# Online Matching from an Economics Viewpoint

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# Online Bipartite Matching

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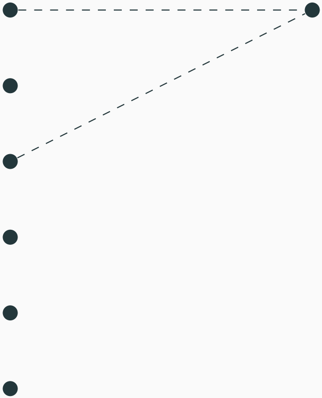

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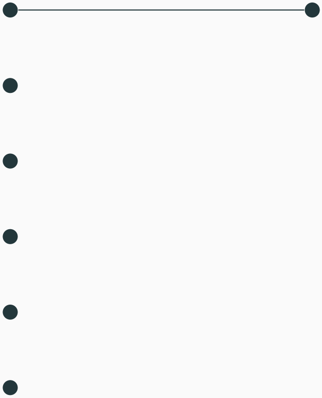

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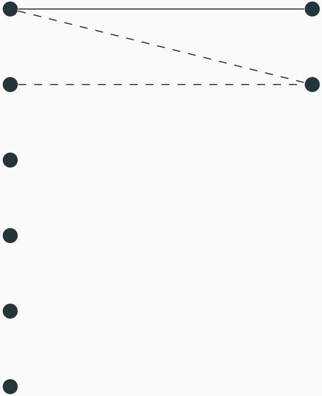

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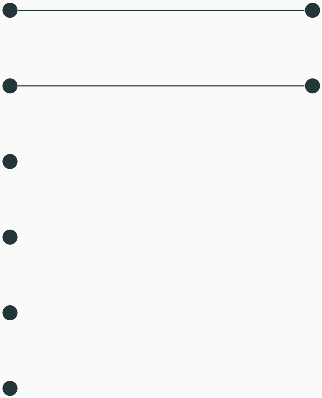

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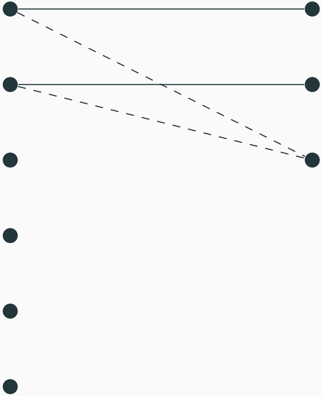

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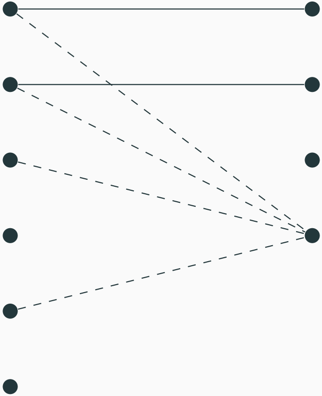

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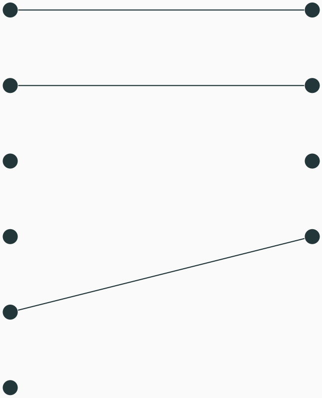

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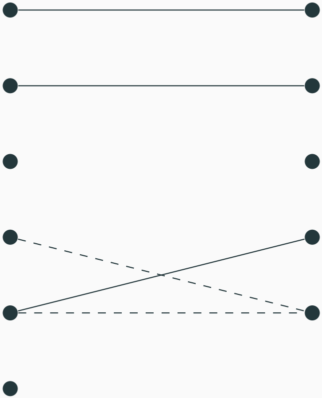

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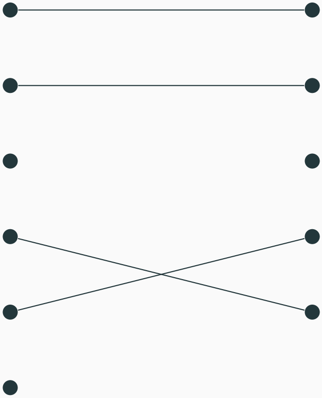

