Cardinal-Utility Matching Markets: The Quest for Envy-Freeness, Pareto-Optimality, and Efficient Computability

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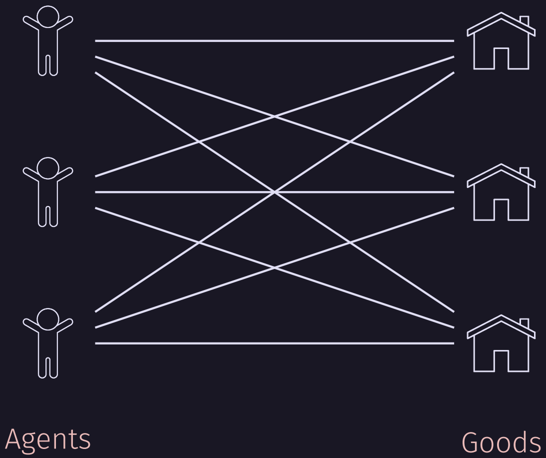
Theory Seminar

February 16, 2024

Cardinal-Utility Matching Markets

Problem Setting



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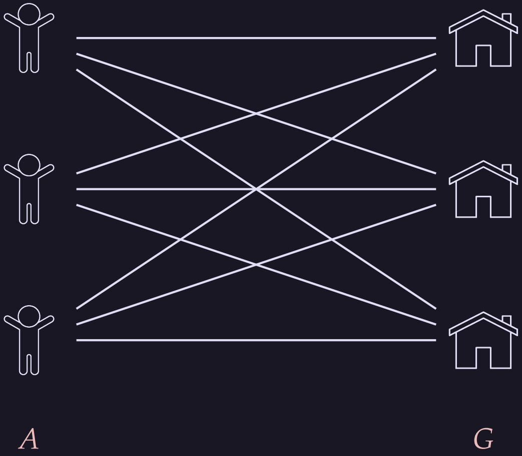

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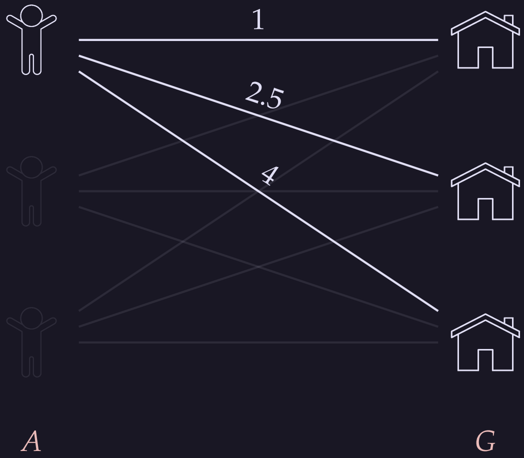

Why Cardinal

Question

Why cardinal utilities instead of ordinal?

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Theorem (Immorlica et al. 2017)

There are matching markets in which cardinal mechanisms can improve the utility of all agents by a $\theta(\log(n))$-factor over ordinal mechanisms.

Question

Why do we allow fractional matchings?

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1. Without, we cannot be fair.

Why Fractional

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1. Without, we cannot be fair.
2. Birkhoff-von-Neumann theorem gives polynomial time lottery.

Definition (Envy-Freeness)

Agent i envies agent i' in allocation x if $u_i \cdot x_i < u_i \cdot x_{i'}$. x is envy-free (EF) if no agent envies another.

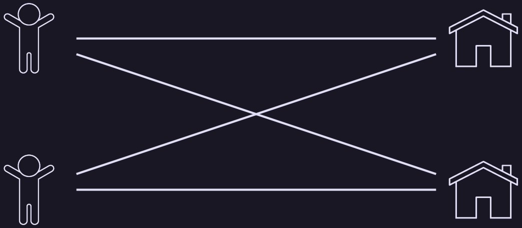
Definition (Utility)

For an agent i, we use

$$u_i \cdot x_i := \sum_{j \in G} u_{ij} x_{ij}$$

to denote the (expected) utility of i.

Envy-freeness alone is trivial: assign goods uniformly!



