Approximation Algorithms for Unsplittable Capacitated Vehicle Routing[1]

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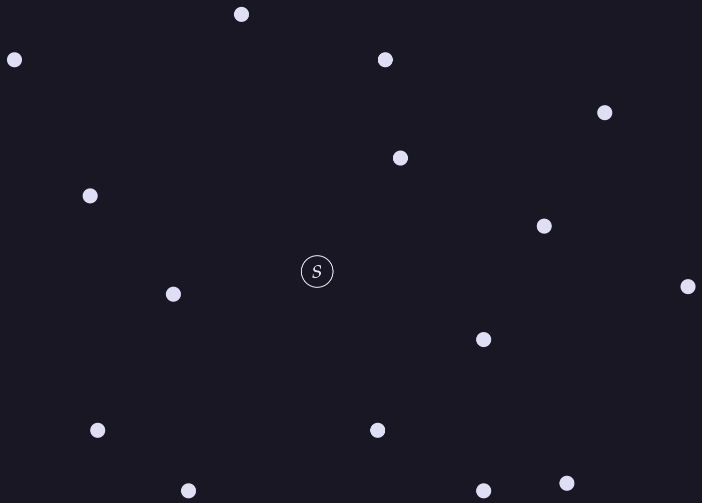
Theory Seminar, April 21, 2023

Department of Computer Science, University of California, Irvine

[1] based on work by Friggstad et al. IPCO 2022

Capacitated Vehicle Routing Problem (CVRP)



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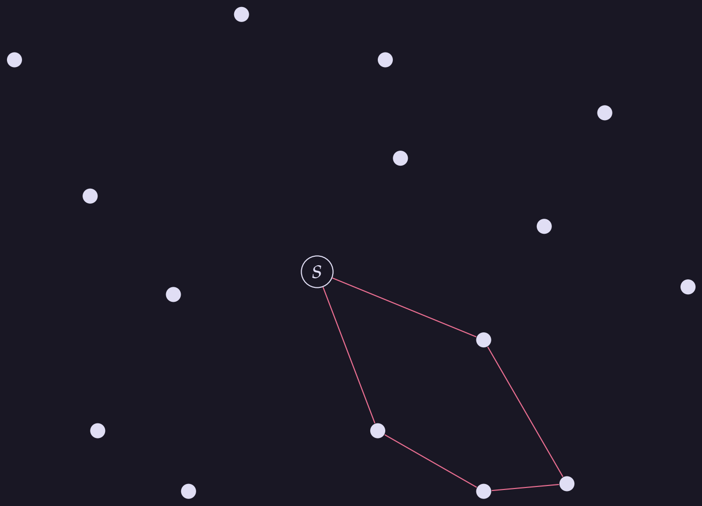

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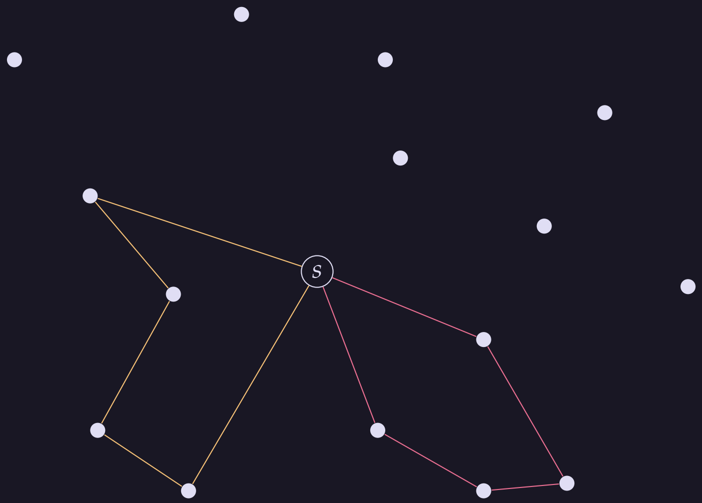

