Operator Formulation of Light Transport II

CS295, Spring 2017

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Announcement

- The deadline for PA1 has been extended to this Thursday (May 4)

Last Lecture

- Monte Carlo Path Tracing III
  - MIS for direct illumination

- Operator formulation of light transport
  - Ray space, throughput measure
  - Functions and operators
  - Operator formulation of the RE

Today's Lecture

- Operator formulation of light transport II
  - Sensors and measurements
  - Adjoint operators and adjoint particle tracing

Recap: Ray Space

- Let $\mathcal{M}$ be the set of surfaces in the scene, then $\mathcal{R} := \mathcal{M} \times \mathbb{S}^2$ is the ray space consisting of all light rays $(\boldsymbol{x}, \boldsymbol{\omega})$ originating from all surface points $\boldsymbol{x} \in \mathcal{M}$

- Let $\boldsymbol{r} := (\boldsymbol{x}, \boldsymbol{\omega})$, then the throughput measure is given by

$$\mathrm{d}\mu(\boldsymbol{r}) = \underbrace{\mathrm{d}\boldsymbol{x}}_{\text{area}} \underbrace{\mathrm{d}\sigma^{\perp}(\boldsymbol{\omega})}_{\substack{\text{projected}\\ \text{solid angle}}} = |\langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle| \, \mathrm{d}\boldsymbol{x} \underbrace{\mathrm{d}\boldsymbol{\omega}}_{\substack{\text{solid}\\ \text{angle}}}$$

Recap: Useful Operators

- For any function $h : \mathcal{R} \mapsto \mathbb{R}$
  - The local scattering operator ***K***:

$$(\boldsymbol{K}h)(\boldsymbol{x}, \boldsymbol{\omega}) := \int_{\Omega_{\boldsymbol{x}}} h(\boldsymbol{x}, \boldsymbol{\omega}_i)\, f_r(\boldsymbol{x}, \boldsymbol{\omega}_i \to \boldsymbol{\omega})\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_i \rangle \, \mathrm{d}\boldsymbol{\omega}_i$$

  - The propagation operator ***G***:

$$(\boldsymbol{G}h)(\boldsymbol{x}, \boldsymbol{\omega}) := \begin{cases} h(\boldsymbol{y}, -\boldsymbol{\omega}) & \text{ray } (\boldsymbol{x}, \boldsymbol{\omega}) \text{ hits a surface at } \boldsymbol{y}, \\ 0 & \text{ray } (\boldsymbol{x}, \boldsymbol{\omega}) \text{ goes to infinity} \end{cases}$$

Recap: Operator Formulation of the RE

- Given the two operators ***K*** and ***G***, we can rewrite the rendering equation (RE) as

$$L = L_e + \boldsymbol{K}\boldsymbol{G}L = L_e + \underbrace{\boldsymbol{T}}_{:=\boldsymbol{K}\boldsymbol{G}} L$$

- Solving the RE is effectively inverting (***I*** - ***T***) where ***I*** denotes the identity operator:

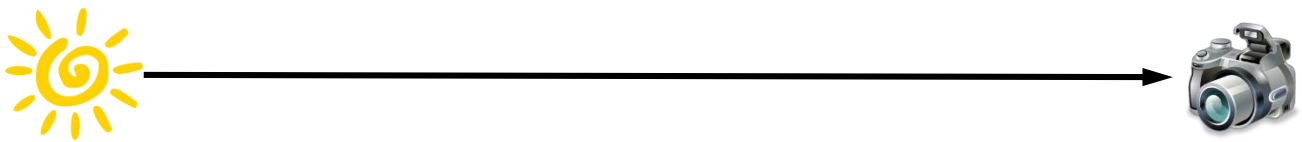
$$L - \boldsymbol{T}L = (\boldsymbol{I} - \boldsymbol{T})L = L_e$$

$$\Downarrow$$

$$L = (\boldsymbol{I} - \boldsymbol{T})^{-1}L_e = L_e + \boldsymbol{T}L_e + \boldsymbol{T}^2 L_e + \ldots$$

Operator Formulation of the RE

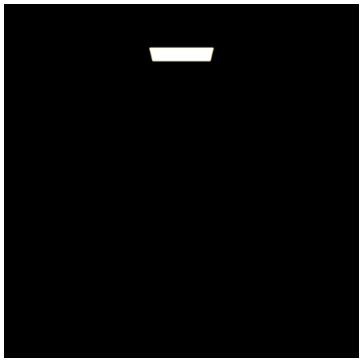
$$L = (\boldsymbol{I} - \boldsymbol{T})^{-1} L_e = L_e + \boldsymbol{T} L_e + \boldsymbol{T}^2 L_e + \ldots$$