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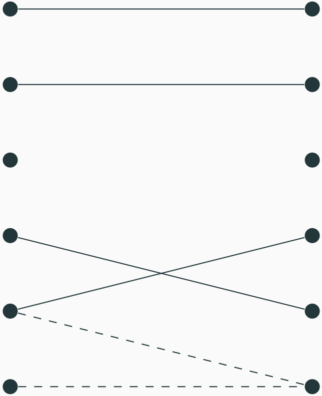

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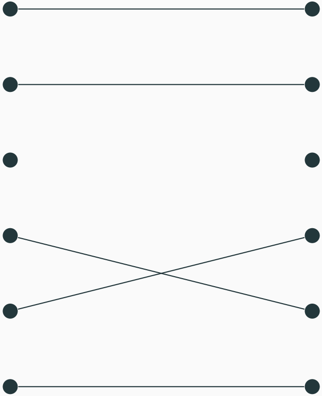

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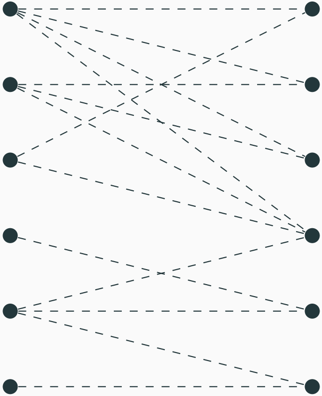

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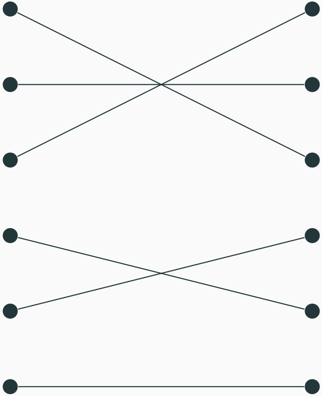

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The goal is to maximize the competitive ratio, i.e.

$$\frac{|M_{\text{online}}|}{\text{OPT}_{\text{offline}}}.$$

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- Water-Filling / Balancing
    - Idea: continuously allocate online vertices to the least-matched offline vertices.
    - Provides fractional solution in a deterministic algorithm.

# History of the Economic Viewpoint

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$\Rightarrow$ Ideas can be extended to WATER-FILLING and many interesting settings!

# Ranking through the Economic Viewpoint

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- Whenever a vertex $i \in B$ arrives, let $N(i)$ be the unmatched neighbors. Match $i$ to $j$ minimizing $\tau(j)$.

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$\Rightarrow$ The resulting algorithm is identical to RANKING no matter what $\mathcal{D}$ is!

# Economic Ranking Example

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0.5

0.4

0.9

0.7

0.3

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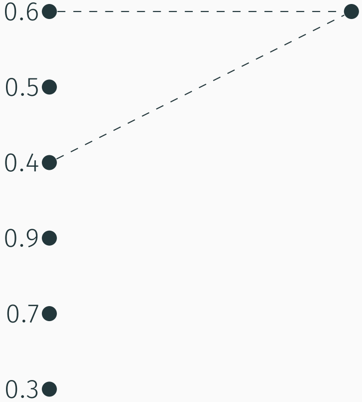

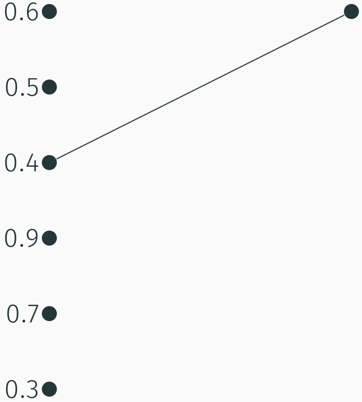
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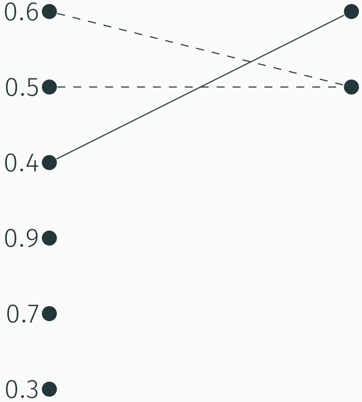

