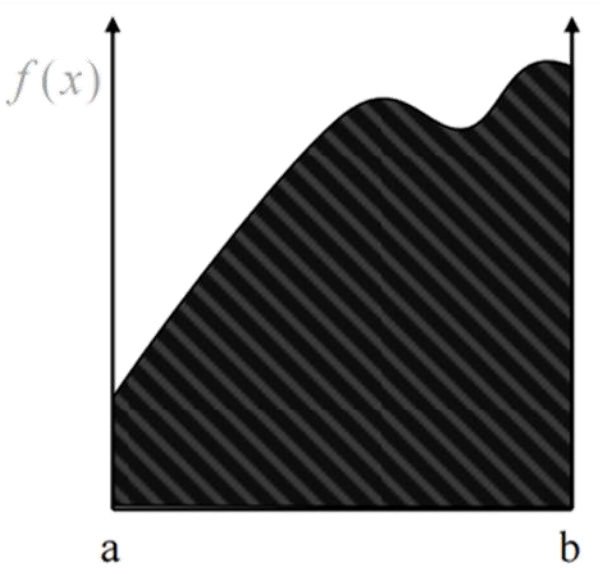


Integral

- f(x): one-dimensional function

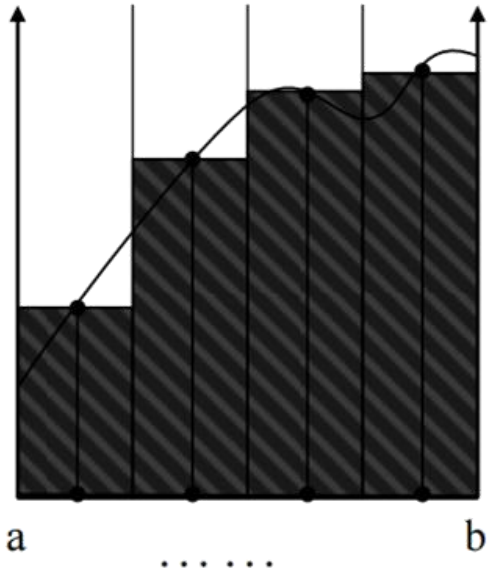
$$I = \int_a^b f(x)\,\mathrm{d}x$$



Deterministic Integration

- Quadrature rules:

$$I = \int_a^b f(x)\,\mathrm{d}x$$
$$\approx \sum_{i=1}^{n} \frac{b-a}{n} f(x_i)$$



- Problem:
  - Scales poorly with high dimensionality (more on this in the next lecture)

Monte Carlo Integration: Overview

- Goal: Estimating $I = \int_a^b f(x)\,\mathrm{d}x$
- Idea: Constructing random variable $\langle I \rangle$
  - Such that $\mathbb{E}[\langle I \rangle] = I$
  - $\langle I \rangle$ is called an **unbiased estimator** of I
- But how?

Monte Carlo Integration

- Let *p*() be any probability density function over [*a*, *b*] and *X* be a random variable with density *p*
- Let $\langle I \rangle := \dfrac{f(X)}{p(X)}$, then:

$$\mathbb{E}[g(X)] = \int_a^b g(x)\, p(x)\, \mathrm{d}x$$

$$\mathbb{E}[\langle I \rangle] = \mathbb{E}\left[\frac{f(X)}{p(X)}\right] = \int_a^b \frac{f(x)}{p(x)} p(x)\, \mathrm{d}x = I$$

- To estimate $\mathbb{E}[\langle I \rangle]$: strong law of large numbers

Monte Carlo Integration

- Goal: to estimate $I = \int_a^b f(x)\,\mathrm{d}x$
  - Pick a probability density function $p(x)$
  - Generate *n* independent samples:

$$x_1, x_2, \ldots, x_n \sim p$$

  - Evaluate $\hat{I}_j := \frac{f(x_j)}{p(x_j)}$ for *j* = 1, 2, …, *n*
  - Return sample mean: $\bar{I} := \frac{1}{n}\sum_{j=1}^{n} \hat{I}_j$

How to pick density function *p*()?

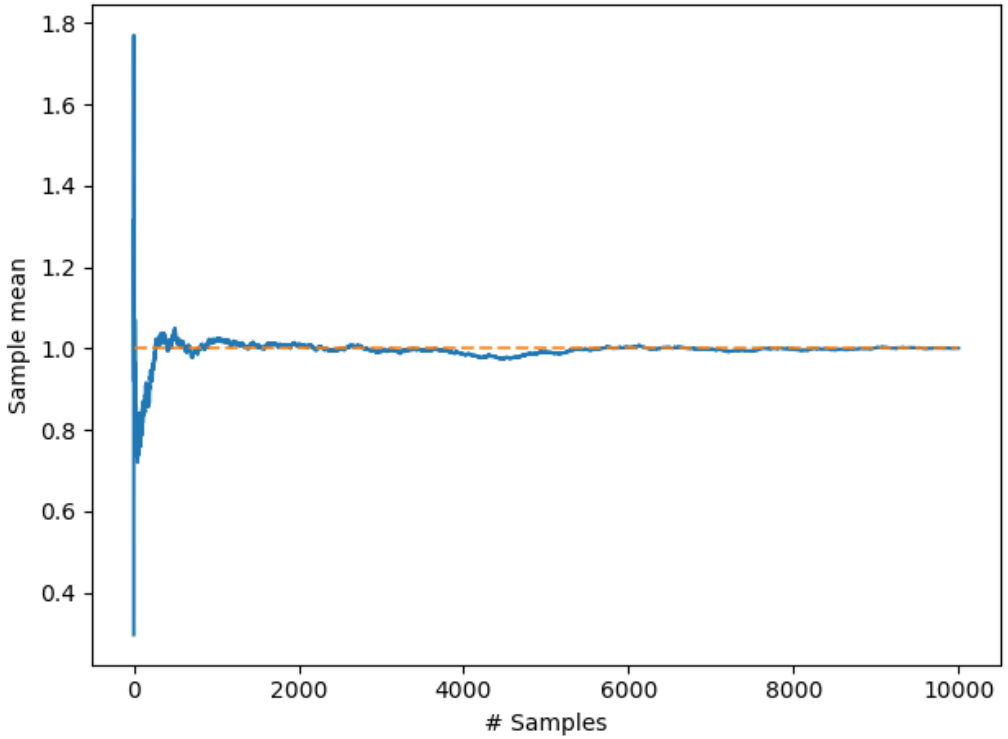
- **In theory**
  - (Almost) anything
- **In practice:**
  - Uniform distributions (almost) always work
    - As long as the domain is *bounded*
  - Choice of *p()* greatly affects the effectiveness (i.e., convergence rate) of the resulting estimator $\langle I \rangle$
  - More in later lectures

Monte Carlo Integration "Hello World"

- Estimating $\int_0^1 5x^4 \,\mathrm{d}x \quad (= \left[x^5\right]_0^1 = 1)$

- **Algorithm:**
  - Draw $x_1, x_2, \ldots, x_n$ from U[0, 1) independently
  - Return

$$\frac{1}{n}\sum_{j=1}^{n}\frac{f(x_j)}{p(x_j)} = \frac{1}{n}\sum_{j=1}^{n}5x_j^4$$

Monte Carlo Integration "Hello World"

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