L

$=$

L_e

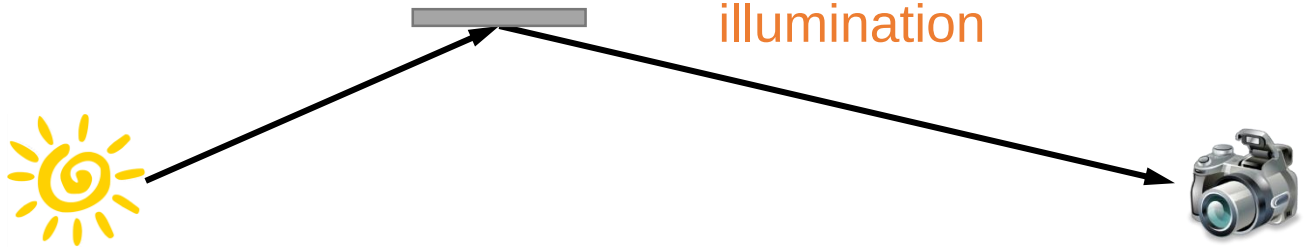


$\ldots$

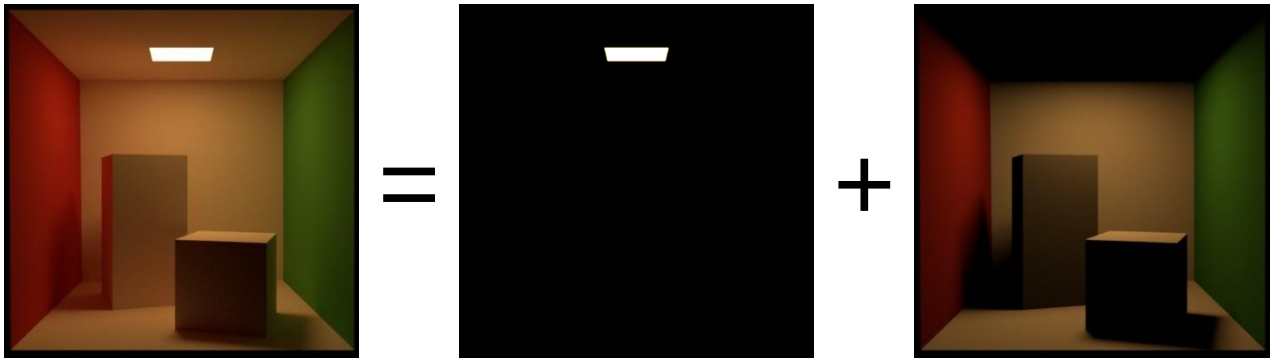
Operator Formulation of the RE

$$L = (\boldsymbol{I} - \boldsymbol{T})^{-1} L_e = L_e + \boldsymbol{T} L_e + \boldsymbol{T}^2 L_e + \ldots$$

Direct illumination



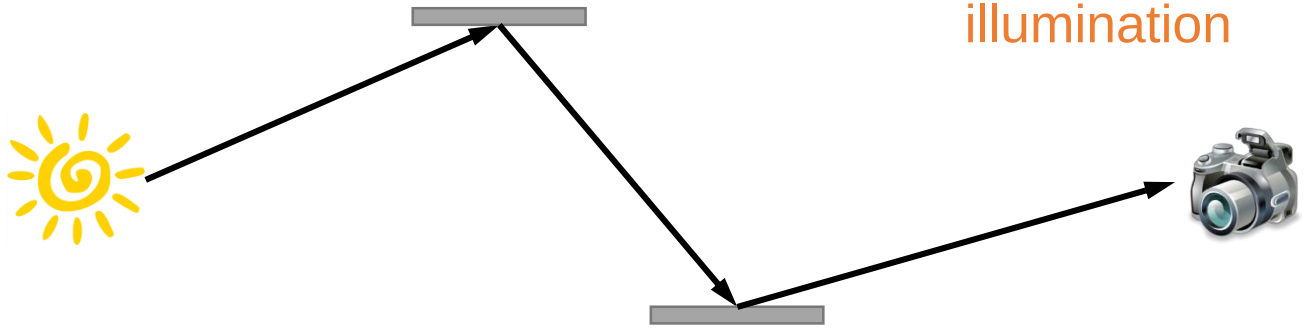
L $\quad$ L_e $\quad$ $\boldsymbol{T}L_e$



