A Fast $(2 + 2/7)$-Approximation Algorithm For Capacitated Cycle Covering

Thorben Tröbst (joint work with Vera Traub)

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IPCO XXI — London, UK (via Zoom)

Capacitated Vehicle Routing and Cycle Covering

Capacitated Vehicle Routing Problem (CVRP)

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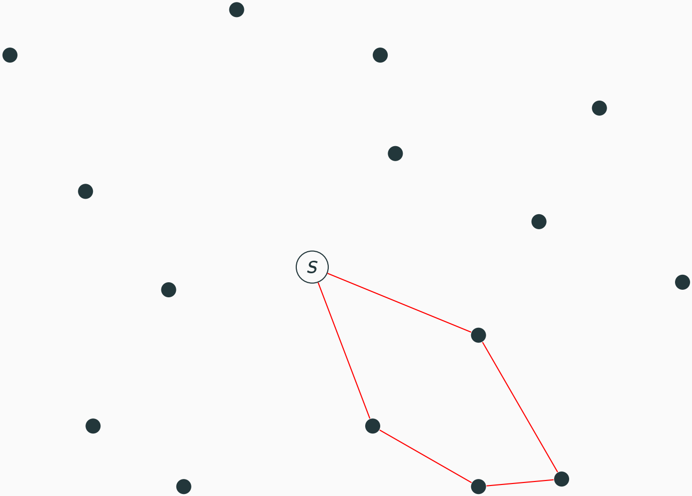

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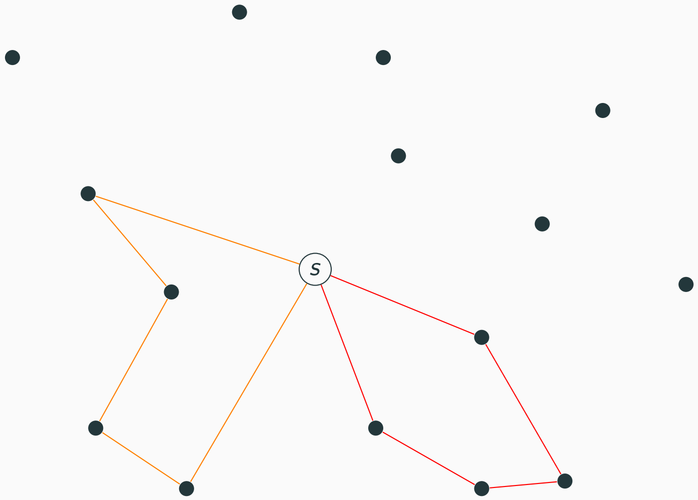

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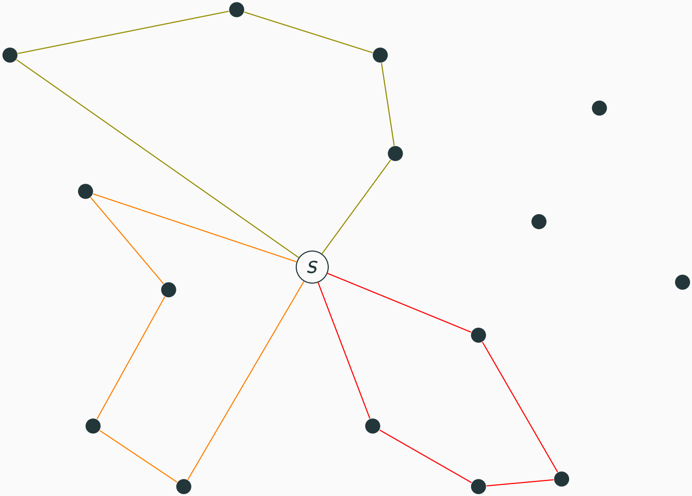