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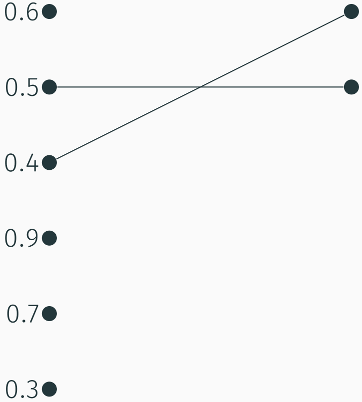

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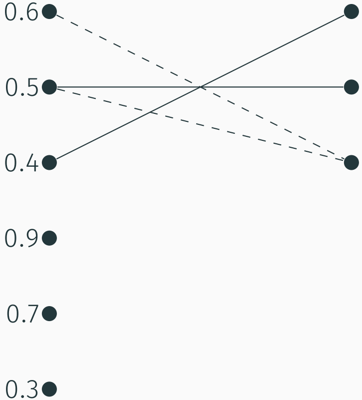

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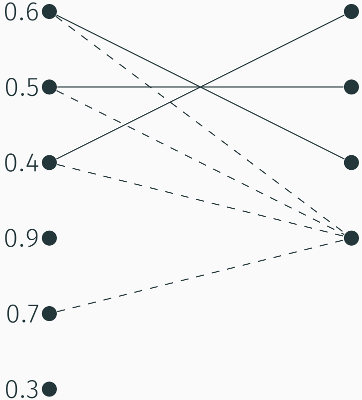
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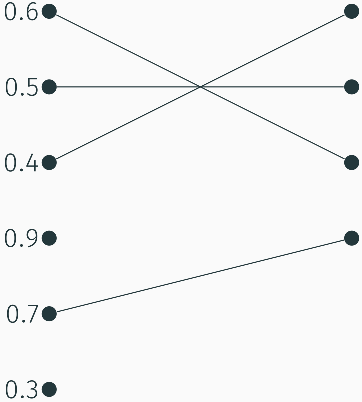

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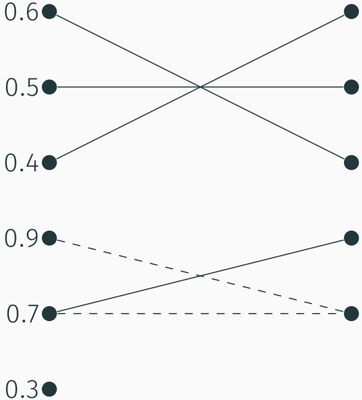

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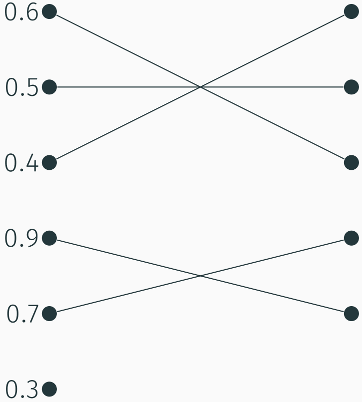

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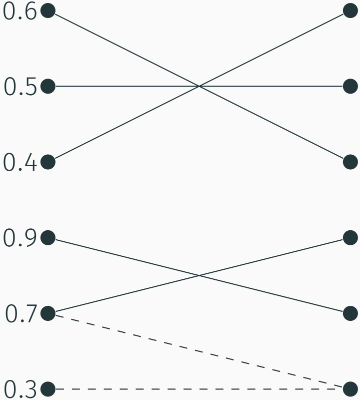

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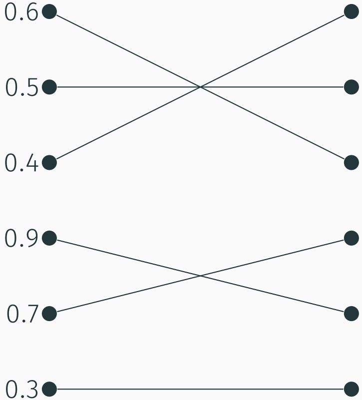

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$$\mathrm{SW} := \sum_{j \in S} r_j + \sum_{i \in B} u_i.$$

$\Rightarrow$ $\mathrm{SW}$ is the size of the matching created by the algorithm!

