Approximately Stable Committee Selection[1]

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Theory Seminar, April 10, 2020

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[1] to appear in STOC 2020

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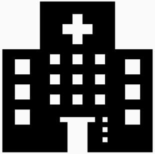
The Committee Selection Problem

Funding Public Goods

Imagine a small community which wants to use local taxes to improve its school, its roads, and its hospital. However, they only have the resources to fund 2 out of 3 projects.



Which projects should they focus on?

Funding Public Goods II

Say they decide to fund the school and the roads.

If a large enough group of taxpayers prefer funding the hospital over the roads, then this selection is not stable.

Formal Problem Statement

We are given

- a finite set V of n voters (assume equal endowments),
- a finite set C of candidates with weights $w : C \to \mathbb{R}_{\geq 0}$, and
- a total, monotonic pre-order $\preccurlyeq_v$ on $\mathcal{P}(C)$ for each voter v.

Question: is there $S \subseteq C$ with $w(S) \leq n$ such that S is stable, i.e. for all other $S' \subseteq C$ we have $V(S, S') < w(S')$? Here

$$V(S, S') := |\{v \in V \mid S \prec_v S'\}|$$

is the number of voters who would rather switch from S to S'.

Approximately Stable Committees

In general, stable committees $S \subseteq C$ with $w(S)$ may not always exist. So we relax the notion of stability:

Definition (Approximate Stability)

Let $S \subseteq C$ be some committee with and $\alpha \geq 1$. Then S is α-approximately stable if for all $S' \subseteq C$ we have

$$V(S, S') < \alpha \cdot w(S').$$

Note: finding α-approximately stable committees $S \subseteq C$ with $w(S) \leq n$ is the same as finding stable committees $S \subseteq C$ with $w(S) \leq \alpha \cdot n$.

Main Result

The main result is the existence of $O(1)$-approximately stable committees. More precisely:

Theorem

There always exists a 32-approximately stable committee $S \subseteq C$ with $w(S) \leq n$.

This was previously only known for select special cases. The proof will use iterative rounding of approximately stable lotteries.

2-Approximately Stable Lotteries

Approximately Stable Lotteries

Definition

A probability distribution Δ on $\mathcal{P}(C)$ is called an α-approximately stable lottery if for any $S' \subseteq \mathcal{P}(C)$ we have

$$\mathbb{E}_{S \sim \Delta}[V(S, S')] < \alpha \cdot w(S').$$

Theorem

There always exists a 2-approximately stable lottery Δ with $\mathbb{P}_{S \sim \Delta}[w(S) \leq n] = 1$.

Proof. Finding a 2-approximately stable lottery Δ with $\mathbb{P}_{S\sim\Delta}[w(S) \leq n] = 1$ is equivalent to finding a (1-approximately) stable lottery Δ with $\mathbb{P}_{S\sim\Delta}[w(S) \leq 2n] = 1$.

So we want

$$\min_{\Delta:\mathbb{P}_{S\sim\Delta}[w(S)\leq 2n]=1} \max_{S'\subseteq C} \mathbb{E}_{S\sim\Delta}[V(S,S') - w(S')] < 0.$$

We can think of this as a zero-sum 2-player game. The defending player picks a committee S with $w(S) \leq 2n$ and the attacking player picks a committee S' to get a payoff of $w(S') - V(S, S')$. Goal: defender wins with mixed strategies.

The Committee Game II

A key observation from game theory is:

$$
\begin{aligned}
&\min_{\Delta:\mathbb{P}_{S\sim\Delta}[w(S)\leq 2n]=1} \max_{S'\subseteq C} \mathbb{E}_{S\sim\Delta}[V(S,S') - w(S')] \\
&= \min_{\Delta:\mathbb{P}_{S\sim\Delta}[w(S)\leq 2n]=1} \max_{\Delta'} \mathbb{E}_{S\sim\Delta, S'\sim\Delta'}[V(S,S') - w(S')] \\
&= \max_{\Delta'} \min_{\Delta:\mathbb{P}_{S\sim\Delta}[w(S)\leq 2n]=1} \mathbb{E}_{S\sim\Delta, S'\sim\Delta'}[V(S,S') - w(S')].
\end{aligned}
$$

This so-called minimax theorem follows essentially from LP duality.

Winning the Committee Game

Given an "attacking" distribution Δ' on $\mathcal{P}(C)$, we wish to construct a "defending" distribution Δ with

$$\mathbb{P}_{S\sim\Delta}[w(S) \leq 2n] = 1$$

$$\mathbb{E}_{S\sim\Delta, S'\sim\Delta'}[V(S, S')] < \beta$$

where

$$\beta := \mathbb{E}_{S'\sim\Delta}[w(S')].$$

Probability Matching

We construct Δ using probability matching.