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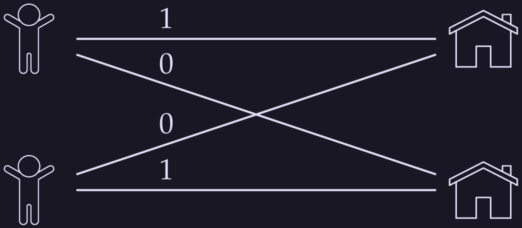

Definition (Pareto-Optimality)

Allocation y is Pareto-better than x, if $u_i \cdot y_i \geq u_i \cdot x_i$ for all i and $u_i \cdot y_i > u_i \cdot x_i$ for at least one i. x is Pareto-optimal (PO) if there is no Pareto-better allocation.

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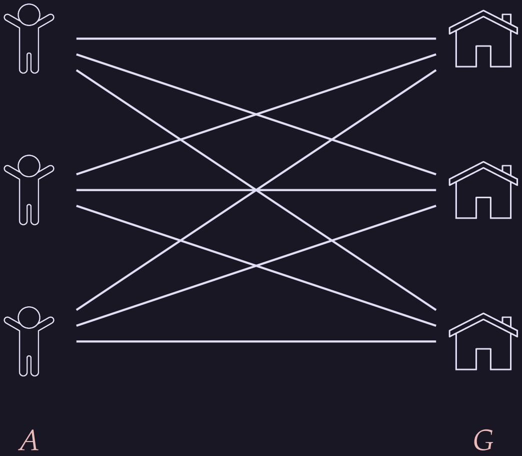
Question

Can we achieve EF and PO at the same time?

Hylland, Zeckhauser 1979 use the power of pricing:



Hylland-Zeckhauser Mechanism

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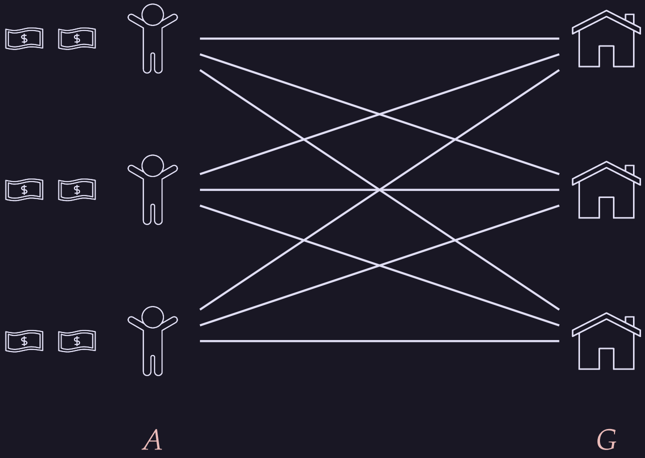

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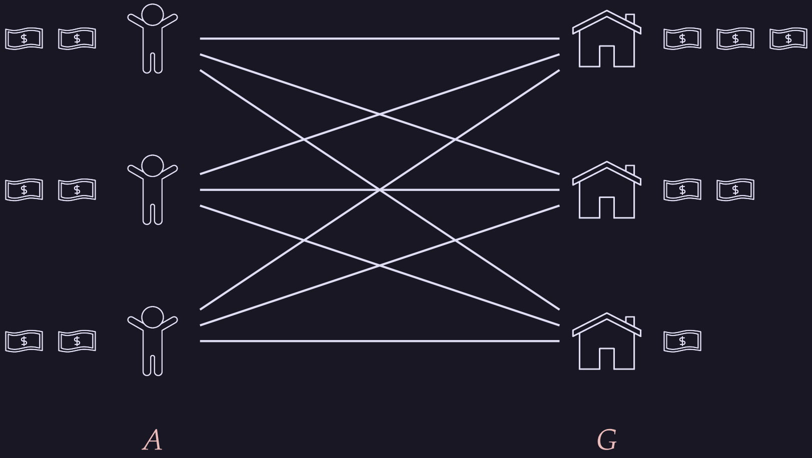

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4. Every agent minimizes expense, i.e. $p \cdot x_i = \min\{p \cdot y \mid \sum_{j \in G} y_j = 1, u_i \cdot y = u_i \cdot x_i\}$.

Theorem (Hylland, Zeckhauser 1979)

An HZ equilibrium always exists. Moreover, if (x, p) is an HZ equilibrium, x is Pareto-optimal and envy-free.

Theorem (He et al. 2018)

The HZ mechanism is incentive-compatible (≈ cannot be gamed by individuals) in the large.

But Wait...

Question

But... how do we actually find an HZ equilibrium?

