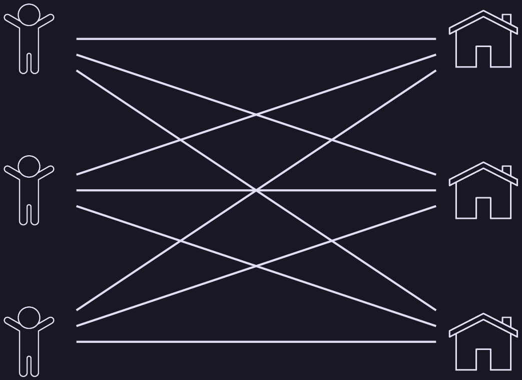
One-Sided Matching Markets with Endowments: Equilibria and Algorithms

Jugal Garg, Thorben Tröbst, Vijay V. Vazirani

AAMAS 2022

One-Sided Matching Markets



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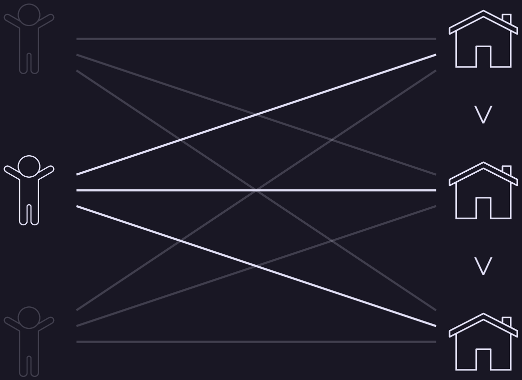

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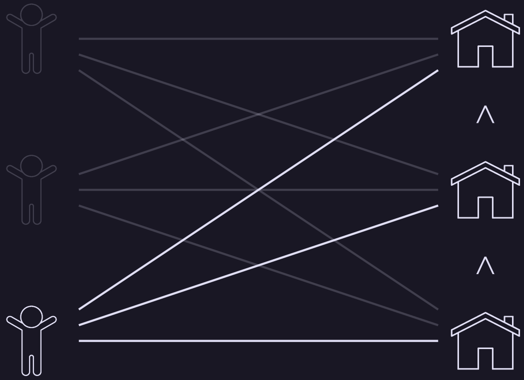

A Classification of One-Sided Matching Markets

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Definition

A linear ADHZ market consists a set A of agents and a set G of goods with $|A| = |G| = n$. Each agent i comes to the market with an endowment $e_{ij} \geq 0$ of each good j and utilities $u_{ij} \geq 0$. The endowment vector e is a fractional (perfect) matching.

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The goal is to find a fractional (perfect) matching x or allocation with desirable fairness properties.

Hylland-Zeckhauser

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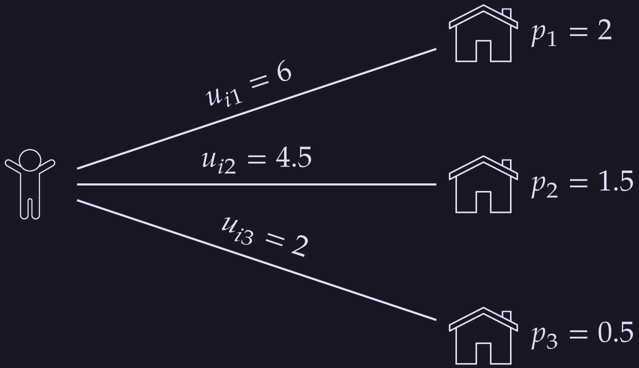
Definition

An HZ equilibrium consists of prices $p_j \geq 0$ for every good and an allocation x, such that every agent gets a cheapest optimal bundle under a budget of 1.

Moreover, if $p_j > 0$, then good j must be fully allocated.

Cheapest Optimal Bundles

In HZ, agents get optimal bundles of goods at given prices. If there are multiple optimal bundles, pick a cheapest one.