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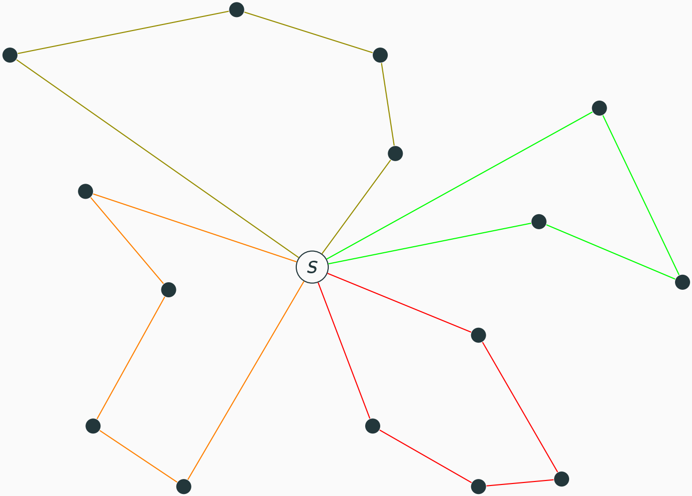

Capacitated Vehicle Routing Problem (CVRP) II

Problem (Capacitated Vehicle Routing)

Input: *a complete graph* $G = (V, E)$ *with* $V = P \cup \{s\}$, *metric edge lengths* $\ell : E \to \mathbb{R}_{\geq 0}$, *and vertex demands* $b : P \to [0, 1]$.

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- General case has not been improved in over 30 years!

Capacitated Cycle Covering Problem (CCCP)

Problem (Capacitated Cycle Covering)

Input: *a complete graph* $G = (V, E)$ *with metric edge lengths* $\ell : E \to \mathbb{R}_{\geq 0}$*, vertex demands* $b : V \to [0, 1]$*, and an opening cost* $\gamma \in \mathbb{R}_{\geq 0}$.

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Instead of a depot we have a fixed opening cost!

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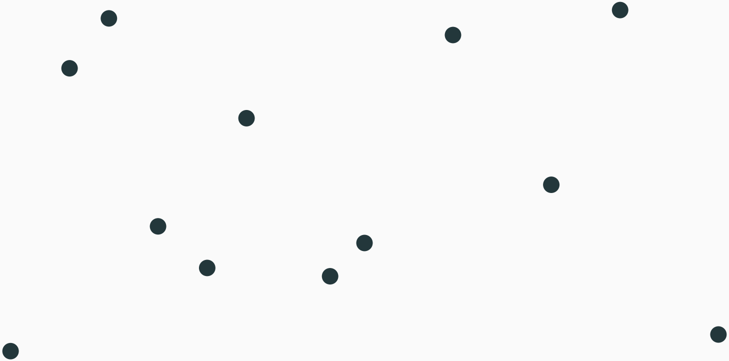

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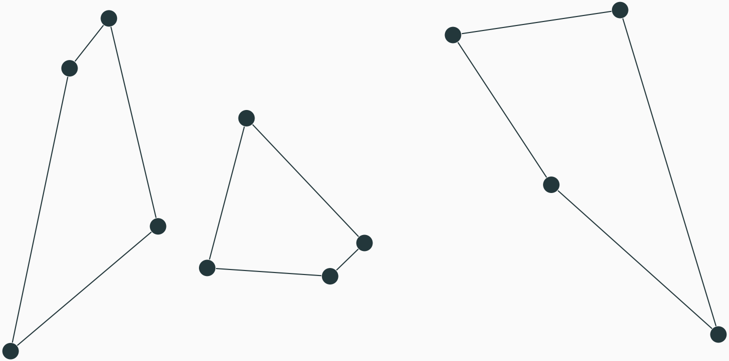

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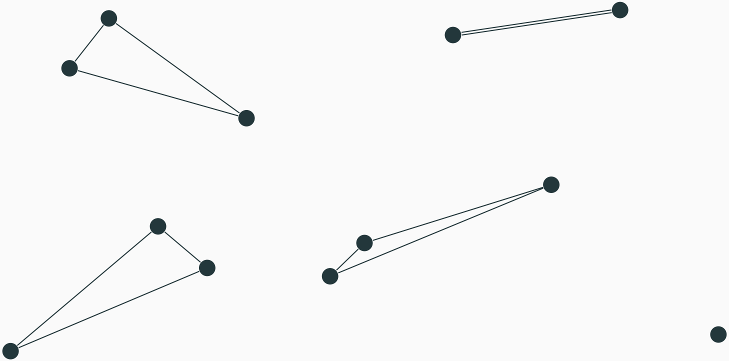

