Fair and Efficient Allocations of Chores under Bivalued Preferences*

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* based on AAAI 2022 paper by Jugal Garg, Aniket Murhekar, and John Qin

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Assume: Linear utilities: u_{ij} for all $i \in N, j \in M$.

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Problem: none of these work for indivisible goods!

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Note: replacing min with max yields stronger EQX / EFX fairness.

Efficiency

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	Phone	Tablet
Alice	10	1
Bob	1	10

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Allocation Alice – Tablet and Bob – Phone is EFX and EQX but obviously bad!

Efficiency Notions

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An allocation is fractionally Pareto-optimal (fPO) if no fractional allocation is weakly better for all agents, and strictly better for at least one agent.

Known Results

Theorem (Barman, Krishnamurthy, Vaish 2018)

EF1 + PO allocations always exist an can be computed in pseudo-polynomial time.

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Theorem (Garg, Murhekar 2021)

EFX + fPO allocations exist under bivalued utilities and can be computed in polynomial time.

Open problem: Do EFX + PO allocations always exist?

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Note: Notions of fairness and efficiency extend to chores!

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EF + PO allocations of divisible chores exist under bivalued utilities and can be computed in strongly polynomial time.

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So our instance looks like:

- n agents N,
- m goods M,
- costs c_{ij} where wlog. $c_{ij} \in \{1, k\}$ for some $k \in \mathbb{N}$.

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Note: If (x, p) is a market equilibrium, then x is fPO!

Market Equilibrium Example

	Dishes	Laundry	Cleaning
Alice	5	1	3
Bob	1	2	4
Charlie	3	1	5

Allocate: Alice – Cleaning, Bob – Dishes, Charlie – Laundry

$$p(\text{Laundry}) = 1, p(\text{Dishes}) = 1, p(\text{Cleaning}) = 3$$

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Note: If all e_i are the same, then this is very fair. This is possible for divisible goods but no algorithm exists.

Market equilibria have a fairness notion too:

Definition

(x, p) is price envy-free up to one chore (pEF1) if
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Lemma

If (x, p) is pEF1 then x is EF1.

General Strategy

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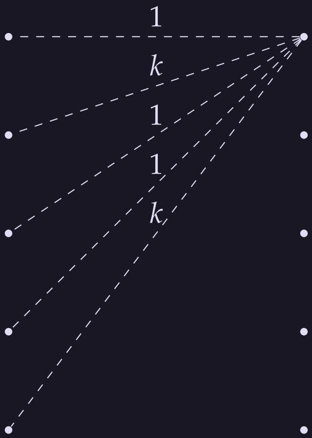
- Try to funnel chores from the big earner to the least earner.
- If this is not possible, raise prices.

Phase 1: Initial Equilibrium



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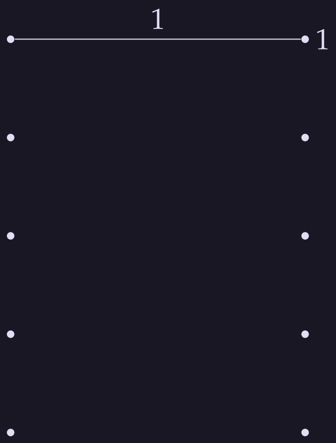

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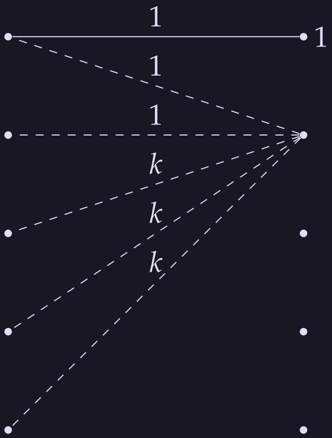

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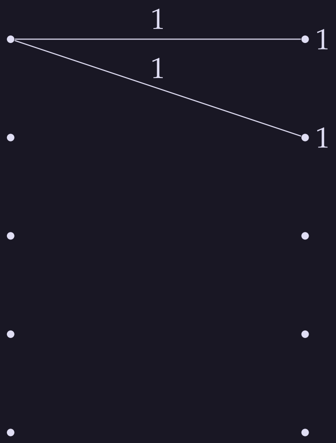

