One-Sided Matching Markets with Endowments: Equilibria and Algorithms

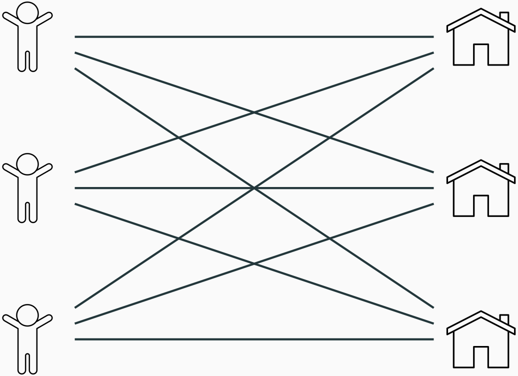
Thorben Tröbst (joint work with Jugal Garg and Vijay Vazirani)

April 23, 2021

CS Theory Seminar, UC Irvine

Introduction

One-Sided Matching Markets



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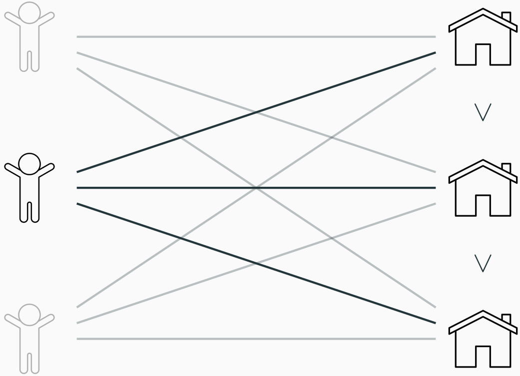

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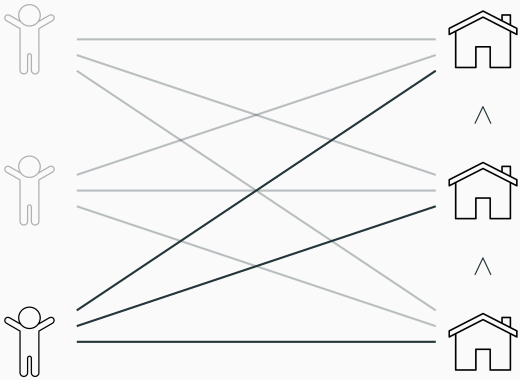

A Classification of One-Sided Matching Markets

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Formal Setup

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Definition

A linear **ADHZ market** consists a set A of **agents** and a set G of **goods** with $|A| = |G| = n$. Each agent i comes to the market with an **endowment** $e_{ij} \geq 0$ of each good j and **utilities** $u_{ij} \geq 0$. The endowment vector e is a fractional (perfect) matching.

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The goal is to find a fractional (perfect) matching x or **allocation** with desirable fairness properties.

A Note on Integrality

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$\Rightarrow$ We will focus on fractional allocations.

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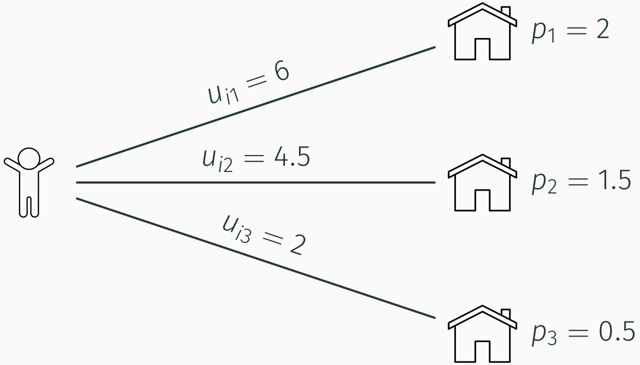
An HZ equilibrium consists of **prices** $p_j \geq 0$ for every good and an allocation x, such that every agent gets a **cheapest optimal bundle** under a **budget** of 1.

Moreover, if $p_j > 0$, then good j must be **fully allocated**.