# Analysis of the Algorithm

**Lemma**

*Let $(j, i) \in E$ be arbitrary and fix all the prices except for $p_j$. Let $u^*$ be the utility of buyer $i$ if seller $j$ were removed. Then:*

1. *No matter what $p_j$ is, $u_i \geq u^*$.*
2. *If $1 - p_j > u^*$, then $j$ will be sold.*

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1. If new sellers enter the market, the utilities of all buyers can only increase. This is called monotonicity of the market.

2. If $j$ was not matched, then $i$ would buy $j$ when they arrive to the market. $\square$

# Analysis of the Algorithm II

Assume that prices are chosen such that $p_j = e^{y_j - 1}$ where $y_j \in [0, 1]$ is uniformly distributed.

**Lemma**

*Let $(j, i) \in E$ be arbitrary. Then*

$$\mathbb{E}[r_j + u_i] \geq 1 - \frac{1}{e}.$$

# Analysis of the Algorithm III

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We know that $j$ will be sold if $1 - p_j > u^*$. Let $y^* := 1 + \ln(1 - u^*)$, then

$$\mathbb{E}[r_j] \geq \int_0^{y^*} e^{y_j - 1} \, \mathrm{d}y_j$$
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# Analysis of the Algorithm III

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Thus $\mathbb{E}[r_j + u_i] \geq 1 - \frac{1}{e}$. $\square$

# Analysis of the Algorithm IV

**Theorem**

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$$\geq \sum_{(j,i) \in M} \left(1 - \frac{1}{e}\right) = \left(1 - \frac{1}{e}\right) |M|. \quad \square$$

# Water-Filling through Continuous Pricing

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$\Rightarrow$ The size of a maximum fractional matching is equal to the size of a maximum integral matching.

$\Rightarrow$ Computing good fractional matchings online is easier than integral ones.

# Economic Water-Filling

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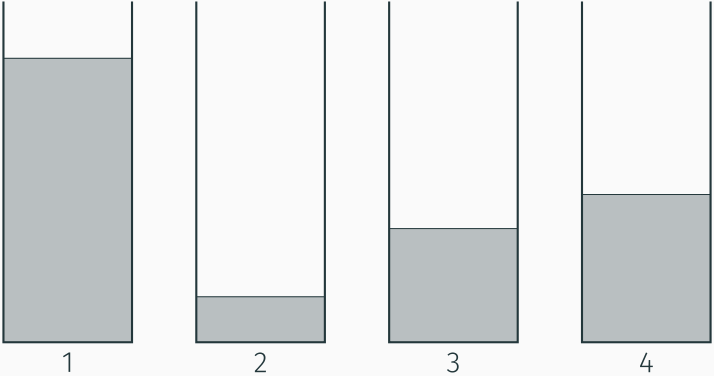
It turns out that $g(w) := e^{w-1}$ is the optimal choice for $g$.

# Water-Filling Example



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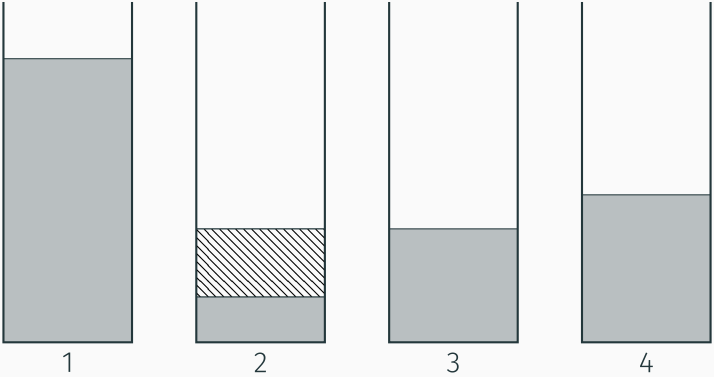