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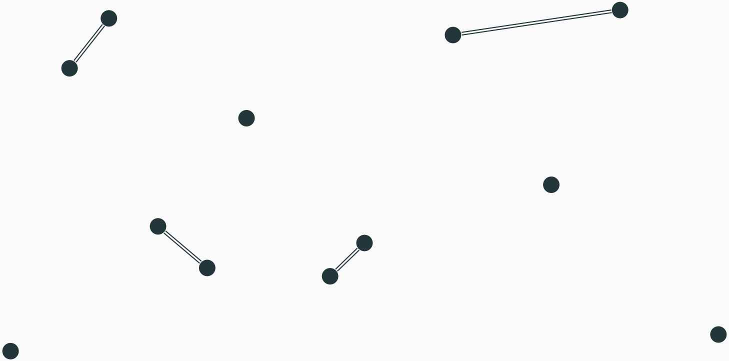

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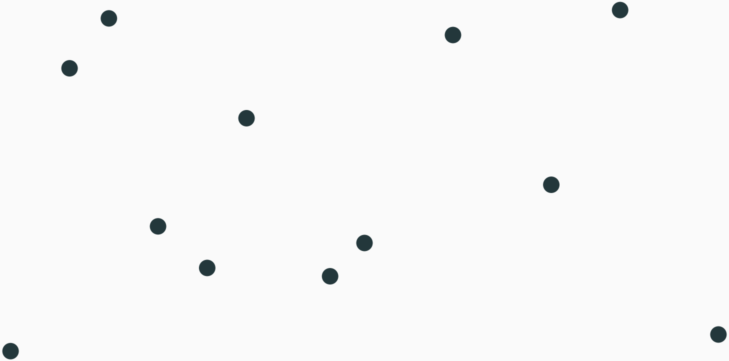

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Related Work

To the best of our knowledge, this precise problem formulation is new. However, there are several related problems (aside from the CVRP) in the literature:

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- Min-max cycle cover and bounded cycle cover (Even et al. 2004, Yu et al. 2019) with approximation guarantees of 5 and $4 + 4/7$ respectively
- Capacitated min-max cycle cover (Das et al. 2019) with an approximation guarantee of > 500

Main Result

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- **To compute a good starting forest, we introduce a tree cover LP and show how to solve it.**
- **We use the (highly structured) LP solution and apply randomized rounding to get our starting forest.**
- The procedure can be derandomized.

Converting Forests into Cycles

Overview

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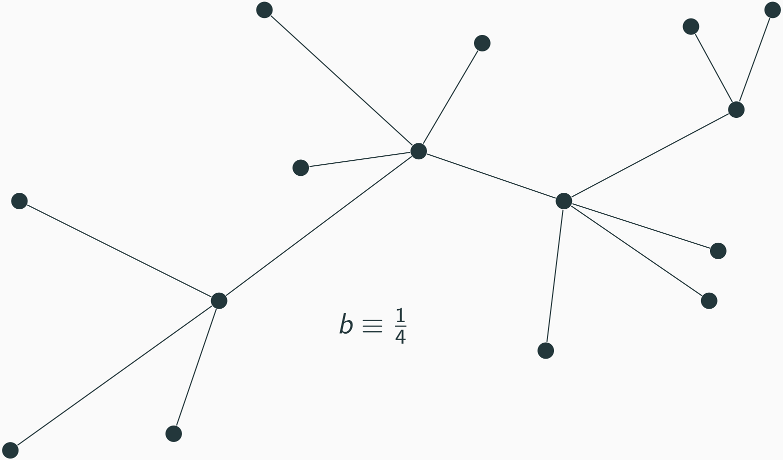
We want to maintain that on average $b(C_i) \geq \frac{1}{2}$ so we do not increase the number of components too much.