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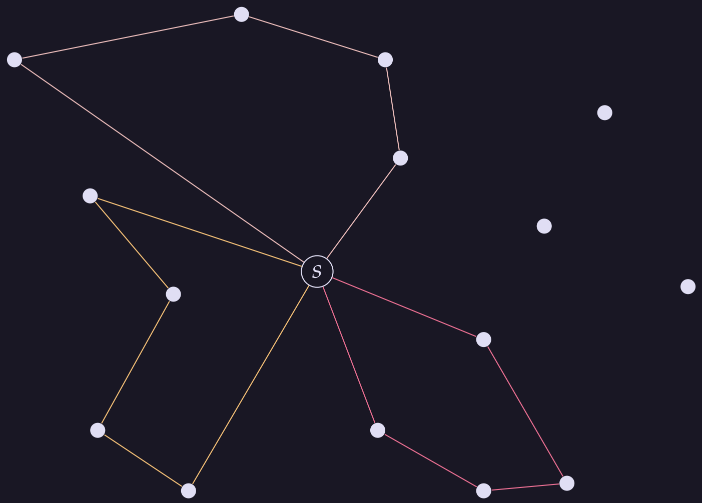

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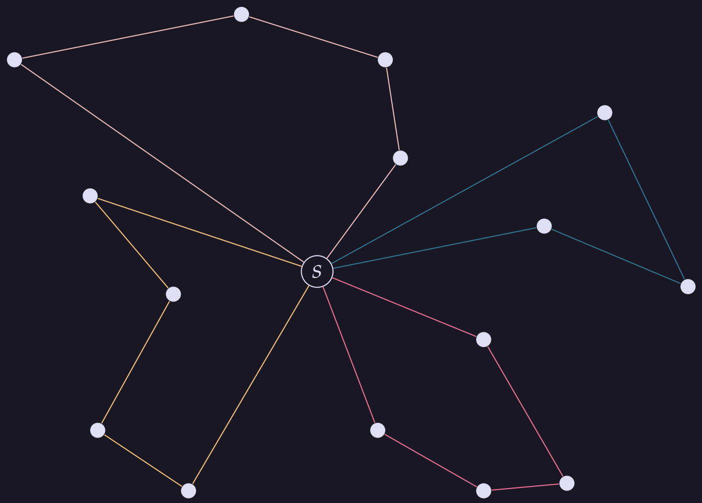

Capacitated Vehicle Routing Problem (CVRP) II

Problem (Capacitated Vehicle Routing)

Input: *a complete graph* $G = (V, E)$ *with* $V = P \cup \{s\}$*, metric edge lengths* $d : E \to \mathbb{R}_{\geq 0}$*, and vertex demands* $b : P \to [0, 1]$.

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- $\sum_{i=1}^{k} d(C_i)$ *is minimum.*

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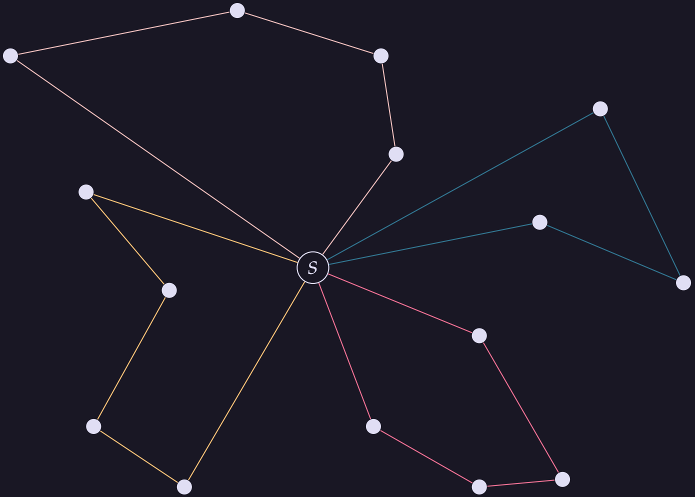
- General case improved by ϵ (Blauth et al. 2021)
- Current best: ≈ 3.2 (Friggstad et al. 2022)