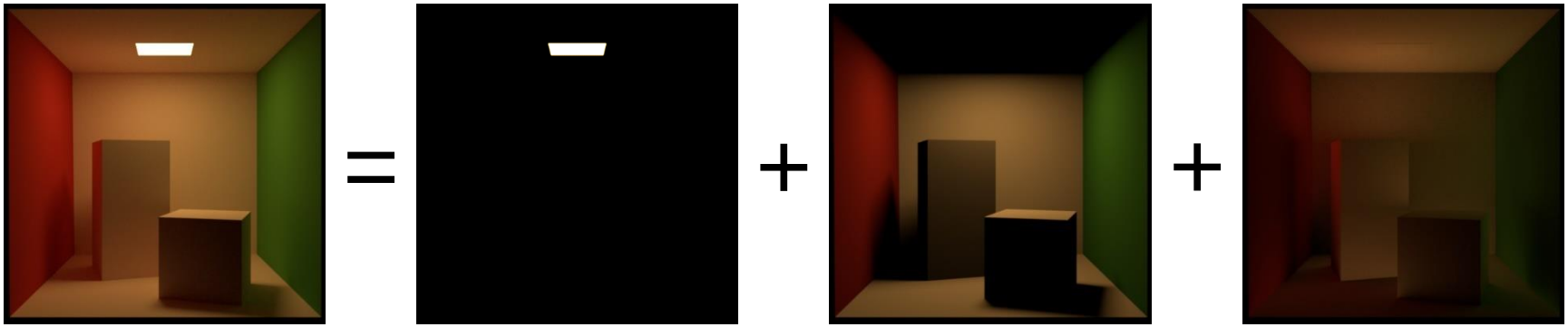
Operator Formulation of the RE

$$L = (\boldsymbol{I} - \boldsymbol{T})^{-1} L_e = L_e + \boldsymbol{T} L_e + \boxed{\boldsymbol{T}^2 L_e} + \ldots$$

2-bounce indirect illumination



L = L_e + $\boldsymbol{T}L_e$ + $\boldsymbol{T}^2L_e$ ...



Sensors and Measurements

- Up to this point, we have been considering estimating individual radiance values $L(\boldsymbol{x}, \boldsymbol{\omega})$ or $\mathrm{Li}(\boldsymbol{x}, \boldsymbol{\omega})$
- In practice, we often need to estimate the response of some sensor described by the measurement function $W_e : \mathcal{R} \mapsto \mathbb{R}$

$$
\begin{aligned}
I = \langle W_e, L_i \rangle &:= \int_{\mathcal{R}} W_e(\boldsymbol{r})\, L_i(\boldsymbol{r})\, \mathrm{d}\mu(\boldsymbol{r}) \\
&= \int_{\mathcal{M}} \int_{\Omega_{\boldsymbol{x}}} W_e(\boldsymbol{x}, \boldsymbol{\omega})\, L_i(\boldsymbol{x}, \boldsymbol{\omega}) \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle\, \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}\boldsymbol{x}
\end{aligned}
$$

Sensors and Measurements

- $I = \langle W_e, L_i \rangle$ can be estimated using MC integration:

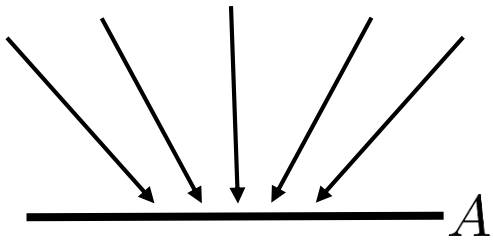
$$\langle I \rangle = \frac{W_e(\boldsymbol{x}, \boldsymbol{\omega})\, L_i(\boldsymbol{x}, \boldsymbol{\omega})\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}{p_0(\boldsymbol{x}, \boldsymbol{\omega})},$$

  where p_0 is a probability density over all rays

- In practice, it is desirable to have $p_0 \propto W_e$ or $p_0(\boldsymbol{x}, \boldsymbol{\omega}) \propto W_e(\boldsymbol{x}, \boldsymbol{\omega})\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle$

- After drawing $\boldsymbol{r} \sim p_0$, $L_i(\boldsymbol{r})$ can in turn be estimated using path tracing

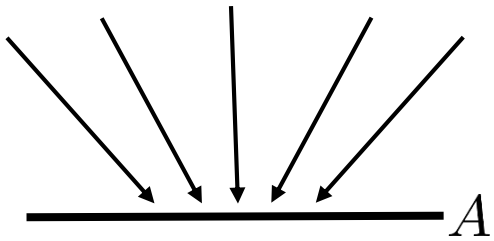
Example: Irradiance Meter



$$
\begin{aligned}
I &= \int_A \int_{\Omega_{\boldsymbol{x}}} L_i(\boldsymbol{x}, \boldsymbol{\omega}) \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle \, \mathrm{d}\boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x} \\
&= \int_{\mathcal{M}} \int_{\Omega_{\boldsymbol{x}}} \underbrace{\mathbb{1}[\boldsymbol{x} \in A]}_{=W_e(\boldsymbol{x}, \boldsymbol{\omega})} L_i(\boldsymbol{x}, \boldsymbol{\omega}) \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle \, \mathrm{d}\boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x}
\end{aligned}
$$

$$
p_0(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A] \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}{|A| \pi}, \quad \langle I \rangle = |A| \, \pi \, L_i(\boldsymbol{x}, \boldsymbol{\omega})
$$