Tree Splitting



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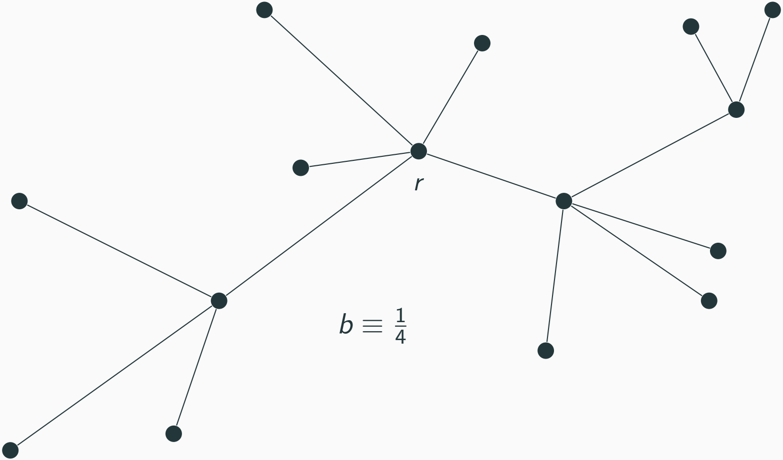

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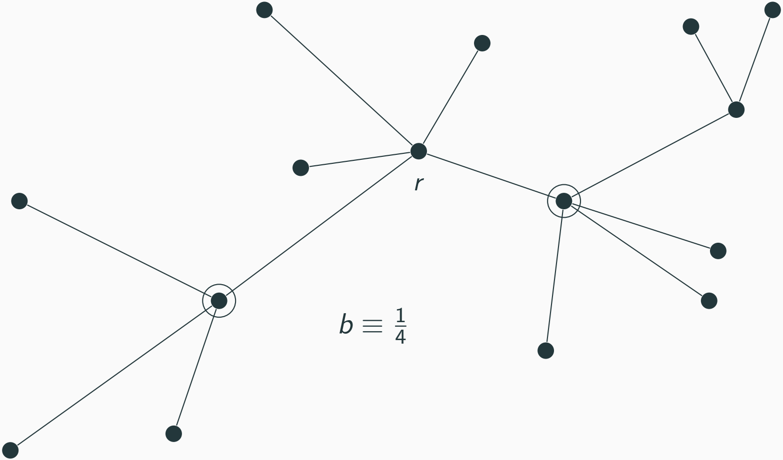

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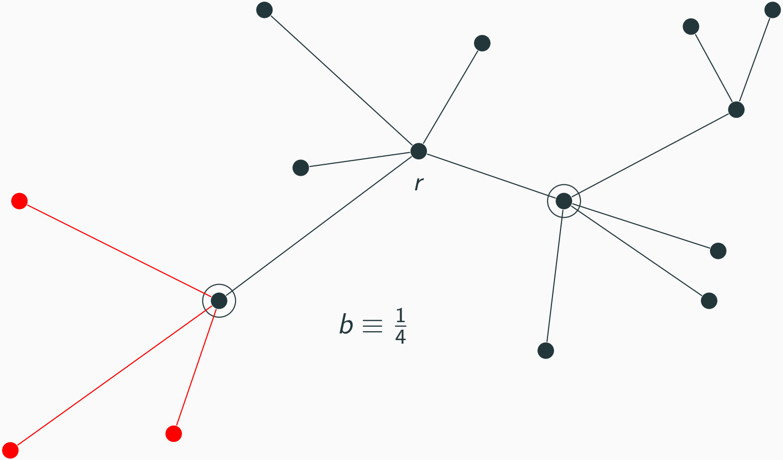

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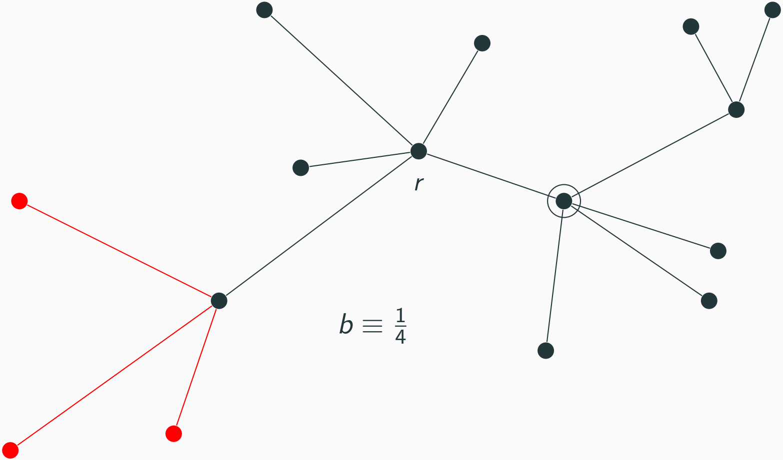