Classic Lower Bounds

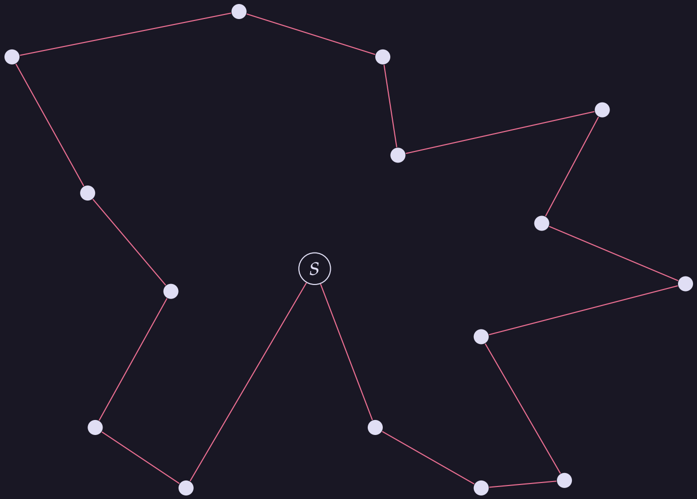
TSP

Clearly, a lower bound for the CVRP is a TSP solution:



TSP

Clearly, a lower bound for the CVRP is a TSP solution:



Radial Weighted Distance

A less obvious lower bound is:

Theorem

The optimal solution of the CVRP is lower bounded by:

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Consider some C_i in OPT. For each $p \in P(C_i)$, clearly $d(C_i) \geq 2d(s,p)$ by the triangle inequality. So $d(C_i) \geq \sum_{p \in P(C_i)} 2b(p)d(s,p)$ since $b(C_i) \leq 1$. □

Tour Partitioning

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Theorem

There is a polynomial time $(\alpha+2)$-approximation algorithm for the CVRP where α is the best approximation ratio for the TSP.

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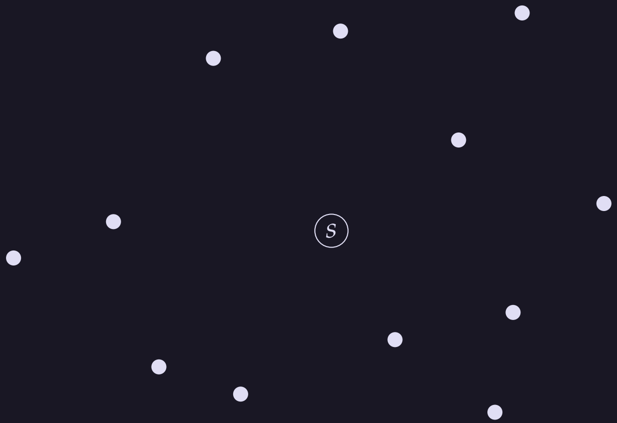
Tour Partitioning Example



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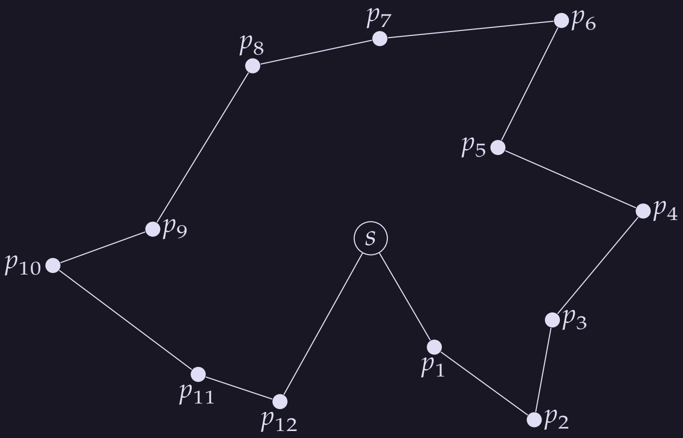

