ICS6B Assignment 3

Due: Thursday, 26th April, 2018 by 7am on Gradescope

1 Methods of Proof

1. Direct Proofs Prove the following statements using direct proofs.

   (a) Show that if there exist three prime numbers whose product is 17 times their sum, then the sum of their squares is 654.

   (b) Show that there is a unique value of x such that \( f(x) = 0 \), where \( f: \mathbb{R} \mapsto \mathbb{R}, x \mapsto x^3 - 8 \).

2. Contradiction Prove the following statements using proofs by contradiction.

   (a) You are on an island of mythical creatures consisting of elves (who always speak the truth) and dark wizards (who always lie). Suppose you came across three natives - Wayne, Sri and Pedro and you were reliably informed that one of those three was a potions master. They make the following statements:

      Wayne: Sri is not both a dark wizard and a potions master.

      Sri: Either Wayne is a dark wizard or I am not a potions master.

      Pedro: The potions master is a dark wizard.

      i. Show (by contradiction) that Pedro is not the potions master.

      ii. Show (by contradiction) that Wayne must be an elf.

      iii. Show (by contradiction) that Sri is not the potions master.

      iv. (Bonus) What are the real identities of the natives Pedro, Wayne and Sri? State clearly whether each one is an elf, dark wizard and/or potions master. Show your work.

   (b) Define \( f: \mathbb{R} \mapsto \mathbb{R}, x \mapsto x^2 + ax + b \). Suppose \( a, b \in \mathbb{Z} \), i.e. \( a \) and \( b \) are integers. Suppose also that \( |b| \leq 800 \) and \( f(120) \) is prime. Show that \( f(x) = 0 \) has no integer roots.

3. Counterexamples Determine whether the following statements are true. If you think it is true, state that it is true and provide some intuition on why you think it’s true (you do NOT need to prove it). However, if you think it is false, provide a counter-example to disprove it.

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(a) Consider any five consecutive integers, each of which is greater than 10. Their product is divisible by 70.

(b) For any integer \( n > 0 \), there exists a prime number between \( n \) and \( 2n \) inclusive.

(c) The sum of all interior angles in a polygon (n-gon) is \( 360^\circ (n - 2) \), \( n > 3 \).

(d) Consider a quadratic function \( f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2 + ax + b \) with real roots. The product of roots of the function is \( b \).

4. **Contra-positives** Prove the following statements by first taking their contra-positive form and then use a direct proof.

   (a) For any integer \( n > 0 \) such that \( n = 4k + 2 \) or \( n = 4k + 3 \), \( n \) is not a perfect square.

   (b) Suppose \( x \) is a positive integer. If \( x \) is even, then \( 7x + 9 \) is odd.

5. **Existence and Constructive Proofs** Consider the function \( f : \mathbb{R} \to \mathbb{R}, x \mapsto 2x^3 + 3x^2 - 11x - 6 \), which has at least one root which is an integer. Prove:

   (a) By construction, that \( f \) has a root between 0 and \(-1\).

   (b) Without construction, that \( f \) has a root between 0 and \(-1\).