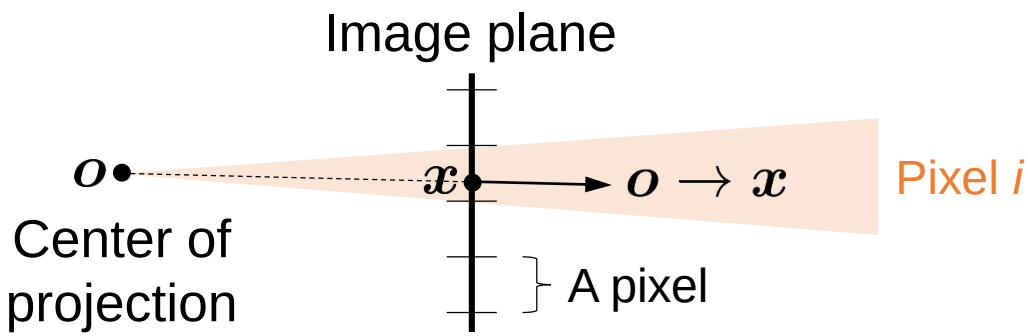
Example: Irradiance Meter



$$p_0(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A]\,\langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}{|A|\pi}, \quad \langle I \rangle = |A|\,\pi\,L_i(\boldsymbol{x}, \boldsymbol{\omega})$$

computeIrradiance():
  x = uniformSampleSensor()
  ω = uniformRandomPSA($\boldsymbol{n}_{\boldsymbol{x}}$)
  return |*A*| * *π* * receivedRadiance(***x***, ***ω***)

Example: Pinhole Camera

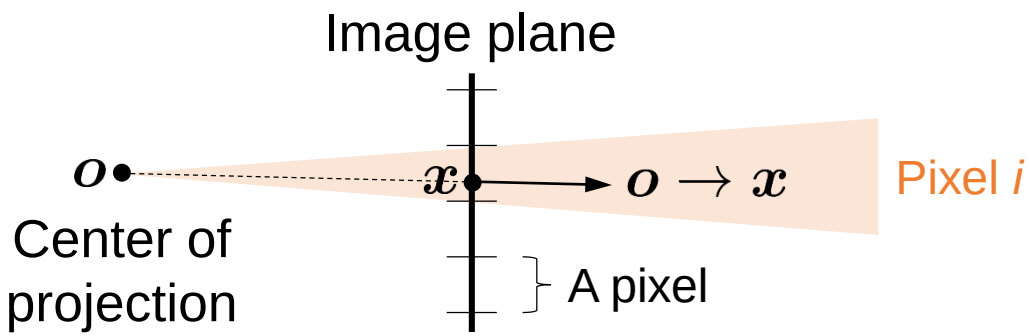


$$I_i = \frac{1}{|A_i|} \int_{A_i} L_i(\boldsymbol{x}, \boldsymbol{o} \to \boldsymbol{x}) \mathrm{d}\boldsymbol{x} \quad \left(\text{where } \boldsymbol{o} \to \boldsymbol{x} := \frac{\boldsymbol{x}-\boldsymbol{o}}{\|\boldsymbol{x}-\boldsymbol{o}\|}\right)$$

$$= \int_{\mathcal{M}} \int_{\Omega_{\boldsymbol{x}}} \underbrace{\frac{\mathbb{1}[\boldsymbol{x} \in A_i]\, \delta_{\boldsymbol{o}\to\boldsymbol{x}}(\boldsymbol{\omega})}{|A_i| \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}}_{=W_e(\boldsymbol{x},\boldsymbol{\omega})} L_i(\boldsymbol{x}, \boldsymbol{\omega}) \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle \, \mathrm{d}\boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x}$$

$$p_0(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A_i]\, \delta_{\boldsymbol{o}\to\boldsymbol{x}}(\boldsymbol{\omega})}{|A_i|}, \quad \langle I \rangle = L_i(\boldsymbol{x}, \boldsymbol{o} \to \boldsymbol{x})$$

Example: Pinhole Camera



$$p_0(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A_i]\, \delta_{\boldsymbol{o} \to \boldsymbol{x}}(\boldsymbol{\omega})}{|A_i|}, \quad \langle I \rangle = L_i(\boldsymbol{x}, \boldsymbol{o} \to \boldsymbol{x})$$

computePixelIntensity(*i*):
  x = uniformSamplePixel(*i*)
  ω = normalize(***x*** - ***o***)
  return receivedRadiance(***x***, ***ω***)

Adjoint Operators

- The adjoint of an operator ***H*** is denoted ***H****, and is defined by the property that

$$\langle \boldsymbol{H}^* f, g \rangle = \langle f, \boldsymbol{H} g \rangle \quad \text{for all } f,\ g$$

- If ***H*** = ***H****, then ***H*** is called self-adjoint
- The propagation operator ***G*** is self-adjoint

Adjoint Operators

- The local scattering operator ***K*** satisfies that

$$(\boldsymbol{K}h)(\boldsymbol{x},\boldsymbol{\omega}) = \int_{\Omega_{\boldsymbol{x}}} h(\boldsymbol{x},\boldsymbol{\omega}_i)\, f_r(\boldsymbol{x},\boldsymbol{\omega}_i \to \boldsymbol{\omega})\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_i \rangle \, \mathrm{d}\boldsymbol{\omega}_i$$

$$(\boldsymbol{K}^* h)(\boldsymbol{x},\boldsymbol{\omega}) = \int_{\Omega_{\boldsymbol{x}}} h(\boldsymbol{x},\boldsymbol{\omega}_o)\, f_r(\boldsymbol{x},\boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle \, \mathrm{d}\boldsymbol{\omega}_o$$