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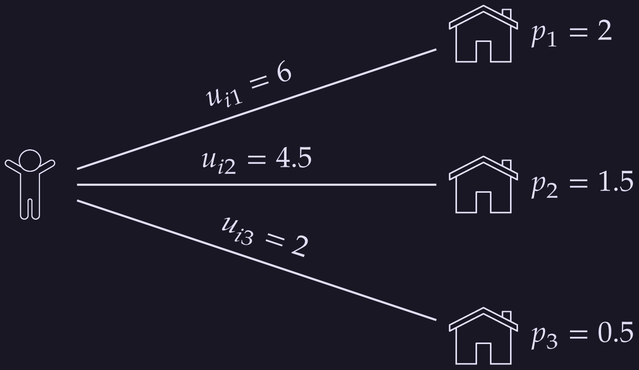

$$x_{i1} = 0.5, x_{i2} = 0, x_{i3} = 0 \Rightarrow \mathbf{u_i = 3}$$

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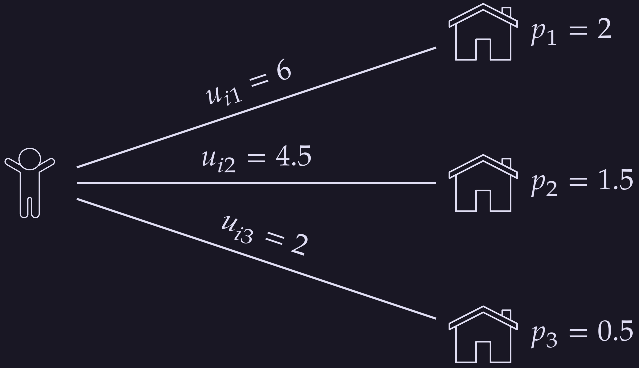

$$x_{i1} = 0, x_{i2} = 0.\overline{6}, x_{i3} = 0 \Rightarrow \mathbf{u_i} = \mathbf{3}$$

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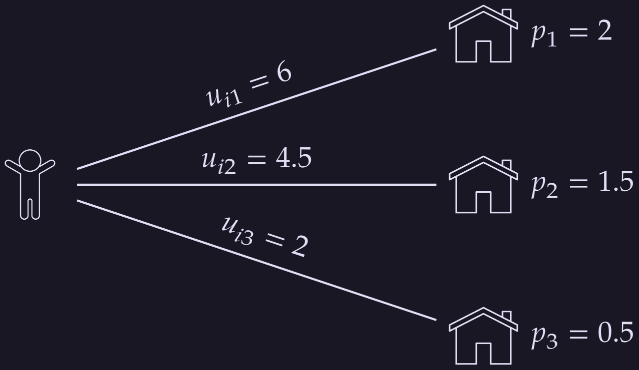

$$x_{i1} = 0, x_{i2} = 0, x_{i3} = 1 \Rightarrow \mathbf{u_i} = \mathbf{2}$$

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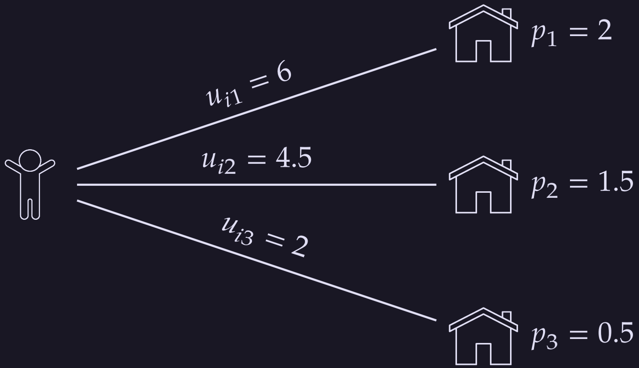

$$x_{i1} = 0, x_{i2} = 0.5, x_{i3} = 0.5 \Rightarrow \mathbf{u_i} = \mathbf{3.25}$$

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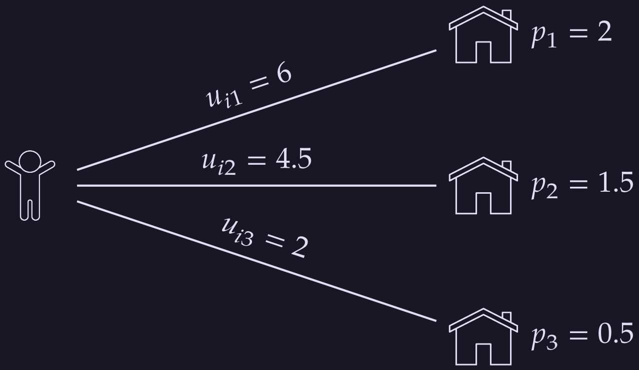

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- constant number of goods / agents (Devanur and Kannan 2008, Alaei et al. 2017) and
- $\{0, 1\}$-utilities (Vazirani and Yannakakis 2021).