Imagine the attacker just hands us a single committee $S' \subseteq C$ with $w(S') \leq n$. How do we construct $S \subseteq C$ with $w(S) \leq n$ and $V(S, S') < w(S')$?

Probability Matching

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Easy: just set $S = S'$, then $V(S, S') = 0$. Probability matching is the technique of playing $\Delta \approx \Delta'$.

Probability Matching II

For each $S \subseteq C$, let $X_S \in \{0, 1\}$ be a Bernoulli random variable with success probability

$$p_S := \mathbb{P}[X_S = 1] = \min\left\{1, \frac{n}{\beta}\mathbb{P}_{S' \sim \Delta'}[S' = S]\right\}.$$

Then we let Δ be the distribution which picks the set

$$\bigcup_{\substack{S \subseteq C \\ X_S = 1}} S.$$

In other words: we independently pick sets according to how likely they are in Δ' and take the union of these sets.

Weight of Δ is Bounded

Recall that we want $\mathbb{P}_{S\sim\mathbf{\Delta}}[w(S) \leq 2n] = 1$. We do not quite have that, but:

$$\begin{aligned}
\mathbb{E}_{S\sim\mathbf{\Delta}}[w(S)] &\leq \sum_{S\subseteq C} w(S)\mathbb{P}[X_S = 1] \\
&\leq \sum_{S\subseteq C} w(S)\frac{n}{\beta}\mathbb{P}_{S'\sim\mathbf{\Delta}'}[S' = S] \\
&= \frac{n}{\beta}\mathbb{E}_{S'\sim\mathbf{\Delta}'}[w(S')] \\
&= n.
\end{aligned}$$

Per Voter Guarantee

Recall that we want to prove

$$\mathbb{E}_{S \sim \Delta, S' \sim \Delta'}[V(S, S')] < \beta.$$

By linearity of expectation, it suffices to show this for each voter v independently, i.e.

$$\mathbb{P}_{S \sim \Delta, S' \sim \Delta'}[S \prec_v S'] < \frac{\beta}{n}.$$

Per Voter Guarantee II

So fix some $v \in V$ and let $S_1 \preccurlyeq_v \cdots \preccurlyeq_v S_t$ enumerate $\mathcal{P}(C)$ in the preference order of v. Then

$$
\begin{aligned}
&\mathbb{P}_{S\sim\Delta, S'\sim\Delta'}[S \prec_v S'] \\
&\leq \sum_{i=1}^{t} \mathbb{P}_{S\sim\Delta, S'\sim\Delta'}[S' = S_i \text{ and } S_1 \not\subseteq S, \ldots, S_i \not\subseteq S] \\
&\leq \sum_{i=1}^{t} \mathbb{P}_{S'\sim\Delta'}[S' = S_i]\mathbb{P}[X_{S_1} = \cdots = X_{S_i} = 0] \\
&\leq \frac{\beta}{n} \sum_{i=1}^{t} \mathbb{P}[X_{S_i} = 1] \prod_{j=1}^{i} \mathbb{P}[X_{S_j} = 0] \\
&< \frac{\beta}{n}.
\end{aligned}
$$

Dependent Rounding

Recall that we only proved $\mathbb{E}_{S\sim\Delta}[w(S)] \leq n$ but we really need $\mathbb{P}_{S\sim\Delta}[w(S) \leq 2n] = 1$. To fix this one can use dependent rounding, i.e. instead of sampling each X_S independently, there is a special distribution Γ on $\{0,1\}^{\mathcal{P}(C)}$ such that

- $\mathbb{P}_{X\sim\Gamma}[X_S = 1] \geq p_S$ for all $S \subseteq C$,
- $\mathbb{P}_{X\sim\Gamma}[\sum_{S\subseteq C} w(S)X_S \leq 2n]$, and
- $\mathbb{P}_{X\sim\Gamma}[X_{S_1} = \cdots = X_{S_t} = 0] \leq \prod_{i=1}^{t} \mathbb{P}_{X\sim\Gamma}[X_{S_1} = 0]$ for all $S_1, \ldots, S_t \subseteq C$.

If we use Γ to define Δ, the theorem follows. $\square$

Rounding Approximately Stable Lotteries

Our Goal

Recall that we have just proved:

Theorem

There always exists a 2-approximately stable lottery Δ with $\mathbb{P}_{S\sim\Delta}[w(S) \leq n] = 1$.

But our goal is to prove:

Theorem

There always exists a 32-approximately stable committee $S \subseteq C$ with $w(S) \leq n$.