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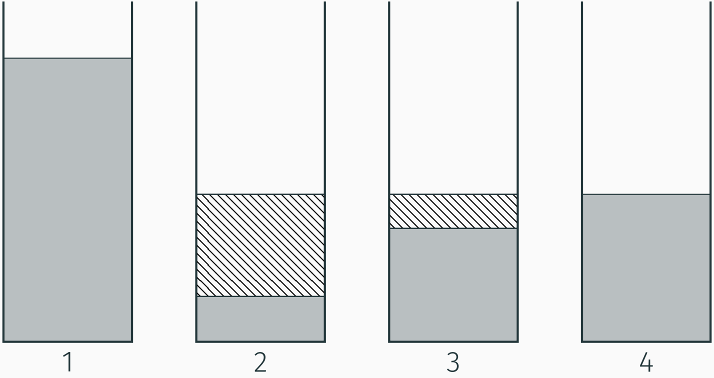

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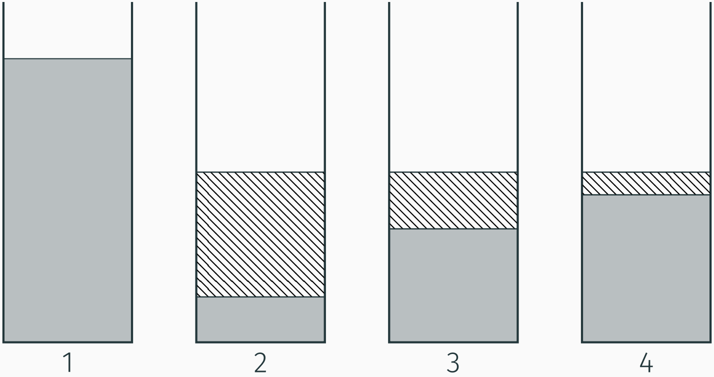

Define revenue and utility as before. Note that social welfare still measures the size of the fractional matching.

**Lemma**

*For any* $(j, i) \in E$*, we have* $r_j + u_i \geq 1 - \frac{1}{e}$ *at the end of the algorithm.*

**Proof.** Let $w$ be the fill-level of seller $j$ at the end of the algorithm. Clearly

$$r_j = \int_0^w e^{t-1}\,\mathrm{d}t = e^{w-1} - \frac{1}{e}.$$

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Thus $r_j + u_i \geq 1 - \frac{1}{e}$. $\square$

## Analysis of WATER-FILLING II

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**Proof.** The proof is identical to Ranking. □

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Both algorithms can be easily generalized to weights on the sellers by scaling both the price and value of each seller by the seller's weight.

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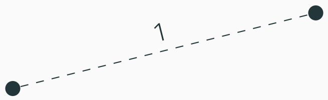

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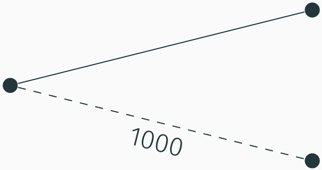
The goal is to maximize the value of matched edges.

Unfortunately, no algorithm has constant competitive ratio:



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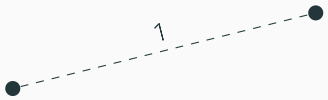

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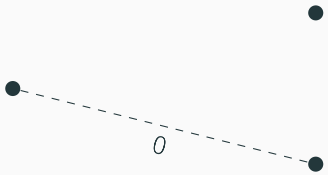

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# Free Disposal

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An online matching problem allows free disposal if the offline vertices are allowed drop previously matched online vertices.

$\Rightarrow$ GREEDY algorithm is now $\frac{1}{2}$-competitive!

$\Rightarrow$ From the economic viewpoint, we allow new buyers to buy out previous buyers!

# Description of the Algorithm

We want to use WATER-FILLING to provide a $(1 - \frac{1}{e})$-competitive algorithm.

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**How do we choose the prices?**

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This is the value contour.