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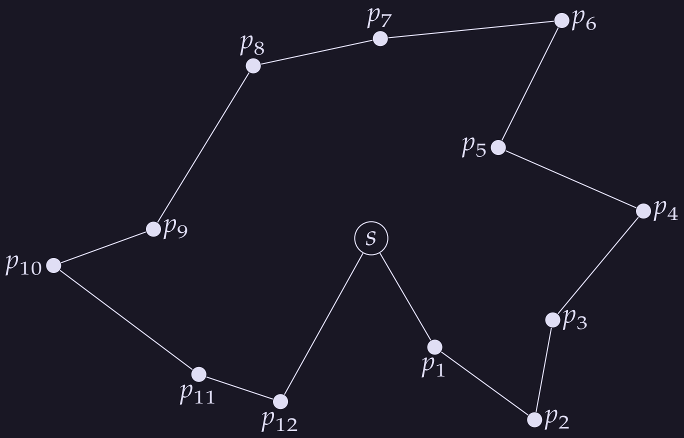

$b(p_1)$	$b(p_2)$	$b(p_3)$	$b(p_4)$	$b(p_5)$	$b(p_6)$	$b(p_7)$	$b(p_8)$	$b(p_9)$	$b(p_{10})$	$b(p_{11})$	$b(p_{12})$

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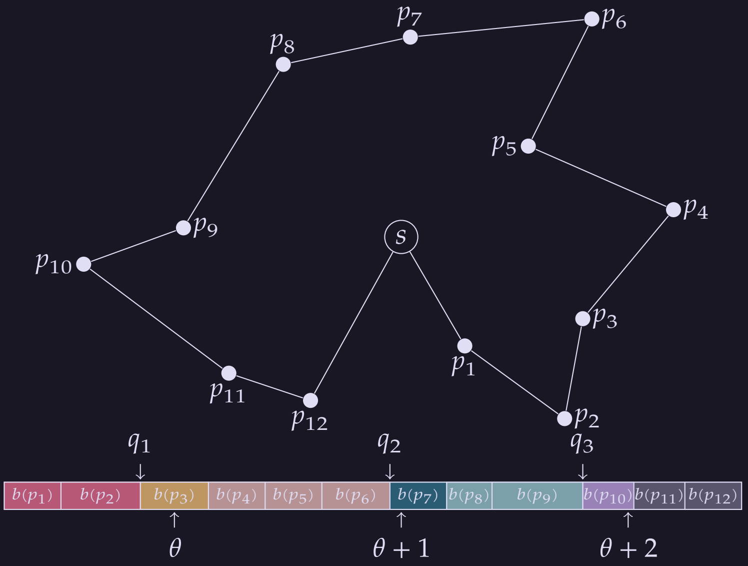

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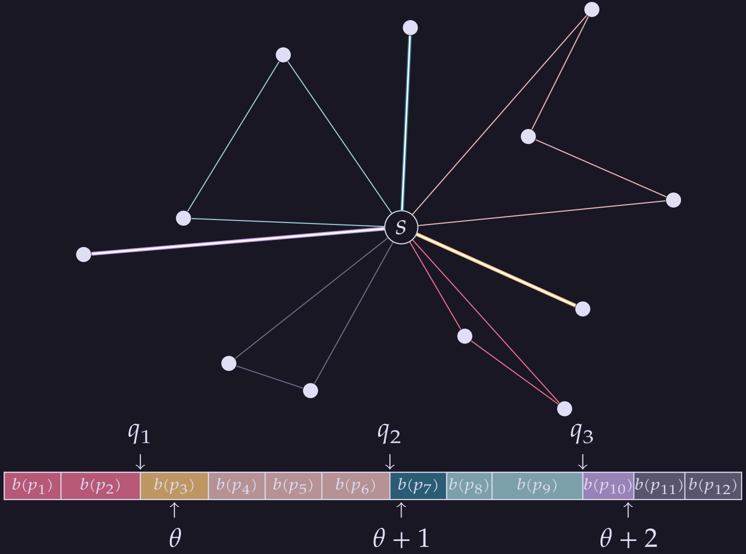

Tour Partitioning Proof

Proof. If C is the TSP solution, and $C_1, \ldots, C_k$ the result of tour partitioning, then:

$$\sum_{i=1}^{k} d(C_i) \leq d(C) + \sum_{i=1}^{k-1} 4d(s, q_i).$$

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$$\mathbb{E}\left[\sum_{i=1}^{k} d(C_i)\right] \leq \alpha \mathrm{TSP} + \sum_{p \in P} 4b(p)d(s, p).$$