Cheapest Optimal Bundles

In HZ, agents get **optimal bundles** of goods at given prices. If there are multiple optimal bundles, pick a **cheapest** one.



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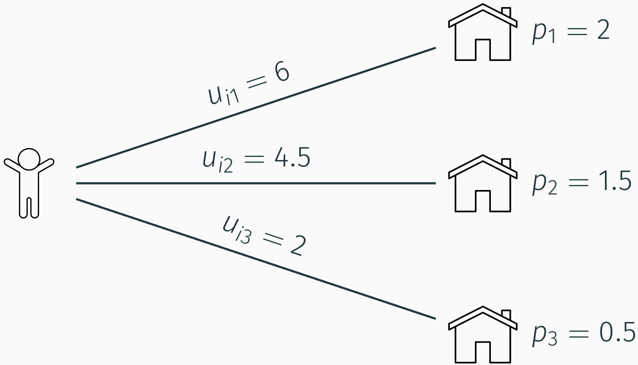

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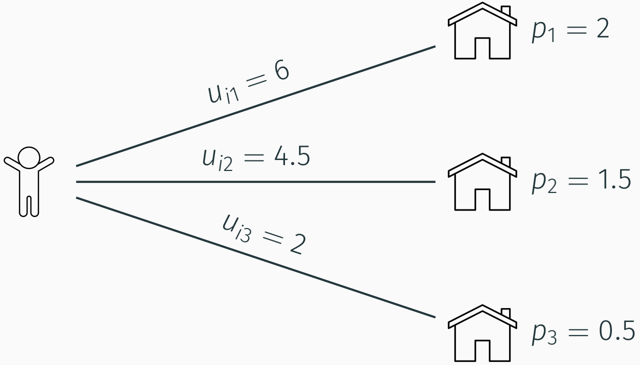

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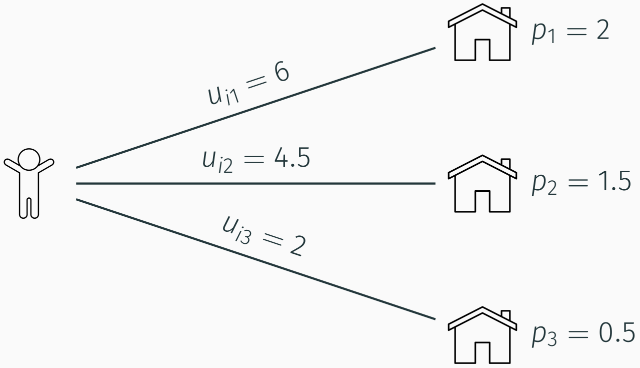

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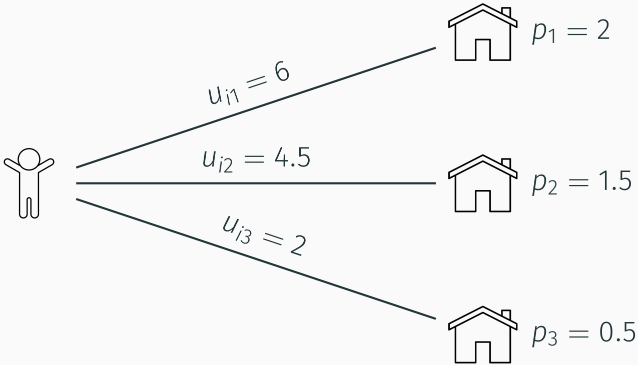

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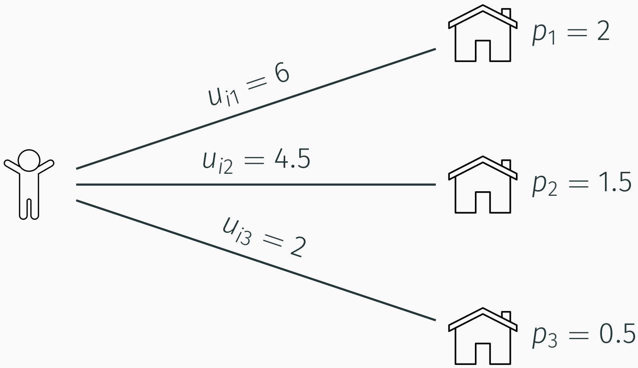

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- $\{0, 1\}$-utilities (Vazirani and Yannakakis 2021).

