One-Sided Matching Markets with Endowments: Equilibria and Algorithms

Thorben Tröbst (joint work with Jugal Garg and Vijay Vazirani)
April 23, 2021

CS Theory Seminar, UC Irvine
Introduction
One-Sided Matching Markets
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One-sided matching markets can be classified on two criteria:

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A Classification of One-Sided Matching Markets

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Cardinal Pref. | Hylland-Zeckhauser | ??? |
Formally, we study the following kind of market:

Definition

A linear ADHZ market consists a set $A$ of agents and a set $G$ of goods with $|A| = |G| = n$. Each agent $i$ comes to the market with an endowment $e_{ij} \geq 0$ of each good $j$ and utilities $u_{ij} \geq 0$.

The endowment vector $e$ is a fractional (perfect) matching. The goal is to find a fractional (perfect) matching or allocation with desirable fairness properties.
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A Note on Integrality

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⇒ We will focus on fractional allocations.
The Hylland-Zeckhauser mechanism works by computing a competitive equilibrium from equal incomes.
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**Definition**

An HZ equilibrium consists of prices $p_j \geq 0$ for every good and an allocation $x$, such that every agent gets a cheapest optimal bundle under a budget of 1.

Moreover, if $p_j > 0$, then good $j$ must be fully allocated.
In HZ, agents get **optimal bundles** of goods at given prices. If there are multiple optimal bundles, pick a **cheapest** one.

\[
u_{i1} = 6 \quad p_1 = 2
\]
\[
u_{i2} = 4.5 \quad p_2 = 1.5
\]
\[
u_{i3} = 2 \quad p_3 = 0.5
\]
Cheapest Optimal Bundles

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\[
\begin{align*}
\mathbf{u}_1 &= 6 \\
\mathbf{u}_2 &= 4.5 \\
\mathbf{u}_3 &= 2
\end{align*}
\]

\[
x_{i1} = 0.5, x_{i2} = 0, x_{i3} = 0 \Rightarrow \mathbf{u}_i = 3
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    p_3 &= 0.5
\end{align*}
\]

\[
x_{i1} = 0, x_{i2} = 0.5, x_{i3} = 0.5 \Rightarrow u_i = 3.25
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\[
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\]

\[
\begin{align*}
    p_1 &= 2, \\
    p_2 &= 1.5, \\
    p_3 &= 0.5
\end{align*}
\]

\[
x_{i1} = 0.3, x_{i2} = 0, x_{i3} = 0.6 \Rightarrow u_i = 3.3
\]
HZ equilibria enjoy many nice properties such as:

- Pareto efficiency,
- envy-freeness, and
- incentive-compatibility in the large.

Sadly, they can be computed only in a few special cases:

- constant number of goods / agents (Devanur and Kannan 2008, Alaei et al. 2017)
- \{0, 1\} utilities (Vazirani and Yannakakis 2021).
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Results on ADHZ

Our results are:

• ADHZ equilibria may not exist (already known to Hylland and Zeckhauser). This holds even with \{0, 1\}-utilities and strong conditions.

• There is a notion of $\epsilon$-approximate ADHZ equilibria which always exist and satisfy approximate individual rationality and core stability.

• We give a combinatorial FPTAS to compute $\epsilon$-approximate ADHZ equilibria for \{0, 1\}-utilities.
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### Definition

Given a convex set $C$ of potential allocations (matchings), the **Nash bargaining point** or **proportionally fair allocation** is

$$\arg \max_{x \in C} \prod_{i \in A} u_i(x).$$
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For $\{0, 1\}$-utilities, these notions coincide!
There is also a natural extension of Nash bargaining to endowments: simply optimize

$$\arg\max_{x \in C} \prod_{i \in A} (u_i(x) - u_i(e)) = \arg\max_{x \in C} \sum_{i \in A} \log(u_i(x) - u_i(e)).$$

This does not coincide with ADHZ even for \(\{0, 1\}\)-utilities. For these utilities, we show:
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- Solutions are always rational, i.e. the above is a rational convex program.
- There is a combinatorial, strongly polynomial time algorithm to compute $x$. 

Hylland-Zeckhauser with Endowments (ADHZ)
Non-Existence of ADHZ Equilibria

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\[
\begin{align*}
 p_a &= 0 \\
p_c &= 0 \\
p_d &= 0
\end{align*}
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Unfortunately, even for $\{0, 1\}$-utilities and strong connectivity assumptions, ADHZ equilibria may not exist:
The problem can be resolved by infusing agents with a small amount of \textit{external budget}. 
Existence of Approximate ADHZ Equilibria

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Echenique et al. (2019) showed that there always exist $\alpha$-slack ADHZ equilibria where

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Existence follows via non-trivial Kakutani fixed point argument.
We define a weaker notion of $\epsilon$-approximate ADHZ, where

$$b_i \in \left[ (1 - \epsilon) \sum_{j \in G} p_j e_{ij}, \epsilon + \sum_{j \in G} p_j e_{ij} \right].$$