  - ***K*** is self-adjoint if $f_r(\boldsymbol{x},\boldsymbol{\omega}_i \to \boldsymbol{\omega}_o) = f_r(\boldsymbol{x},\boldsymbol{\omega}_o \to \boldsymbol{\omega}_i)$ for all $\boldsymbol{\omega}_i, \boldsymbol{\omega}_o \in \Omega_{\boldsymbol{x}}$
  - This is the case for all BRDFs with reciprocity (which holds for all BRDFs we have seen up to this point)

Adjoint Operators

- For any operators ***A***, ***B*** and ***C***:
  - $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$
  - $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$
  - $(\boldsymbol{A} + \boldsymbol{B})^* = \boldsymbol{A}^* + \boldsymbol{B}^*$, $(\boldsymbol{A}\,\boldsymbol{B})^* = \boldsymbol{B}^*\,\boldsymbol{A}^*$, $(\boldsymbol{A}^{-1})^* = (\boldsymbol{A}^*)^{-1}$
- Based on these properties and the fact that ***G*** is self-adjoint, we have

$$(\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{K}\boldsymbol{G})^{-1})^* = (\boldsymbol{I}-\boldsymbol{G}\boldsymbol{K}^*)^{-1}\boldsymbol{G} = \left(\sum_{i=0}^{\infty}(\boldsymbol{G}\boldsymbol{K}^*)^i\right)\boldsymbol{G} = \sum_{i=0}^{\infty}(\boldsymbol{G}\boldsymbol{K}^*)^i\boldsymbol{G}$$

$$= \sum_{i=0}^{\infty}\boldsymbol{G}(\boldsymbol{K}^*\boldsymbol{G})^i = \boldsymbol{G}\left(\sum_{i=0}^{\infty}(\boldsymbol{K}^*\boldsymbol{G})^i\right) = \boldsymbol{G}(\boldsymbol{I}-\boldsymbol{K}^*\boldsymbol{G})^{-1}$$

Adjoint Operators

- $\boldsymbol{G}(\boldsymbol{I} - \boldsymbol{KG})^{-1}$ is self-adjoint when $\boldsymbol{K}$ is self-adjoint
- Therefore,

$$\begin{aligned}\langle W_e,\ L_i\rangle = \langle W_e,\ \boldsymbol{G}L\rangle &= \langle W_e,\ \boldsymbol{G}(\boldsymbol{I} - \boldsymbol{KG})^{-1}L_e\rangle \\ &= \langle \boldsymbol{G}(\boldsymbol{I} - \boldsymbol{K}^*\boldsymbol{G})^{-1}W_e,\ L_e\rangle\end{aligned}$$

- $(\boldsymbol{I} - \boldsymbol{KG})^{-1} L_e$ gives the radiance L (as a function on the ray space) satisfying the RE: $L = \boldsymbol{KG}L + L_e$
- $(\boldsymbol{I} - \boldsymbol{K}^*\boldsymbol{G})^{-1} W_e$ gives another function $W : \mathcal{R} \mapsto \mathbb{R}$ satisfying $W = \boldsymbol{K}^*\boldsymbol{G}W + W_e$

The Importance Transport

- W is called the importance function
  - Computing W requires solving the *important transport* problem given by $W = \boldsymbol{K}^*\boldsymbol{G}W + W_e$
  - The sensor acts as the light source in the light transport problem by *emitting* importance W_e
  - Any algorithm that estimates L for the original light transport problem (e.g., path tracing) can be adapted to estimate W

Light vs. Importance Transport

Light transport

- **Transport equation**
  $L = \boldsymbol{KG}L + L_e$
- **Central quantity**
  L (radiance)
- **Source term**
  L_e (radiance emitted by light sources)
- **Measurements**
  $I = \langle W_e, \boldsymbol{G}L \rangle = \langle W_e, L_i \rangle$

Importance transport

- **Transport equation**
  $W = \boldsymbol{K}^*\boldsymbol{G}W + W_e$
- **Central quantity**
  W (importance)
- **Source term**
  W_e (importance "emitted" by sensors)
- **Measurements**
  $I = \langle \boldsymbol{G}W, L_e \rangle = \langle W_i, L_e \rangle$

Same results!

Adjoint Particle Tracing

- **Goal:** estimating

$$I = \langle \underbrace{W_i}_{:=\boldsymbol{G}W}, L_e \rangle = \int_{\mathcal{M}} \int_{\Omega_{\boldsymbol{x}}} W_i(\boldsymbol{x}, \boldsymbol{\omega})\, L_e(\boldsymbol{x}, \boldsymbol{\omega})\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle \, \mathrm{d}\boldsymbol{\omega}\, \mathrm{d}\boldsymbol{x}$$

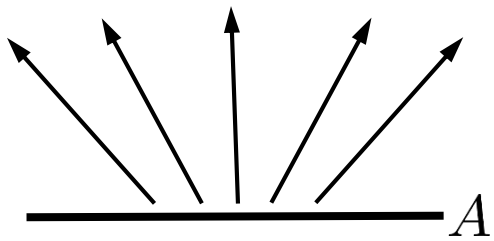
- **Initial** step:
  - Draw ***x*** and ***ω*** from some probability density $p_0^* \propto L_e$ or $p_0^*(\boldsymbol{x}, \boldsymbol{\omega}) \propto L_e(\boldsymbol{x}, \boldsymbol{\omega}) \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle$
  - Then,

$$\langle I \rangle = \frac{L_e(\boldsymbol{x}, \boldsymbol{\omega})\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}{p_0^*(\boldsymbol{x}, \boldsymbol{\omega})} W_i(\boldsymbol{x}, \boldsymbol{\omega})$$