HZ with Endowments

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- ADHZ equilibria may not exist (already known to Hylland and Zeckhauser). This holds even with $\{0, 1\}$-utilities and strong conditions.
- There is a notion of ϵ-approximate ADHZ equilibria which always exist and satisfy approximate individual rationality and core stability.
- We give a combinatorial FPTAS to compute ϵ-approximate ADHZ equilibria for $\{0, 1\}$-utilities.

Alternative Solution Concept: Nash Bargaining

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For $\{0, 1\}$-utilities, these notions coincide!

Nash Bargaining with Endowments

There is also a natural extension of Nash bargaining to endowments: simply optimize

$$\arg\max_{x \in C} \prod_{i \in A} (u_i(x) - u_i(e)) = \arg\max_{x \in C} \sum_{i \in A} \log(u_i(x) - u_i(e)).$$

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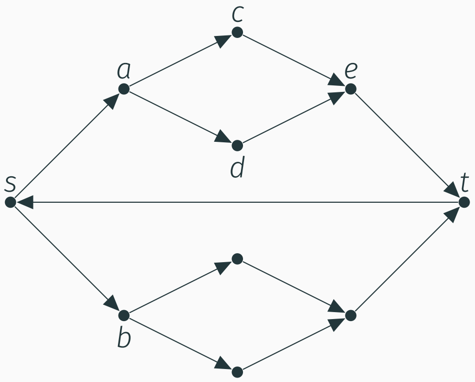
- Solutions are always rational, i.e. the above is a **rational convex program**.
- There is a combinatorial, strongly polynomial time algorithm to compute x.

Hylland-Zeckhauser with Endowments (ADHZ)

Non-Existence of ADHZ Equilibria

Unfortunately, even for $\{0, 1\}$-utilities and strong connectivity assumptions, ADHZ equilibria may not exist:



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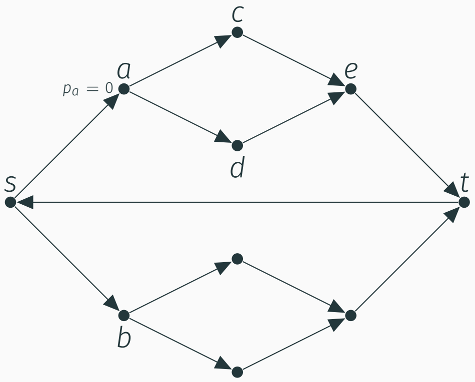

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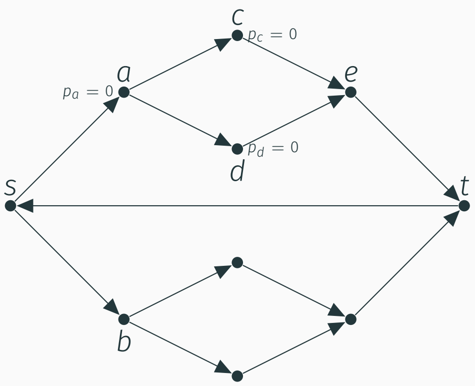

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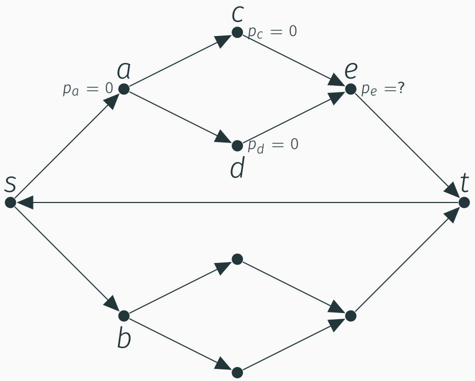

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Existence follows via non-trivial Kakutani fixed point argument.