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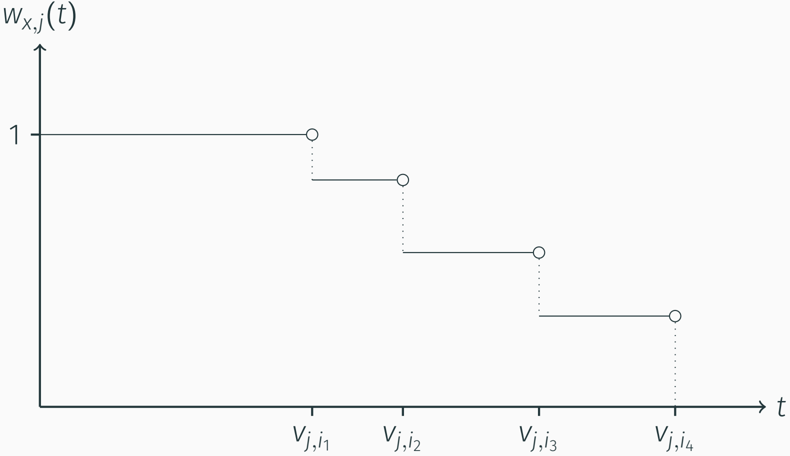

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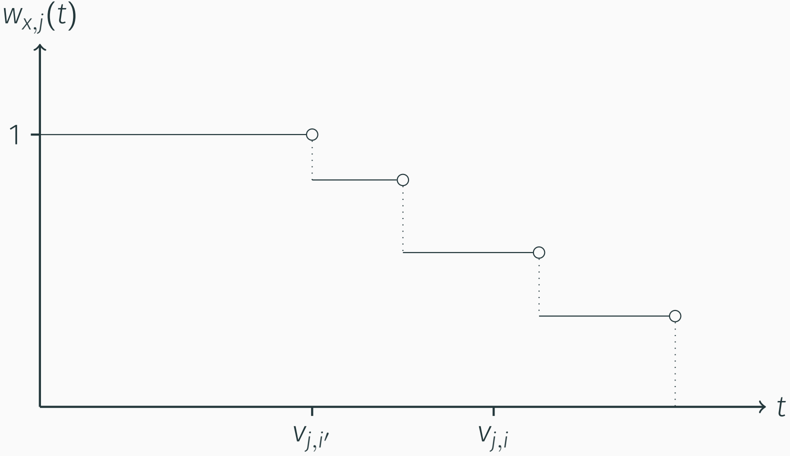

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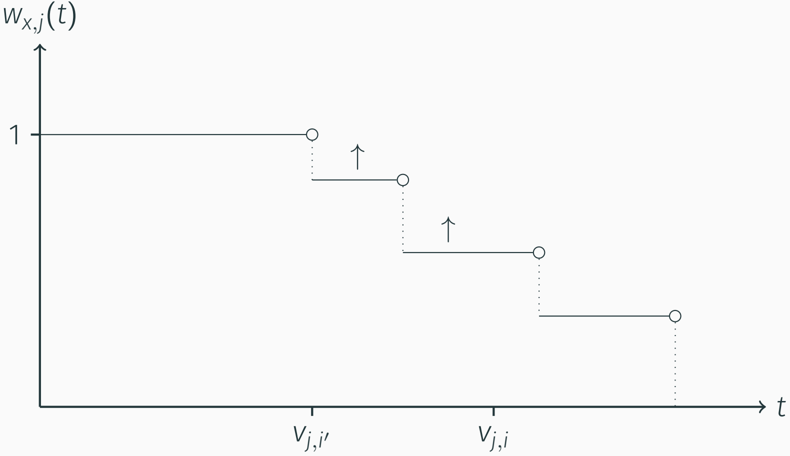

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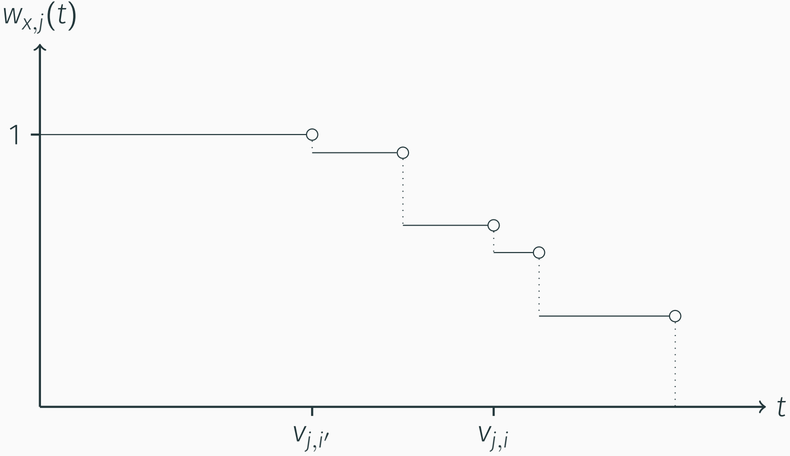