Known after drawing
x and ***ω***

Example: Sampling Area Lights

Area light emitting radiance L_0 into all directions



Similar to irradiance meters in the light transport problem

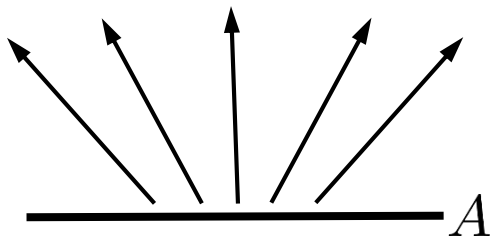
$$I = \int_A \int_{\Omega_{\boldsymbol{x}}} L_0 \, W_i(\boldsymbol{x}, \boldsymbol{\omega}) \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle \, \mathrm{d}\boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x}$$

$$= \int_{\mathcal{M}} \int_{\Omega_{\boldsymbol{x}}} \underbrace{L_0 \mathbb{1}[\boldsymbol{x} \in A]}_{=L_e(\boldsymbol{x},\boldsymbol{\omega})} W_i(\boldsymbol{x}, \boldsymbol{\omega}) \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle \, \mathrm{d}\boldsymbol{\omega} \, \mathrm{d}\boldsymbol{x}$$

$$p_0^*(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A] \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}{|A| \pi}, \quad \langle I \rangle = |A| \, L_0 \, \pi \, W_i(\boldsymbol{x}, \boldsymbol{\omega})$$

Example: Sampling Area Lights

Area light emitting radiance L_0 into all directions



Similar to irradiance meters in the light transport problem

$$p_0^*(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A] \, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega} \rangle}{|A| \pi}, \quad \langle I \rangle = |A| \, L_0 \, \pi \, W_i(\boldsymbol{x}, \boldsymbol{\omega})$$

computeMeasurement():
  x = uniformSampleLight()
  ω = uniformRandomPSA($\boldsymbol{n}_{\boldsymbol{x}}$)
  return $|A|$ * L_0 * π * receivedImportance(***x***, ***ω***)

Adjoint Particle Tracing

- **Main** step: estimating $W(\boldsymbol{x}, \boldsymbol{\omega})$
  - This can be done using an algorithm equivalent to path tracing since

$$W(\boldsymbol{x}, \boldsymbol{\omega}) = W_e(\boldsymbol{x}, \boldsymbol{\omega}) + \underbrace{\int_{\Omega_{\boldsymbol{x}}} W_i(\boldsymbol{x}, \boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle \, \mathrm{d}\boldsymbol{\omega}_o}_{=:\, W_r(\boldsymbol{x}, \boldsymbol{\omega})}$$

  - We call this algorithm adjoint particle tracing to emphasize that it solves the adjoint (importance transfer) problem

Path vs. Adjoint Particle Tracing

```
# Path tracing (version 0.1)
radiance(x, ω):
    rad = emittedRadiance(x, ω)
    ω_i = uniformRandomPSA(n_x)
    y = RayTrace(x, ω_i)
    rad += π * radiance(y, -ω_i) * brdf(x, ω_i, ω)
    return rad

# Adjoint particle tracing (version 0.1)
importance(x, ω):
    imp = emittedImportance(x, ω)
    ω_o = uniformRandomPSA(n_x)
    y = RayTrace(x, ω_o)
    imp += π * importance(y, -ω_o) * brdf(x, ω, ω_o)
    return imp
```

Next-Event Estimation

- Similar to path tracing, *next-event estimation* can drastically improve the convergence rate

- Same idea: separating direct & indirect

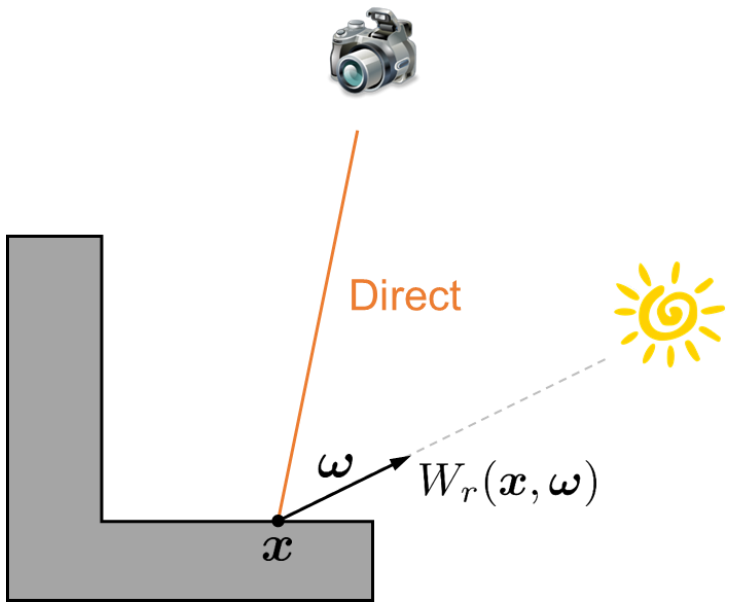
Path tracing: $$L_r(\boldsymbol{x}, \boldsymbol{\omega}) = L_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) + L_r^{\text{indirect}}(\boldsymbol{x}, \boldsymbol{\omega})$$