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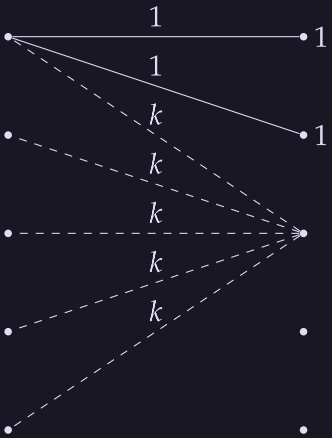

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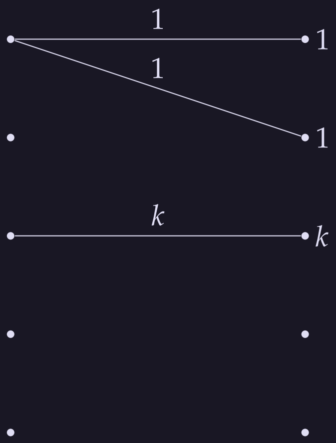

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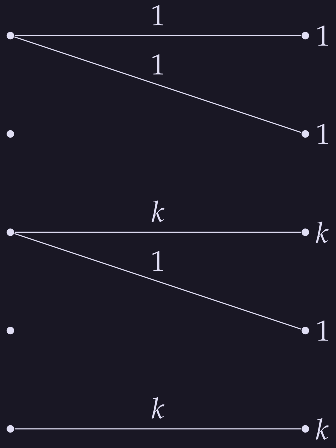
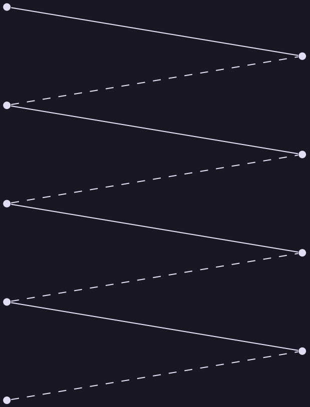
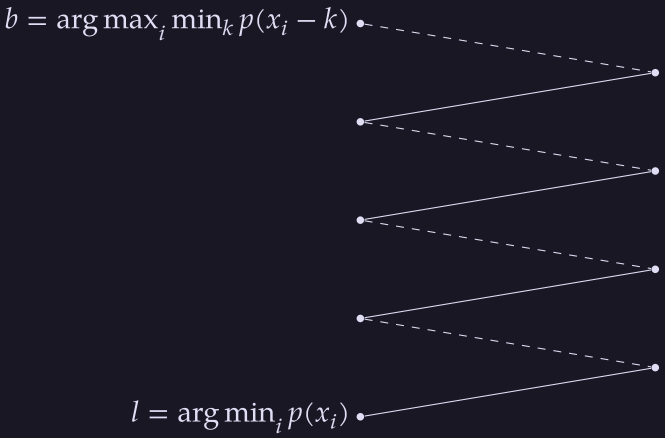


Phase 1: Initial Equilibrium II

$b = \arg\max_i \min_k p(x_i - k)$



$l = \arg\min_i p(x_i)$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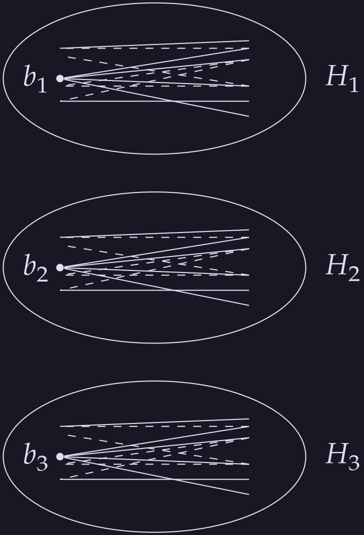
3. If yes, funnel chores from b to l and go back to 1.
4. If no, remove the component of b from the graph and go back to 1.