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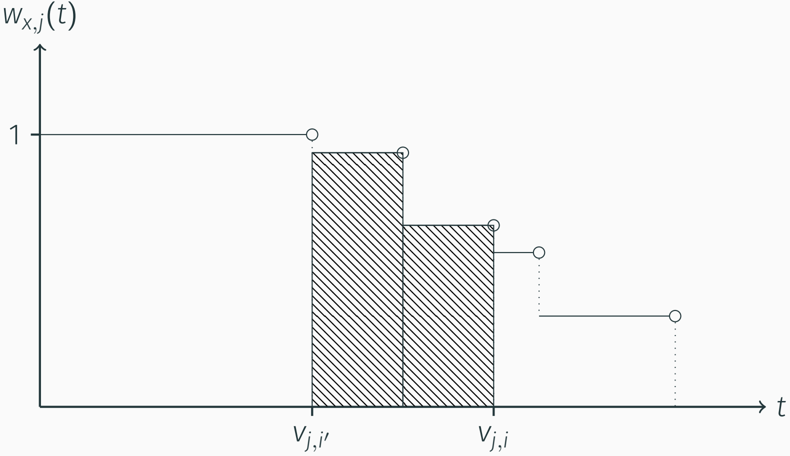

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However, buyer $i$ must also buy out $i'$, so the total price is

$$v_{j,i'} + \int_{v_{j,i'}}^{v_{j,i}} e^{w_{x,j}(t)-1}\,\mathrm{d}t = \int_{0}^{v_{j,i}} e^{w_{x,j}(t)-1}\,\mathrm{d}t.$$

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Note that due to buyouts, utilities never decrease!

# Analysis of the Algorithm

**Lemma**

*For any $(j, i) \in E$, we have $r_j + u_i \geq \left(1 - \frac{1}{e}\right) v_{j,i}$ at the end of the algorithm.*

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**Proof.** Let $x$ be the allocation at the end of the algorithm. Then the price of $j$ for $i$ would have been at most

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at any point in the algorithm.

**Lemma**

*For any* $(j, i) \in E$*, we have* $r_j + u_i \geq \left(1 - \frac{1}{e}\right) v_{j,i}$ *at the end of the algorithm.*

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$$\Rightarrow u_i \geq v_{j,i} - \int_0^{v_{j,i}} e^{w_{x,j}(t)-1}\,\mathrm{d}t.$$

# Analysis of the Algorithm II

However, one can see that

$$r_j = \int_0^\infty \int_0^{w_{x,j}(t)} e^{s-1}\,\mathrm{d}s\,\mathrm{d}t = \int_0^\infty \left( e^{w_{x,j}(t)-1} - \frac{1}{e} \right) \mathrm{d}t.$$

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$$\begin{aligned}\Rightarrow r_j + u_i &\geq v_{j,i} - \int_0^{v_{j,i}} e^{w_{x,j}(t)-1}\,\mathrm{d}t + \int_0^\infty \left(e^{w_{x,j}(t)-1} - \frac{1}{e}\right)\,\mathrm{d}t \\ &\geq v_{j,i} - \int_0^{v_{j,i}} \frac{1}{e} = \left(1 - \frac{1}{e}\right) v_{j,i}. \quad \square\end{aligned}$$

# Analysis of the Algorithm III

**Theorem**

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\mathrm{SW} &= \sum_{j \in S} r_j + \sum_{i \in B} u_i \\
&\geq \sum_{(j,i) \in M} r_j + u_i \\
&\geq \sum_{(j,i) \in M} \left(1 - \frac{1}{e}\right) v_{j,i} = \left(1 - \frac{1}{e}\right) v(M). \quad \square
\end{aligned}$$

# Conclusion

# Final Remarks

The economic viewpoint is a powerful and simple way to think about online matching problems.

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There are still many open problems in online matching theory, especially when it comes to general graphs, wait costs, stochasticity, etc.

I hope that beyond being a useful tool for exposition, the economic viewpoint may inspire solutions to new variants of online matching!

Thank You!