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For each p, $\mathbb{P}[p = q_i \text{ for some } i] = b(p)$. So

$$\mathbb{E}\left[\sum_{i=1}^{k} d(C_i)\right] \leq \alpha \mathrm{TSP} + \sum_{p \in P} 4b(p)d(s,p).$$

So there is a partition with distance $\leq (\alpha + 2)\mathrm{OPT}$. □

A 3.25-Approximation

Main Theorem

Our goal is to show:

Theorem (Friggstad et. al 2022)

There is a polynomial time $(\alpha + 1.75)$-approximation algorithm for the CVRP where α is the best approximation ratio for the TSP.

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There is a polynomial time $(\alpha + 1.75)$*-approximation algorithm for the CVRP where* α *is the best approximation ratio for the TSP.*

We will need a more fine-grained tour partitioning result!

The δ-Tank Lemma

Lemma

Let C be a cycle on V and $\delta \in [0,1)$, then C can be partitioned into $C_1, \dots, C_k$ with

$$\sum_{i=1}^{k} d(C_i) \leq d(C) + \frac{1}{1-\delta} D_{\leq\delta} + \frac{2}{1-\delta} D_{>\delta} - \frac{\delta}{1-\delta} D'_{>\delta}.$$

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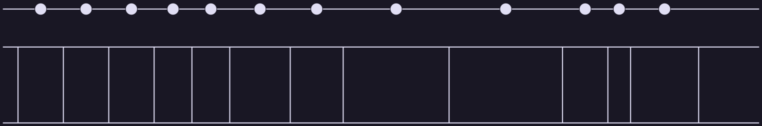
$$D_{\leq \delta} := \sum_{p \in P, b(p) \leq \delta} 2b(p)d(s,p), \quad D_{>\delta} := \sum_{p \in P, b(p) > \delta} 2b(p)d(s,p).$$

$$D'_{>\delta} := \sum_{p \in P, b(p) > \delta} 2d(s,p)$$

Proof of δ-Tank Lemma



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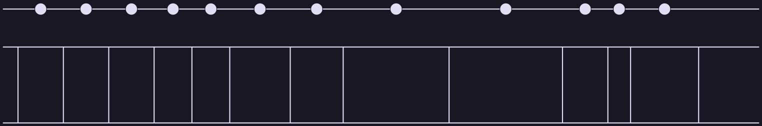

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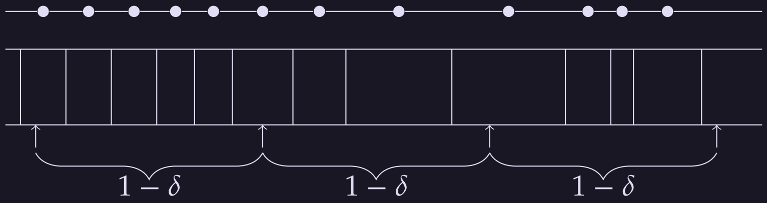

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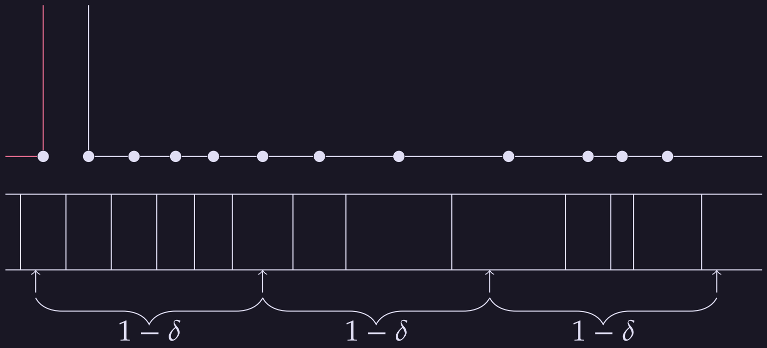