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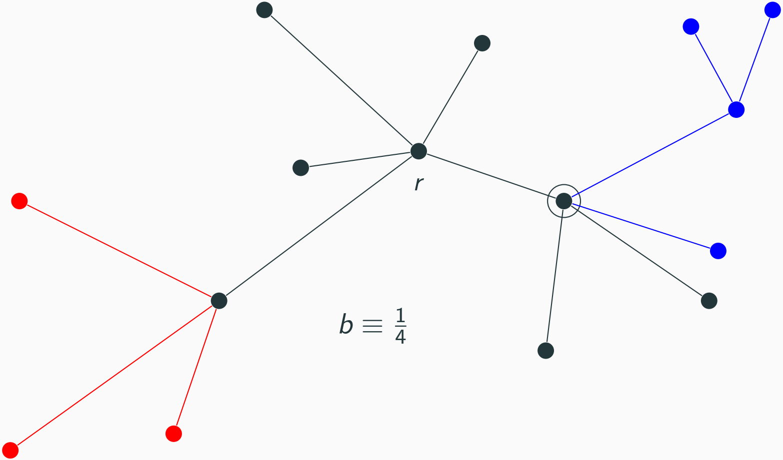

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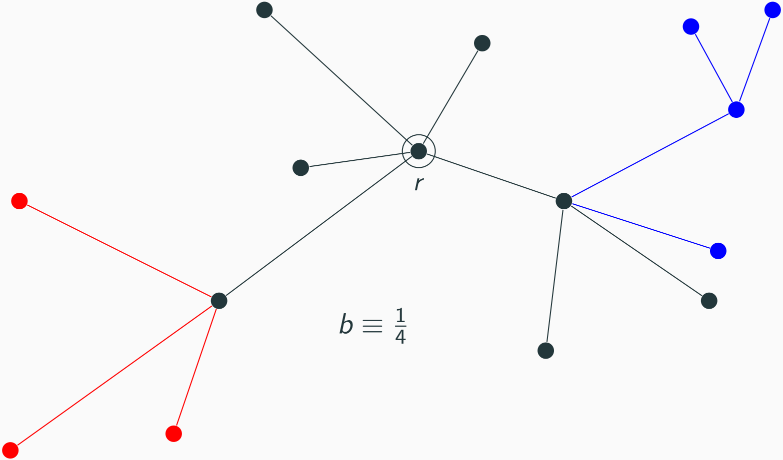

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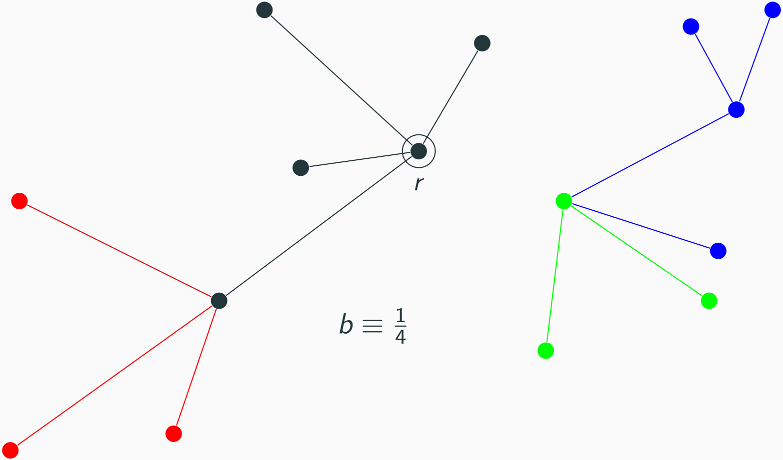