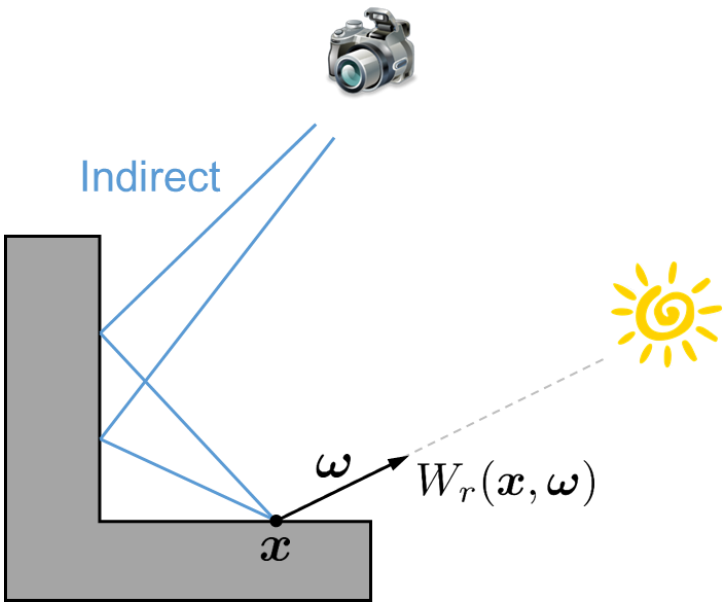
Adjoint particle tracing: $$W_r(\boldsymbol{x}, \boldsymbol{\omega}) = W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) + W_r^{\text{indirect}}(\boldsymbol{x}, \boldsymbol{\omega})$$

Direct and Indirect Importance

$$W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) = \int_{\Omega_{\boldsymbol{x}}} W_e(\boldsymbol{y}, -\boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle \, \mathrm{d}\boldsymbol{\omega}_o$$

$$W_r^{\text{indirect}}(\boldsymbol{x}, \boldsymbol{\omega}) = \int_{\Omega_{\boldsymbol{x}}} W_r(\boldsymbol{y}, -\boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle \, \mathrm{d}\boldsymbol{\omega}_o$$





Evaluating Direct Importance

- Sensor sampling

$$W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) = \int_A W_i^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, G(\boldsymbol{x} \leftrightarrow \boldsymbol{y})\, \mathrm{d}\boldsymbol{y}$$

$$\boldsymbol{\omega}_o = \boldsymbol{x} \to \boldsymbol{y}$$

$$\langle W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) \rangle_{\text{sensor}} = \frac{W_i^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, G(\boldsymbol{x} \leftrightarrow \boldsymbol{y})}{p_{\text{sensor}}(\boldsymbol{y})}$$

- p_{sensor} is usually picked as the uniform distribution over the surface A of the sensor. Namely,

$$p_{\text{sensor}}(\boldsymbol{y}) \equiv \frac{1}{|A|} \quad \text{for all } \boldsymbol{y} \in A$$

Evaluating Direct Importance

- BRDF sampling

$$W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) = \int_{\Omega_{\boldsymbol{x}}} W_i^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle \, \mathrm{d}\boldsymbol{\omega}_o$$

$$\Downarrow$$

$$\langle W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) \rangle_{\text{BRDF}} = \frac{W_i^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_o)\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle}{p_{\text{BRDF}}(\boldsymbol{\omega}_o)}$$

  - Desirable to have $p_{\text{BRDF}}(\boldsymbol{\omega}_o) \propto f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_o \rangle$ or $p_{\text{BRDF}}(\boldsymbol{\omega}_o) \propto f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_o)$

Evaluating Direct Importance

- Similar to evaluating direct illumination, the two sampling methods can be combined using multiple importance sampling (MIS)

$$\langle W_r^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}) \rangle_{\text{balance}} = \frac{W_i^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_{o1})\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_{o1}) \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_{o1} \rangle}{p_{\text{sensor}}(\boldsymbol{\omega}_{o1}) + p_{\text{BRDF}}(\boldsymbol{\omega}_{o1})} + \frac{W_i^{\text{direct}}(\boldsymbol{x}, \boldsymbol{\omega}_{o2})\, f_r(\boldsymbol{x}, \boldsymbol{\omega} \to \boldsymbol{\omega}_{o2}) \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_{o2} \rangle}{p_{\text{sensor}}(\boldsymbol{\omega}_{o2}) + p_{\text{BRDF}}(\boldsymbol{\omega}_{o2})}$$