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- $(1+\epsilon)$-approximately core stable.

Computing ϵ-Approximate ADHZ Equilibria

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- We show that **iterating this algorithm** converges to an ϵ-approximate ADHZ equilibrium in $O(\frac{n}{\epsilon} \log(\frac{n}{\epsilon}))$ iterations.

Our algorithm works similar to the one by Vazirani and Yannakakis for the uniform budget case (and DPSV):

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3. Raise all prices in G' uniformly until some $S \subseteq G'$ goes **tight**, i.e.

$$\sum_{i \in \Gamma(S)} \min\{b_i, p_i^*\} = \sum_{j \in S} p_j$$

where $p_i^* := \min\{p_j \mid u_{ij} = 1\}$ and $\Gamma(S)$ consists of all $i \in A \setminus A'$ s.t. there is some $j \in S$ with $p_j = p_i^*$ and $u_{ij} = 1$.

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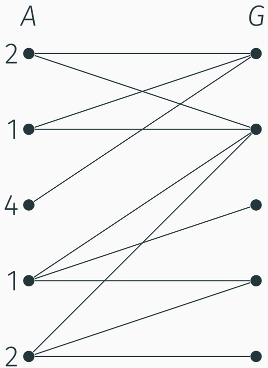
4. Set $G' := G' \setminus S$ and $A' := A \cup \Gamma(S)$ and go back to 3 if $G' \neq \emptyset$.
5. Finally, use a max flow to find an equilibrium allocation (as in DPSV).

Example for HZ with Non-Uniform Budgets



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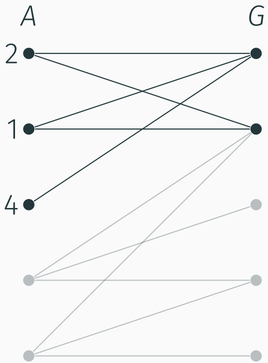

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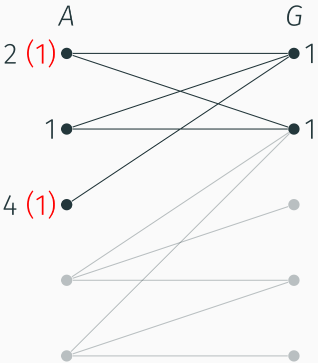

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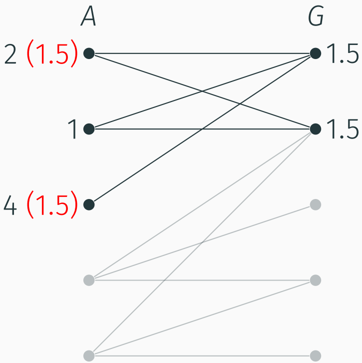

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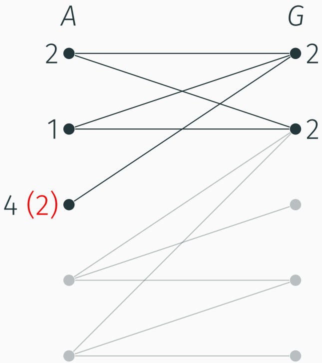

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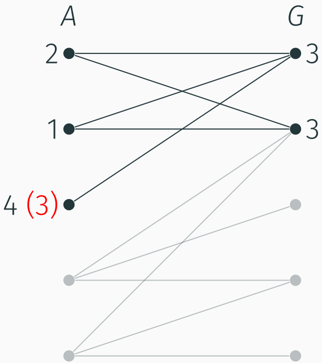