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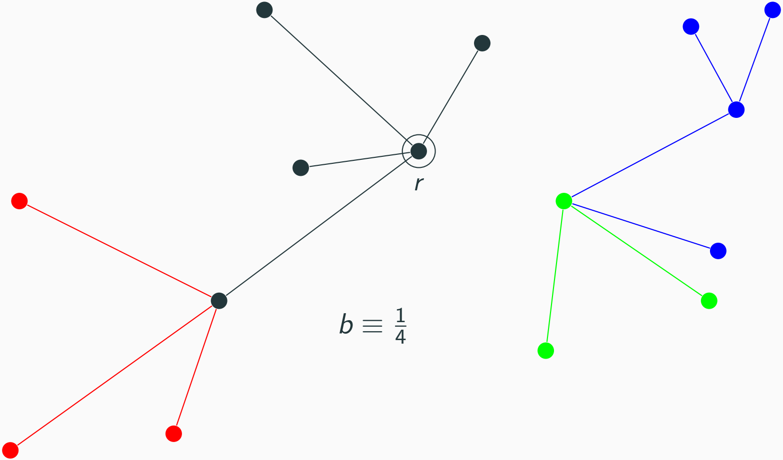

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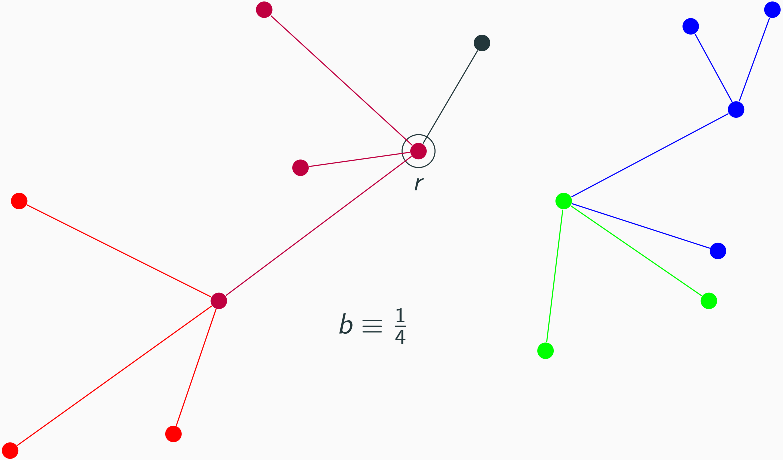

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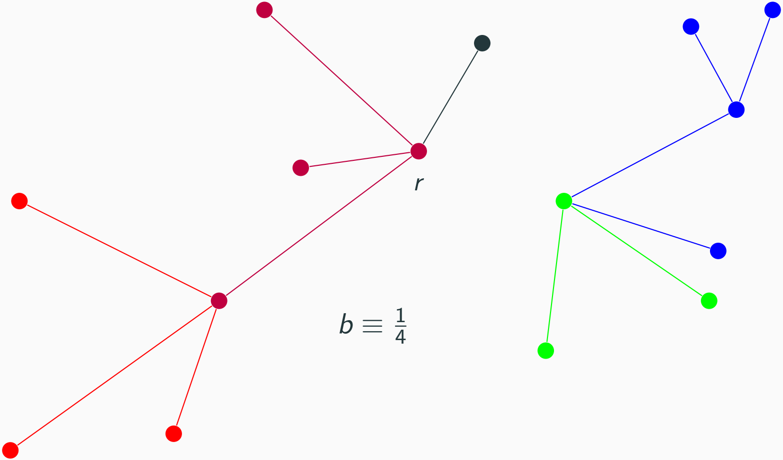