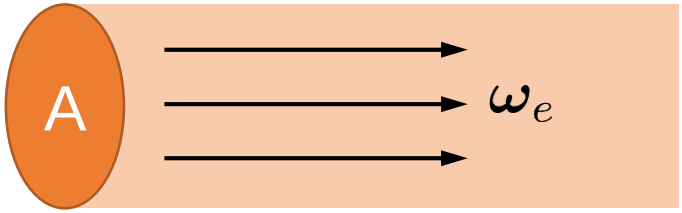
Path vs. Adjoint Particle Tracing

- Since $I = \langle W_e, L_i \rangle = \langle W_i, L_e \rangle$, both algorithms lead to the same answer
- Which one to use depends on the light source and measurement functions (namely, L_e and W_e)

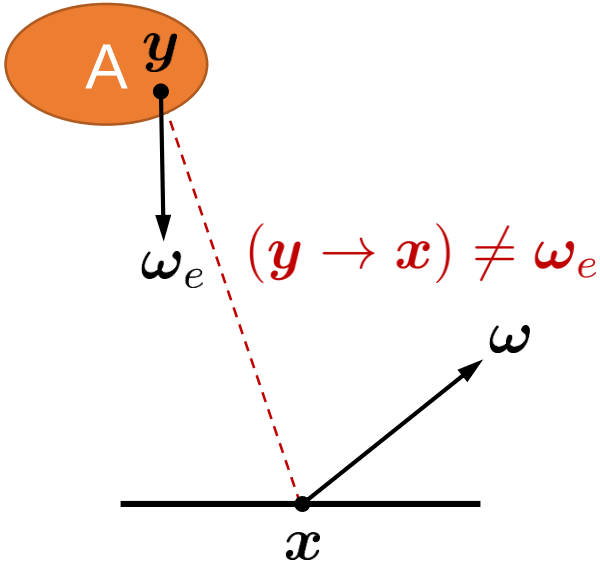
Challenging Light Sources

- Collimated beam

$$L_e(\boldsymbol{x}, \boldsymbol{\omega}) = \mathbb{1}[\boldsymbol{x} \in A]\, \delta_{\boldsymbol{\omega}_e}(\boldsymbol{\omega})$$



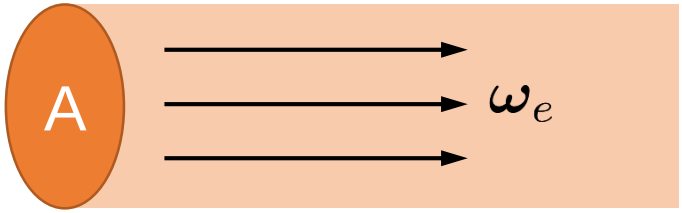
- Problem:
  - Neither light source nor BRDF sampling can give a point ***y*** on *A* with $(\boldsymbol{y} \to \boldsymbol{x})$ precisely aligned with $\boldsymbol{\omega}_e$
  - Can never hit *A* precisely from direction $-\boldsymbol{\omega}_e$



Challenging Light Sources

- Collimated beam

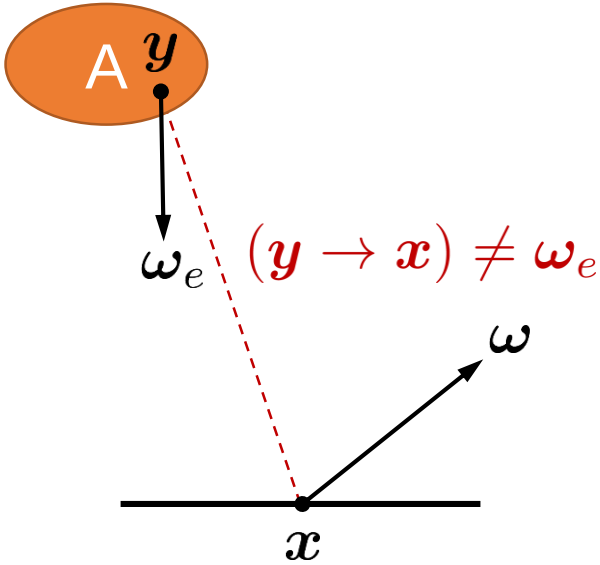
$$L_e(\boldsymbol{x}, \boldsymbol{\omega}) = \mathbb{1}[\boldsymbol{x} \in A]\, \delta_{\boldsymbol{\omega}_e}(\boldsymbol{\omega})$$



- In this case, we should solve the adjoint problem instead

$$p_0^*(\boldsymbol{x}, \boldsymbol{\omega}) = \frac{\mathbb{1}[\boldsymbol{x} \in A]\, \delta_{\boldsymbol{\omega}_e}(\boldsymbol{\omega})}{|A|},$$

$$\langle I \rangle = |A|\, W_i(\boldsymbol{x}, \boldsymbol{\omega}_e)\, \langle \boldsymbol{n}_{\boldsymbol{x}}, \boldsymbol{\omega}_e \rangle$$



Next Lecture

- Path integral formulation