Good and Bad Committees

Definition (Good Committee)

Let Δ be some distribution over committees and $v \in V$ some voter. Then a committee $S' \subseteq C$ is good for v if

$$\mathbb{P}_{S \sim \Delta}[S \prec_v S'] \geq \frac{1}{4}.$$

Definition (Bad Committee)

Let Δ be some distribution over committees and $v \in V$ some voter. Then a committee $S' \subseteq C$ is bad for v if

$$\mathbb{P}_{S \sim \Delta}[S \preccurlyeq_v S'] \leq \frac{1}{4}.$$

Bounds on Good and Bad Committees

Lemma

Let Δ be 2-approximately stable. Then

- *for all $S' \subseteq C$, there are less than $8w(S')$ voters v such that S' is good for v and*
- *there exists $S \in \mathrm{supp}(\Delta)$ such that S is not bad for at least $\frac{3}{4}n$ voters.*

Bounds on Good and Bad Committees II

Proof. Let $S' \subseteq C$ be arbitrary and let $G \subseteq V$ be those voters for which S' is good. Then

$$\begin{aligned}
\mathbb{E}_{S\sim\Delta}[V(S,S')] &= \sum_{v\in V} \mathbb{P}_{S\sim\Delta}[S \prec_v S'] \\
&\geq \sum_{v\in G} \mathbb{P}_{S\sim\Delta}[S \prec_v S'] \\
&\geq \frac{|G|}{4}.
\end{aligned}$$

But by 2-approximate stability of Δ, we know that

$$\mathbb{E}_{S\sim\Delta}[V(S,S')] < 2w(S')$$

and so $|G| < 8w(S')$.

Bounds on Good and Bad Committees III

For any $S \subseteq C$, let $N(S) \subseteq V$ be the set of voters for which S is not bad. We claim $\mathbb{E}_{S\sim\Delta}[|N(S)|] \geq \frac{3}{4}n$.

Let v be any voter and let $\hat{S} \in \mathrm{supp}(\Delta)$ be the best bad set for v. Then

$$\begin{aligned}\mathbb{P}_{S\sim\Delta}[v \in N(S)] &= \mathbb{P}_{S\sim\Delta}[\hat{S} \prec_v S] \\ &= 1 - \mathbb{P}_{S\sim\Delta}[S \preccurlyeq_v \hat{S}] \\ &\geq \frac{3}{4}.\end{aligned}$$

This shows the claim by linearity of expectation and thus the lemma using the probabilistic method. □

Proof of Main Theorem. We run the following algorithm:

1. Set $i := 0$, $V_0 := V$, and $w_0 := 2w$.
2. Let Δ be a 2-approximately stable lottery wrt. V_i such that $\mathbb{P}_{S \sim \Delta}[w_i(S) \leq |V_i|] = 1$.
3. Pick $S_i \in \mathrm{supp}(\Delta)$ such that $|N(S_i)| \geq \frac{3}{4}|V_i|$.
4. If $N(S_i) \neq V_i$, set $V_{i+1} := V_i \setminus N(S_i)$, $w_{i+1} := \frac{1}{2}w_i$, and $i := i + 1$. Then go back to step 2.
5. Return $S := S_1 \cup \cdots \cup S_i$.

Iterative Rounding Analysis

Let $S' \subseteq C$ be arbitrary. Then

$$\begin{aligned} V(S, S') &\leq \sum_{i=0}^{\infty} V_i(S_i, S') \\ &\leq \sum_{i=0}^{\infty} 8w_i(S') \\ &\leq 8w(S) \sum_{i=0}^{\infty} 2^{1-i} \\ &= 32w(S). \end{aligned}$$

So S is indeed 32-approximately stable.

Iterative Rounding Analysis II

We still need to show that $w(S) \leq n$. But observe that

$$\begin{aligned} w(S) &= \sum_{i=0}^{\infty} w(S_i) \\ &= \sum_{i=0}^{\infty} 2^{i-1} w_i(S_i) \\ &\leq \sum_{i=0}^{\infty} 2^{i-1} |V_i| \\ &\leq \sum_{i=0}^{\infty} 2^{i-1} \frac{n}{4^i} \\ &= n. \end{aligned}$$

So S satisfies both conditions of the theorem. □

Open Questions

There are several open questions relating to the committee selection problem:

- Does there always exist a 1-approximately stable lottery? The authors conjecture this to be true and prove it for some simple cases.
- Is there always a stable committee if the preferences are induced by "approval sets" for each voter?
- For which preference structures can we efficiently compute approximately stable lotteries? Recall that the existence proof was not constructive.
- Is there always a 2-approximately stable committee? The authors conjecture this to be true also.

Thank You!

Icon Credit

Icon Credit:

- Hospital icon by Becris
- Road icon by Sergey Krivoy
- School icon by b farias

from the Noun Project.