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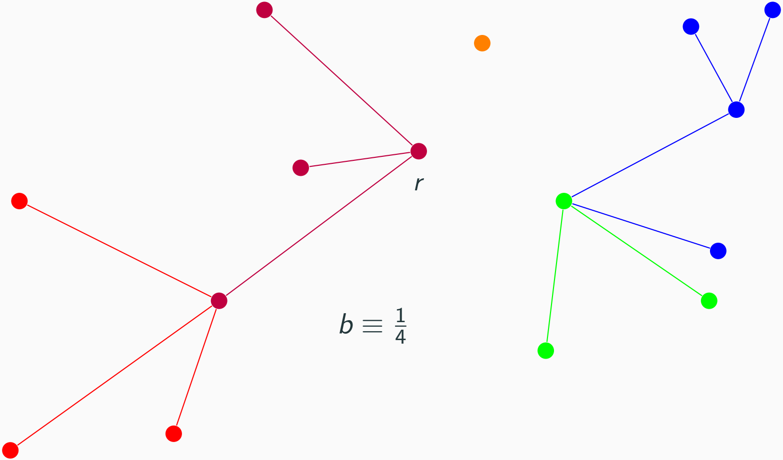

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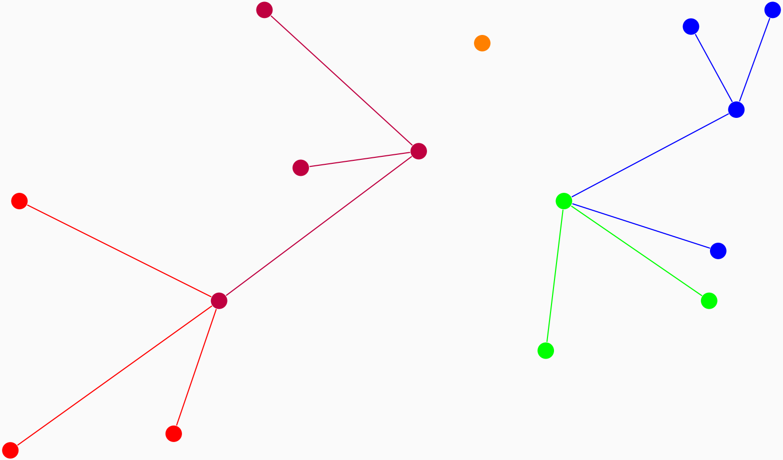
Trees to Cycles



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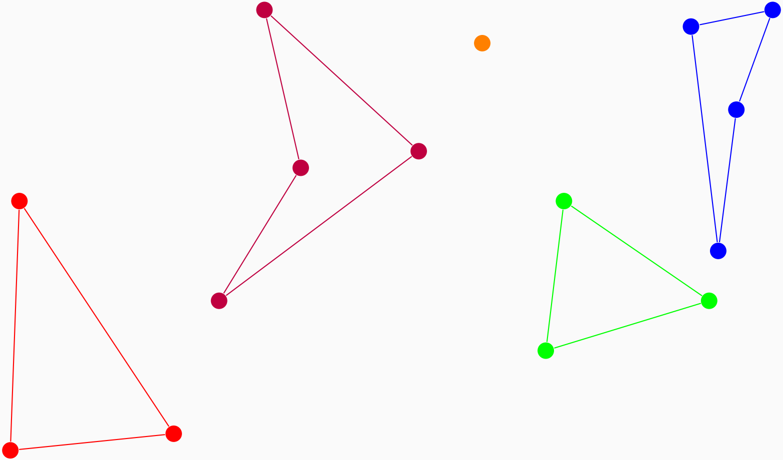

Tree Splitting Lemma

Lemma (Tree Splitting)

Let (V, F) be a forest. Then we can compute in linear time a feasible solution $C_1, \ldots, C_k$ to the CCCP with cost bounded by

$$2\ell(F) + \gamma \cdot \sum_{A \in \mathcal{C}(F)} u(A)$$

where $u : 2^V \to \mathbb{R}_{\geq 0}$ is given by

$$u(A) := \begin{cases} 1 & \text{if } b(A) \leq 1, \\ 2b(A) & \text{if } b(A) > 1. \end{cases}$$

Tree Cover LP

The Tree Cover LP

To obtain a lower bound on the cost of an optimum solution to the CCCP, we use the following linear program.

$$
\begin{aligned}
\min \quad & \ell(x) + \gamma(|V| - x(E)) \\
\text{s.t.} \quad & x(E[A]) \leq |A| - \max\{1, b(A)\} \quad \forall \emptyset \neq A \subseteq V, \\
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Note: integral solutions of this LP are forests, not cycle covers! However, we still have $\mathrm{LP} \leq \mathrm{OPT}$.