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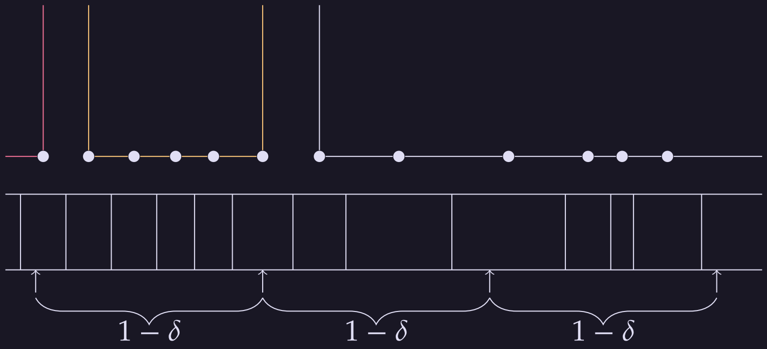

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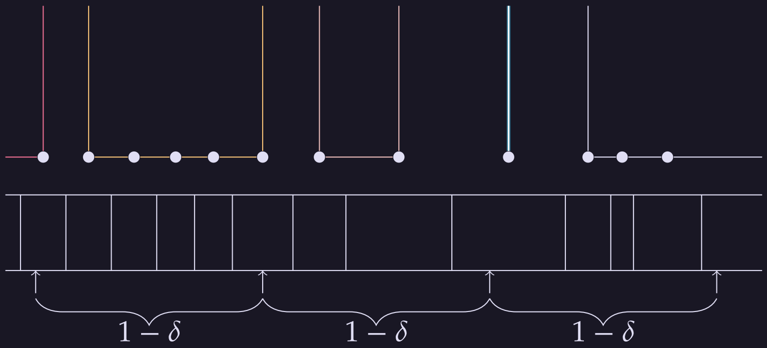

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   - Cost: $4d(s,p)$
   - Probability: $\frac{b(p)-\delta}{1-\delta}$

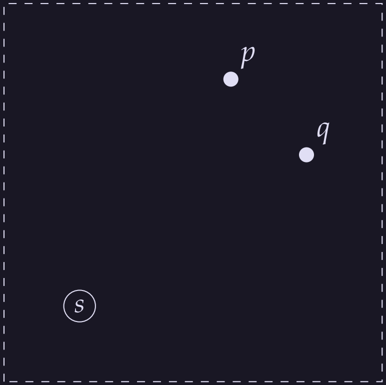
□

δ-Tank Algorithm

Algorithm 1: δ-Tank Algorithm

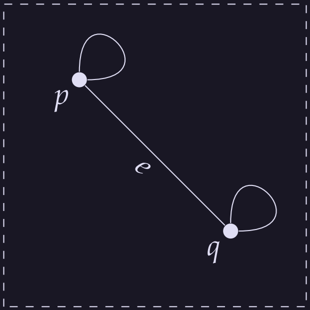
```
For the first solution:
    Match up $p \in P_{>\frac{1}{3}}$ via min-cost perfect matching into $T'$.
    Compute a TSP tour $A$ on $\{s\} \cup P_{\leq\frac{1}{3}}$.
    Apply $\delta$-tank lemma with $\delta = \frac{1}{3}$ to $A$ to get $T''$.
    Let $T = T' \cup T''$ be the solution.
For the second solution:
    Compute a TSP tour $A$ on $V$.
    Apply $\delta$-tank lemma with $\delta = \frac{1}{3}$ to get $F$.
return better of $T$ and $F$
```

Matching Step



$\Rightarrow$

$$w(e) = d(s,p) + d(p,q) + d(q,s)$$



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For the second solution:

$$d(F) \leq \alpha \cdot \mathrm{OPT} + \frac{3}{2}D_{\leq\frac{1}{3}} + 3D_{>\frac{1}{3}} - \frac{1}{2}D'_{>\frac{1}{3}}.$$

Proof of δ-Tank Algorithm II

Now combine:

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&\leq \frac{2\alpha \cdot \mathrm{OPT} + 3D + \frac{1}{2}d(T')}{2}
\end{aligned}
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&\leq \frac{2\alpha \cdot \mathrm{OPT} + 3D + \frac{1}{2}d(T')}{2} \\
&\leq (\alpha + 1.75)\mathrm{OPT}.
\end{aligned}$$

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Clearly everything was polynomial time! □

A 3.194-Approximation

Result

We can actually do slightly better than the previous section:

Theorem (Friggstad et. al 2022)

There is a $(\alpha + \ln(2) + \delta)$-approximation algorithm for the CVRP that runs in $n^{O(1/\delta)}$ time where α is the best approximation ratio for the TSP.

Configuration LP

The idea is to replace the matching by a configuration LP:

$$
\begin{aligned}
\min \quad & \sum_{C \in \mathcal{C}} d(C) x_C \\
\text{s.t.} \quad & \sum_{\substack{C \in \mathcal{C} \\ p \in C}} x_C \geq 1 \qquad \forall p \in P_{>\delta}, \\
& \qquad x \geq 0.
\end{aligned}
$$

The Configuration LP Algorithm

Algorithm 2: Configuration LP Algorithm

```
Solve the configuration LP to get $x^\star$.
Let $T := \emptyset$.
for $C \in \mathcal{C}$ do
    With probability $\min\{1, \ln(2) x_C\}$ add $C$ to $T$.
Compute a TSP tour $A$ on $V \quad P(T)$.
Apply $\delta$-tank lemma to $A$ to get $T'$.
return $T \cup T'$
```

Proof of the Configuration LP Algorithm

Proof. First note $\mathbb{E}[d(T)] \leq \ln(2)\mathrm{OPT}$ and:

$$\mathbb{P}[p \text{ uncovered by } T] = \prod_{C \in \mathcal{C}} (1 - \ln(2) x_C) \leq e^{-\ln(2)} = \frac{1}{2}.$$

Recall by δ-tank lemma ($\hat{D}$ counts only uncovered parcels):

$$d(T') \leq \alpha \cdot \mathrm{OPT} + \frac{1}{1-\delta}\hat{D}_{\leq \delta} + \frac{2}{1-\delta}\hat{D}_{>\delta}.$$

Thus:

$$\mathbb{E}[d(T')] \leq \alpha \cdot \mathrm{OPT} + \frac{1}{1-\delta}D_{\leq \delta} + \frac{2}{1-\delta}\frac{1}{2}D_{>\delta}.$$

Proof of the Configuration LP Algorithm II

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$$\mathbb{E}[d(T) + d(T')] \leq \ln(2)\mathrm{OPT} + \alpha\mathrm{OPT} + \frac{1}{1-\delta}D_{\leq\delta} + \frac{1}{1-\delta}D_{>\delta}$$

Proof of the Configuration LP Algorithm II

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$$\begin{aligned}\mathbb{E}[d(T) + d(T')] &\leq \ln(2)\mathrm{OPT} + \alpha\mathrm{OPT} + \frac{1}{1-\delta}D_{\leq\delta} + \frac{1}{1-\delta}D_{>\delta} \\ &\leq \left(\ln(2) + \alpha + \frac{1}{1-\delta}\right)\mathrm{OPT}.\end{aligned}$$

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Now combine:

$$\begin{aligned}\mathbb{E}[d(T)+d(T')] &\le \ln(2)\mathrm{OPT} + \alpha\mathrm{OPT} + \frac{1}{1-\delta}D_{\le\delta} + \frac{1}{1-\delta}D_{>\delta} \\ &\le \left(\ln(2) + \alpha + \frac{1}{1-\delta}\right)\mathrm{OPT}.\end{aligned}$$

Note: the running time is $n^{O(1/\delta)}$. The algorithm can be derandomized via method of conditional expectation. □

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- For the splittable and equal demand cases, can we do better than $2.5 - \epsilon$?
- This algorithm bounds against a natural LP. We know the integrality gap is ≥ 2. Can we get a better bound?
- For the Euclidean plane, all cases have $2 + \epsilon$ ratios. Can we do better?

Thank your for listening!