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Computing $\epsilon$-Approximate ADHZ Equilibria

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Our algorithm works similar to the one by Vazirani and Yannakakis for the uniform budget case (and DPSV):
Algorithm for HZ with Non-Uniform Budgets

1. Compute a minimum vertex cover $A' \cup G'$ in the bipartite graph of utility 1 edges.
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1. Compute a minimum vertex cover $A' \cup G'$ in the bipartite graph of utility 1 edges.
2. Set $p_j := \min_{i \in A \setminus A'} b_i$ for all $j \in G'$.
3. Raise all prices in $G'$ uniformly until some $S \subseteq G'$ goes **tight**, i.e.

\[
\sum_{i \in \Gamma(S)} \min\{b_i, p^*_i\} = \sum_{j \in S} p_j
\]

where $p^*_i := \min\{p_j \mid u_{ij} = 1\}$ and $\Gamma(S)$ consists of all $i \in A \setminus A'$ s.t. there is some $j \in S$ with $p_j = p^*_i$ and $u_{ij} = 1$.

4. Set $G' := G' \setminus S$ and $A' := A' \cup \Gamma(S)$ and go back to 3 if $G' \neq \emptyset$.

5. Finally, use a max flow to find an equilibrium allocation (as in DPSV).
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Example for HZ with Non-Uniform Budgets
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![Diagram](image-url)
Example for HZ with Non-Uniform Budgets
Example for HZ with Non-Uniform Budgets
Example for HZ with Non-Uniform Budgets
Example for HZ with Non-Uniform Budgets

A

2

G

3

1

3

4 (3)


Example for HZ with Non-Uniform Budgets
Let $\alpha \in (0, 1)$ and consider the following iteration:
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\[ b^{(2)} := \alpha + (1 - \alpha) e \cdot p^{(1)} \]

\[ b^{(1)} := \alpha \]
Let $\alpha \in (0, 1)$ and consider the following iteration:

\[
b^{(1)} := \alpha
\]

\[
p^{(1)} := \text{HZ prices for } b^{(1)}
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Let $\alpha \in (0, 1)$ and consider the following iteration:

\[
b^{(3)} := \alpha + (1 - \alpha)e \cdot p^{(2)}
\]

\[
p^{(3)} := \text{HZ prices for } b^{(3)}
\]
One can show:

- Total prices and budgets are bounded by $n$ at all times.
- Our algorithm for HZ with non-uniform budgets behaves monotonically, i.e. prices and budgets are non-decreasing during iteration.

$\Rightarrow b(k)$ and $p(k)$ both converge, in the limit we get an $\alpha$-slack equilibrium.

$\Rightarrow$ If one uses $\alpha := \epsilon^2$, then one gets an $\epsilon$-approximate ADHZ equilibrium in $O(n \epsilon \log(n \epsilon))$ phases.
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Iteration Yields \(\epsilon\)-Approximate ADHZ Equilibrium II

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⇒ $b^{(k)}$ and $p^{(k)}$ both converge, in the limit we get an $\alpha$-slack equilibrium.

⇒ If one uses $\alpha := \frac{\epsilon}{2}$, then one gets an $\epsilon$-approximate ADHZ equilibrium in $O\left(\frac{n}{\epsilon} \log\left(\frac{n}{\epsilon}\right)\right)$ phases. □
Nash-Bargaining with Endowments
Recall that the **Nash bargaining point** is the solution to

\[
\begin{align*}
\max_{X} & \quad \sum_{i \in A} \log(u_i(x) - c_i) \\
\text{s.t.} & \quad \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\
& \quad \sum_{j \in A} x_{ij} \leq 1 \quad \forall i \in A, \\
& \quad x \geq 0.
\end{align*}
\]

where \( c_i := u_i(e) \).
Recall that the **Nash bargaining point** is the solution to

\[
\max_X \sum_{i \in A} \log(u_i(x) - c_i)
\]

subject to

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\]

\[
x \geq 0.
\]

where \(c_i := u_i(e)\).

It turns out that \(x\) can also be seen as a kind of market equilibrium!
For \( \{0, 1\} \)-utilities, \( x \) may be characterized as an equilibrium where \( b_i = 1 + c_i p_i^* \) where \( p_i^* := \min\{p_j \mid u_{ij} = 1\} \).
Nash Bargaining Algorithm

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3. Raise all prices in \( G' \) uniformly until some \( S \subseteq G' \) goes tight, i.e.

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\sum_{i \in \Gamma(S)} (1 + c_i p_i^*) = \sum_{j \in S} p_j. 
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4. Set \( G' := G' \setminus S \) and \( A' := A \cup \Gamma(S) \) and go back to 3 if \( G' \neq \emptyset \).

Finally, use a max flow to find an equilibrium allocation (as in DPSV).
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Rationality of general utilities is unknown but likely irrational.
Thank You!