Phase 1: Initial Equilibrium IV



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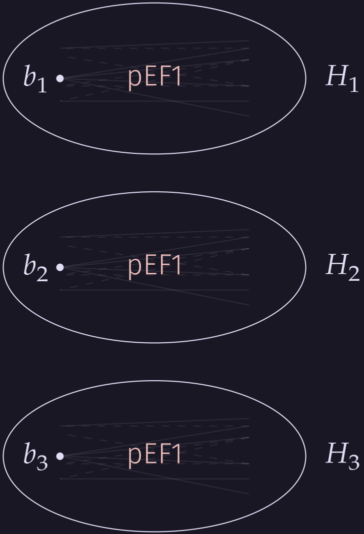
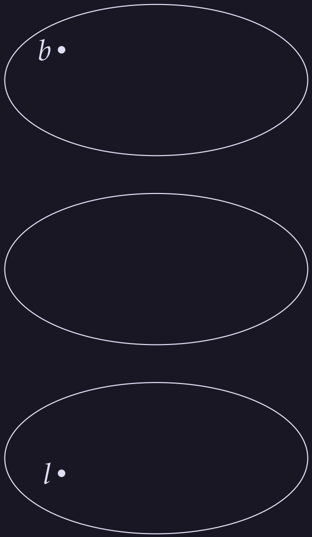


Phase 2: Raise Prices



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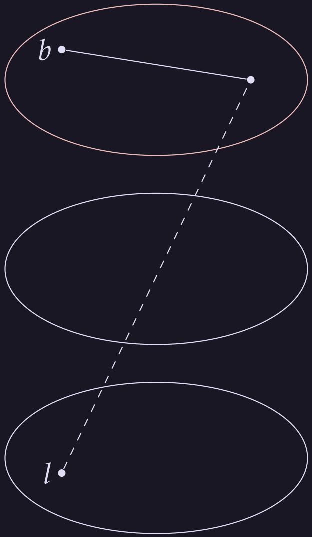

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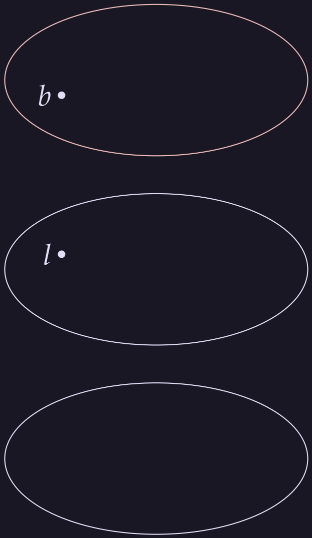

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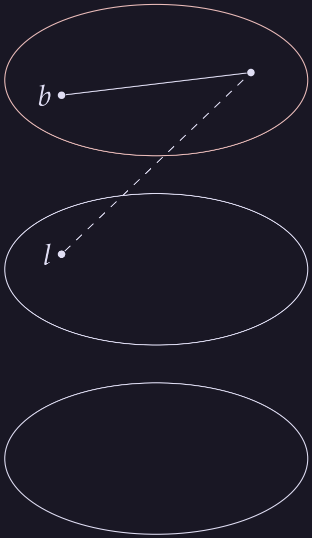

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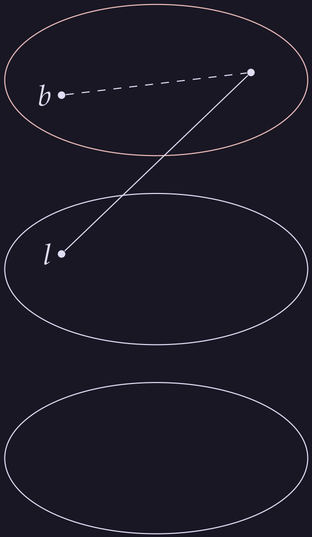

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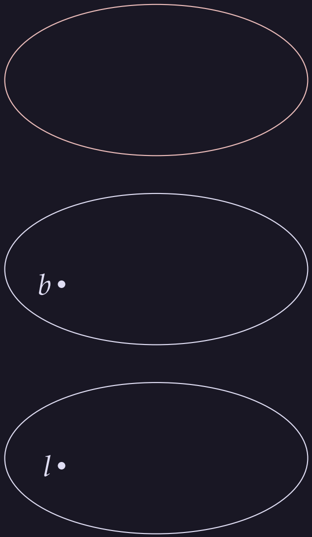