Connection to the CVRP

This LP is inspired by the following LP relaxation of the CVRP.

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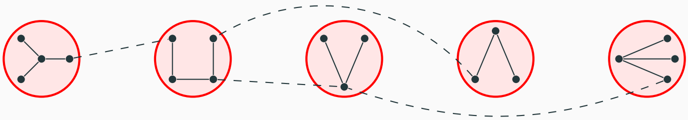
\end{aligned}$$

The integrality gap of this CVRP LP is at most 3.5 matching the best-known approximation algorithms.

Structure of Solutions

Since the LP optimizes over a polymatroid, each connected component in the support of an extreme point solution has some nice structure:



The fractional edges satisfy $x_e \geq 1 - b(R)$ where R is the right red component incident to e.

Randomized Rounding

Our Goal

Recall: we now want to find a forest (V, F) with

$$2\ell(F) + \sum_{A \in \mathcal{C}(F)} u(A)$$

bounded by $(2 + 2/7)\mathrm{LP}$ where

$$u(A) := \begin{cases} 1 & \text{if } b(A) \leq 1, \\ 2b(A) & \text{if } b(A) > 1. \end{cases}$$

Randomized Rounding

This can be achieved with via randomized rounding:

Theorem (Randomized rounding)

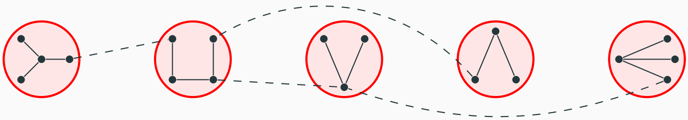
Let x be an extreme point solution of the tree cover LP. Define a random edge set $F \subseteq E$ by independently picking each edge e with probability $\min\{1, \frac{8}{7} x_e\}$. *Then*

$$\mathbb{E}\left[\sum_{A \in \mathcal{C}(F)} u(A)\right] \leq \left(2 + \frac{2}{7}\right)(|V| - x(E))$$

and $\mathbb{E}[2\ell(F)] \leq (2 + 2/7)\,\ell(x)$.

Dropping Edges

Consider some connected component T in $\mathrm{supp}(x)$ and assume $b(T) > 1$ so $u(T) = 2b(T)$ whereas $|V(T)| - x(E(T)) = b(T)$.



Dropping an edge e increases the sum of u's by at most $1 - 2b(R)$ where R is the right red component.

Dropping Edges II

Recall that $x_e \geq 1 - b(R)$ and so

$$\begin{aligned}\mathbb{E}\left[\Delta \sum_{A \text{ conn. comp.}} u(A)\right] &\leq \max\left\{1 - \frac{8}{7}x_e, 0\right\}(1 - 2b(R)) \\ &\leq \max\left\{1 - \frac{8}{7}(1 - b(R)), 0\right\}(1 - 2b(R)) \\ &\leq \frac{2}{7}b(R).\end{aligned}$$

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$$\leq \frac{2}{7}b(R).$$

Since all sets R are disjoint, we get

$$\mathbb{E}\left[\sum_{A \in \mathcal{C}(F)} u(A)\right] \leq \left(2 + \frac{2}{7}\right) b(V) = \left(2 + \frac{2}{7}\right)(|V| - x(E)).$$

Conclusion

Tightness Results

Theorem

The analysis of our algorithm is tight, i.e. there are instances where it computes solutions of cost $(2 + 2/7 - \epsilon)\mathrm{OPT}$.

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Theorem

The gap between the tree covering LP and the optimum capacitated cycle covering solution can be more than $2 + 62/11745 \approx 2.005$.

Open Questions

Open questions:

- What is the gap between the tree cover LP and the CCCP?
- Provide non-trivial lower bounds for the CVRP LP shown earlier (might not exist).
- Can a similar tree-first approach be used for the CVRP?
- **Show that the integrality gap of the CVRP LP is less than 3.5 and/or provide a better approximation algorithm for the CVRP.**

Thank You!