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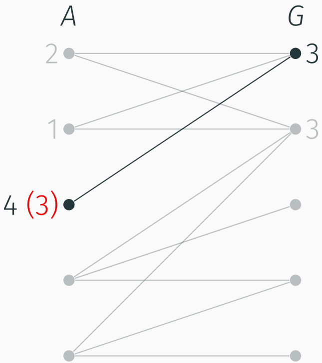

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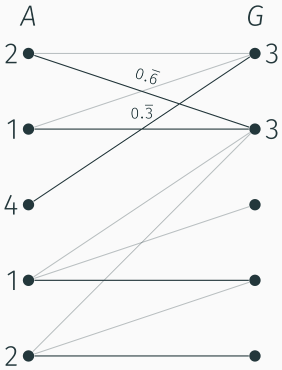

Iteration Yields ϵ-Approximate ADHZ Equilibrium

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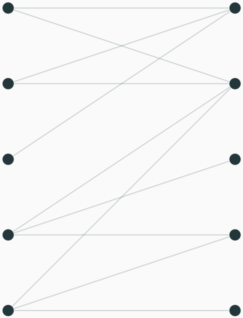

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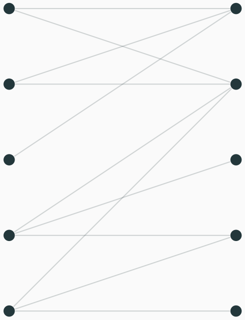

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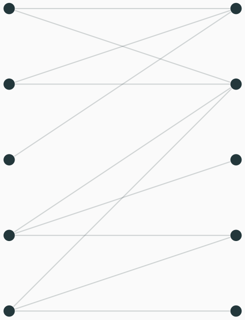

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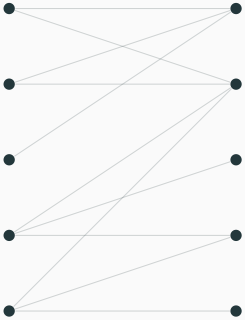

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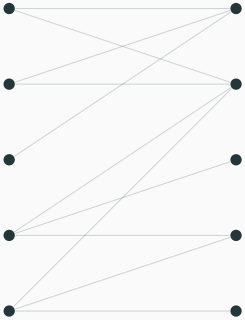

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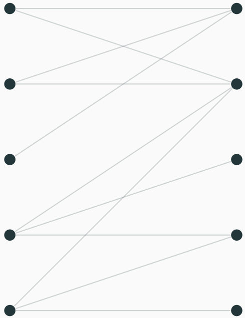

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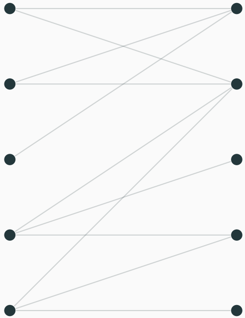

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$\Rightarrow$ $b^{(k)}$ and $p^{(k)}$ both converge, in the limit we get an α-slack equilibrium.

$\Rightarrow$ If one uses $\alpha := \frac{\epsilon}{2}$, then one gets an ϵ-approximate ADHZ equilbrium in $O(\frac{n}{\epsilon}\log(\frac{n}{\epsilon}))$ phases. $\square$

Nash-Bargaining with Endowments

Nash Bargaining as a Market

Recall that the **Nash bargaining point** is the solution to

$$\begin{aligned} \max_{X} \quad & \sum_{i \in A} \log(u_i(x) - c_i) \\ \text{s.t.} \quad & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\ & \sum_{j \in A} x_{ij} \leq 1 \quad \forall i \in A, \\ & x \geq 0. \end{aligned}$$

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It turns out that x can also be seen as a kind of market equilibrium!

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For $\{0,1\}$-utilities, x may be characterized as an equilibrium where $b_i = 1 + c_i p_i^*$ where $p_i^* := \min\{p_j \mid u_{ij} = 1\}$.

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5. Finally, use a max flow to find an equilibrium allocation (as in DPSV).

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Rationality of general utilities is unknown but likely irrational.

Thank You!