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- (weak) core stability.

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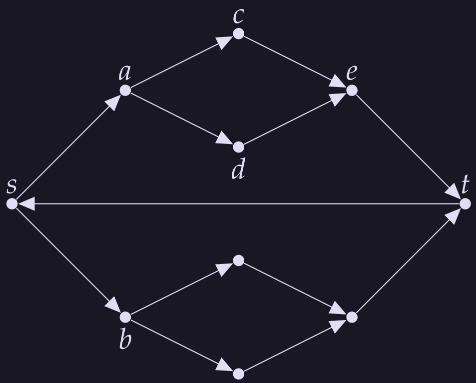

Non-Existence of ADHZ Equilibria

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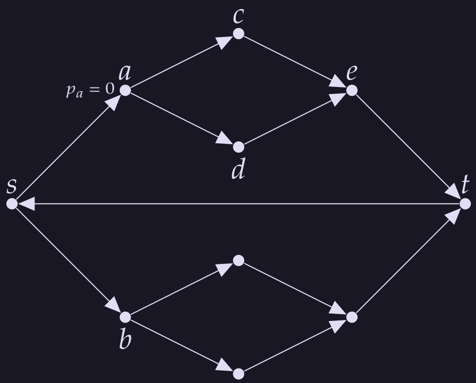

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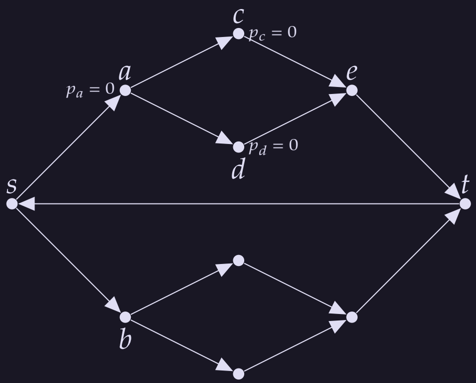

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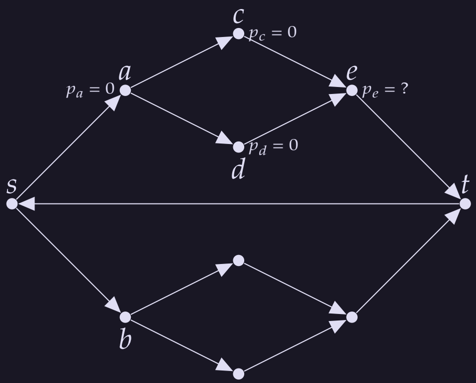

ϵ-Approximate ADHZ Equilibria

We define a weaker notion of ϵ-approximate ADHZ, where

$$b_i \in \left[(1-\epsilon)\sum_{j\in G} p_j e_{ij}, \epsilon + \sum_{j\in G} p_j e_{ij}\right].$$

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Our algorithm works similar to the one by Vazirani and Yannakakis for the uniform budget case (and DPSV):

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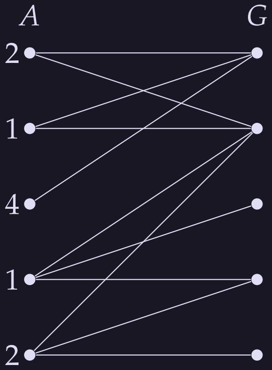

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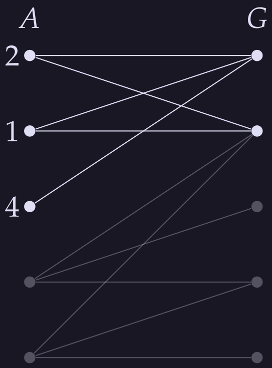

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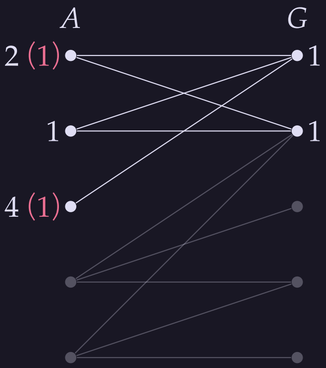