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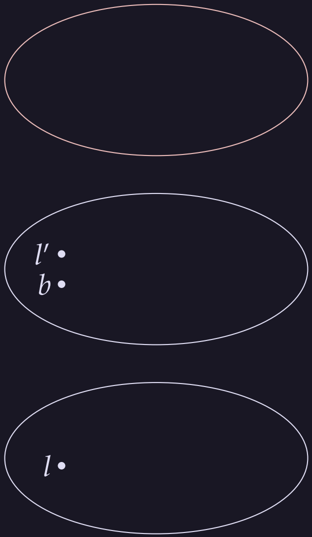
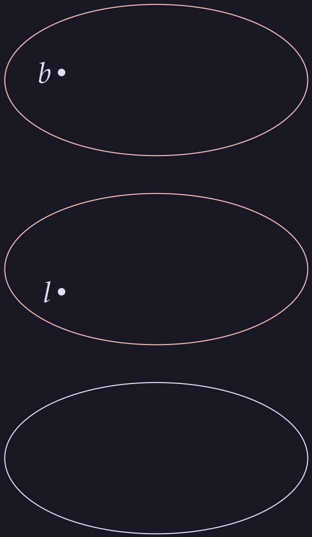


Phase 3: Final Trades



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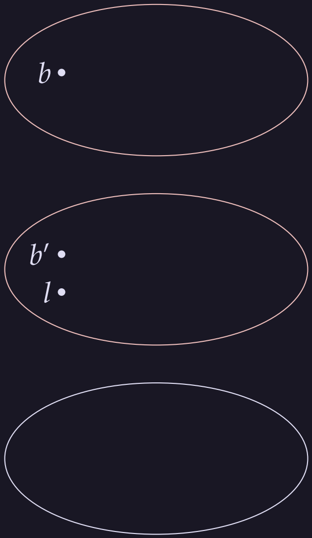

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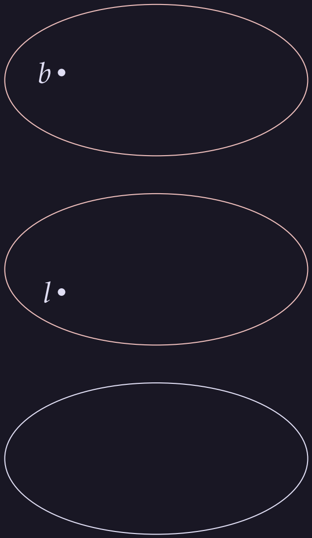

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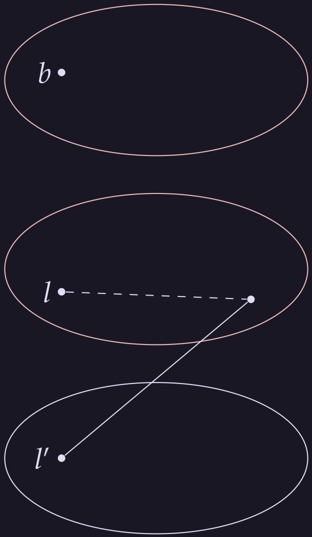

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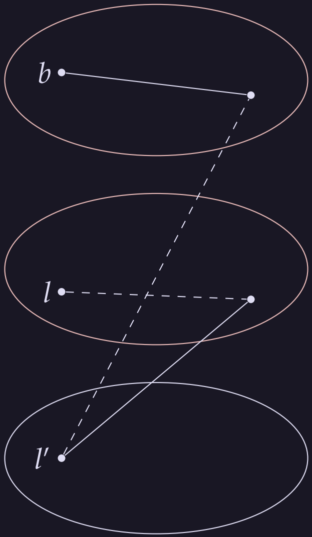

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- Create pEF1 groups $H_1, \dots, H_r$.
- Transfer chores from b to l by successively raising prices in groups $H_1, \dots, H_r$.
- If l ends up in a raised group, transition to phase 3 and trade along alternating paths.

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Can use similar techniques for:

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- Do EF1 + PO allocations always exist?
- If so: can we compute them? If not: is decidability hard?
- EF + PO allocations of divisible chores are known to always exist. Is there a polynomial time algorithm?

THANK YOUR FOR LISTENING!