ICS6B Assignment 5 Solutions

Due: Thursday, 17th May, 2018 by 7am on Gradescope

1 Relations

1. Let $S$ be a set and $R_1$ and $R_2$ are relations on $S$. Prove the following if they are true. If it is false, give a counter-example.

(a) If $R_1$ and $R_2$ are reflexive, is $R_1 \cup R_2$ reflexive?
(b) If $R_1$ is reflexive, is $R_1^{-1}$ reflexive?
(c) If $R_1$ and $R_2$ are symmetric, is $R_1 \circ R_2$ symmetric?
(d) If $R_1$ and $R_2$ are anti-symmetric, is $R_1 \oplus R_2$ anti-symmetric?
(e) If $R_1$ is anti-symmetric, is $R_1^n$ anti-symmetric, $n \in \mathbb{Z}$?

Solution:

(a) Consider any $x \in S$. Since $R_1$ is reflexive, $(x, x) \in R_1$. Similarly, $(x, x) \in R_2$. Hence, $(x, x) \in R_1 \cup R_2$. Thus, $R_1 \cup R_2$ is also reflexive.

(b) Consider any $x \in S$. Since $R_1$ is reflexive, $(x, x) \in R_1$. Reversing the order of elements in $(x, x)$ gives the same tuple. Therefore, $(x, x) \in R_1^{-1}$ and $R_1^{-1}$ is reflexive.

(c) $R_1 \circ R_2$ is not necessarily symmetric. Consider the set $S = \{a, b, c\}$. Let $R_1 = \{(a, b), (b, a)\}$ and $R_2 = \{(b, c), (c, b)\}$. Then, $(a, b) \circ (b, c) = (a, c) \in R_1 \circ R_2$ but $(c, a) \notin R_1 \circ R_2$.

(d) $R_1 \oplus R_2$ is not necessarily anti-symmetric. Consider the set $S = \{a, b, c\}$ with anti-symmetric relations $R_1 = \{(a, b)\}$ and $R_2 = \{(b, a)\}$. $R_1 \oplus R_2 = \{(a, b), (b, a)\}$, which is symmetric but not anti-symmetric.

(e) $R_1^n$ is not necessarily anti-symmetric. Consider the set $S = \{a, b, c, d\}$ with the anti-symmetric relation $R_1 = \{(a, b), (b, c), (c, d), (d, a)\}$. $R_1^2 = \{(a, c), (b, d), (c, a), (d, b)\}$, which is symmetric, but not anti-symmetric.

2. Consider the set of integers $\mathbb{Z}$ with relation $R$ where $xRy$ means $y$ is the remainder when $7$ divides $x^2$.

(a) Write the pairs $(x, y)$ formed by relation $R$ for $x = 1, 2, 3, ..., 14$.

(b) Show that the range of $y$ is $\{0, 1, 2, 3, 4\}$ under this relation.
(c) Find the range of $y$ if $xRy$ is redefined to be the remainder when 5 divides $x^2$.
(d) Is $R$ symmetric?
(e) Is $R$ reflexive?
(f) Is $R$ transitive?

Solution:

(a) $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (7, 0), (8, 1), (9, 2), (10, 3), (11, 4), (12, 5), (13, 6), (14, 0)$.
(b) Any number $x \in \mathbb{Z}$ can be represented as $x = 7n + r$, where $n \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3, 4, 5, 6\}$. Hence, $x^2 = 7q + r^2$, where $q \in \mathbb{Z}$ and $r^2 \in \{0, 1, 4, 9, 16, 25, 36\}$. However, upon dividing the possible values of $r^2$ by 7, we get the remainders $\hat{r} \in \{0, 1, 2, 4\}$.

(c) $y \in \{0, 1, 4\}$.
(d) $R$ is not symmetric
(e) $R$ is not reflexive
(f) If $x \in \mathbb{Z}$, then $(x, r) \in R$, where $x = 7n+r$, as defined in part b. Since $r \in \mathbb{Z}$, then $(r, r) \in R$. Since $(x, r) \in R$, it means $R$ is transitive.

3. Consider a set $A$ with a relation $R$. Show that if $R$ is transitive and symmetric, then $R$ is reflexive. Solution: Suppose $x \in A$. Since $R$ is symmetric, then there exists $y \in A$ such that $(x, y) \in R$ and $(y, x) \in R$. Since $R$ is transitive, it means $(x, x) \in R$. Hence, $R$ is reflexive.

4. Consider the ordered set of elements $S = \{a, b, c, f, d\}$ with the relation $R$ where $xRy$ means if element $x$ occupies position $i$ in $S$, then it is related to the element in position $i + 2$. The position loops around, so from position 5 we go back to 1.

(a) Write the elements of $R$.
(b) Is $R$ symmetric?
(c) Is $R$ anti-symmetric?
(d) Is $R$ reflexive?
(e) Is $R$ transitive?
(f) Redo the above questions for the case where the position of elements do not loop.

Solution:

(a) $R = \{(a, c), (b, f), (c, d), (f, a), (d, b)\}$
(b) $R$ is not symmetric.
(c) $R$ is anti-symmetric.
(d) $R$ is not reflexive.
(e) $R$ is not transitive.
(f) $R = \{(a, c), (b, f), (c, d)\}$. $R$ is anti-symmetric, but not symmetric, reflexive or transitive.