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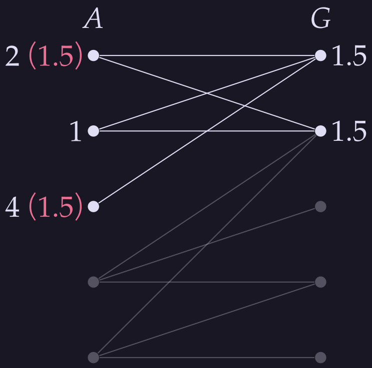

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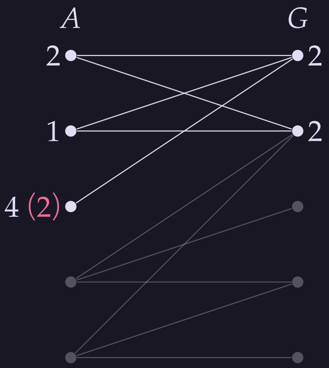

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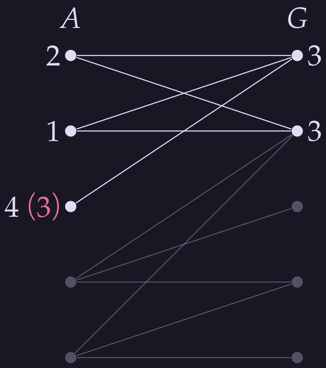

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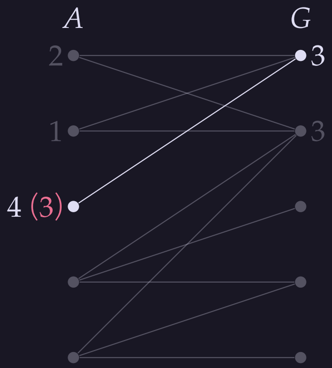

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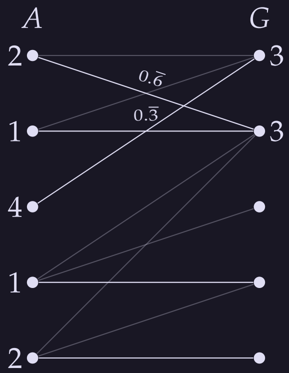

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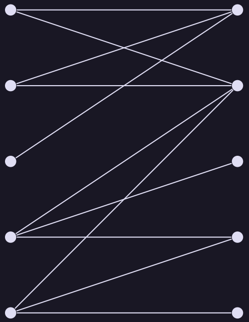

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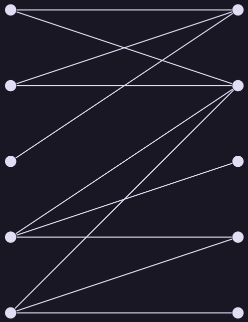

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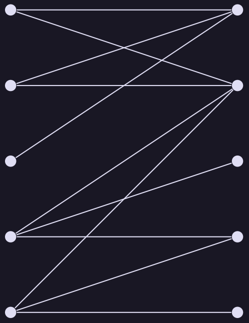

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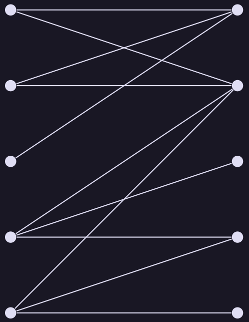

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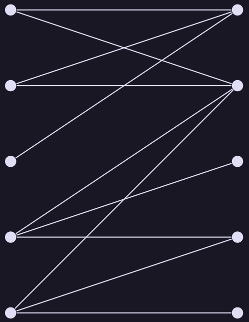

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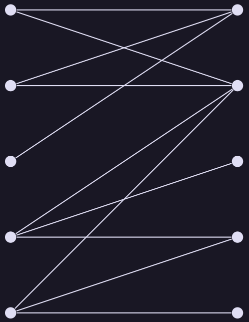

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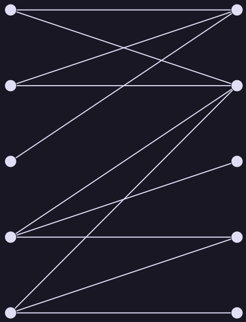

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$\Rightarrow$ If one uses $\alpha := \frac{\epsilon}{2}$, then one gets an ϵ-approximate ADHZ equilbrium in $O(\frac{n}{\epsilon}\log(\frac{n}{\epsilon}))$ phases. $\square$

Thank your for listening!