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1. Hylland-Zeckhauser 1979: Kakutani fixed-point theorem, Scarf's method
2. Alaei et al. 2017: algebraic cell decomposition
3. Vazirani, Yannakakis 2020: DPSV-like algorithm for $\{0, 1\}$-utilities

Intractibility

Theorem (Chen, Chen, Peng, Yannakakis 2022)

The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard when $\epsilon = 1/n^c$ for any constant $c > 0$.

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The problem of computing an ϵ-approximate HZ-equilibrium is PPAD-hard when $\epsilon = 1/n^c$ for any constant $c > 0$.

- PPAD is a class of total search problems with rational solutions.
- Other famous PPAD-complete problems:
  - Nash-equilibrium,
  - Market equilibria with non-linear utilities,
  - Brouwer's fixed-point theorem.

Question

Can we find an envy-free and Pareto-optimal allocation polynomial time?

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Answer

***No**, this is already PPAD-hard!*

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***No**, this is already PPAD-hard!*

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Can we at least get an approximate solution?

Answer

***Yes**, we can get $(2 + \epsilon)$-EF and PO via Nash bargaining!*

PPAD-Hardness

Theorem (Tröbst, Vazirani 2024)

There is a polynomial reduction from $\frac{3}{n}$-approximate HZ to finding EF+PO allocations.

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Strategy:

1. Use the second welfare theorem, to conjure up prices and budgets from Pareto-optimality.

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There is a polynomial reduction from $\frac{3}{n}$-approximate HZ to finding EF+PO allocations.

Strategy:

1. Use the second welfare theorem, to conjure up prices and budgets from Pareto-optimality.
2. Use envy-freeness to show that budgets must be (approximately) equal.

Background

Theorem (Ashlagi, Shi 2016)

In continuum markets, HZ and EF+PO are the same.

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Theorem (Miralles, Pycia 2016)

In large finite markets, HZ and EF+PO need not be approximately the same, even if the markets converge to a continuum market.

Theorem (Second Welfare Theorem)

Under certain conditions, any Pareto-optimal allocation can be supported as a competitive equilibrium for some budgets.

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Under certain conditions, any Pareto-optimal allocation can be supported as a competitive equilibrium for some budgets.

Careful: technically HZ does not satisfy the conditions!

Lemma

Let x be Pareto-optimal, then there are positive $(\alpha_i)_{i \in A}$ such that x maximizes $\sum_{i \in A} \alpha_i u_i \cdot x_i$. α can be found in polynomial time.

Characterization of Pareto-Optimality

Lemma

Let x be Pareto-optimal, then there are positive $(\alpha_i)_{i \in A}$ such that x maximizes $\sum_{i \in A} \alpha_i u_i \cdot x_i$. α can be found in polynomial time.

Proof Sketch. Look at the LP below and apply duality:

$$\begin{aligned}
\max \quad & \sum_{i \in A} u_i \cdot \hat{x}_i & \\
\text{s.t.} \quad & u_i \cdot \hat{x}_i \geq u_i \cdot x_i & \forall i \in A, \\
& \sum_{j \in G} \hat{x}_{ij} = 1 & \forall i \in A, \\
& \sum_{i \in A} \hat{x}_{ij} = 1 & \forall j \in G, \\
& \hat{x}_{ij} \geq 0 & \forall i \in A, j \in G.
\end{aligned}$$

Primal:

$$
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\end{aligned}$$

Dual:

$$\begin{aligned}
\min \quad & \sum_{i\in A} q_i + \sum_{j\in G} p_j \\
\text{s.t.} \quad & q_i + p_j \geq \alpha_i u_{ij} \quad \forall i \in A, j \in G
\end{aligned}$$

Lemma (Optimal Bundles)

For every agent i, x_i is an optimum solution to

$$\begin{aligned} \max \quad & u_i \cdot x_i \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \\ & p \cdot x_i \leq b_i, \\ & x_i \geq 0. \end{aligned}$$

where $b_i := \alpha_i u_i \cdot x_i - q_i$.

Equal Budgets From Envy-Freeness

Lemma

Let $i, i' \in A$ *such that* $u_i = u_{i'}$*. Assume that neither* i *nor* i' *is satiated. Then* $b_i = b_{i'}$*.*

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Proof. Assume otherwise, wlog. $b_i > b_{i'}$.

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Both agents agree, x_i is an optimal bundle at budget b_i.

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i' is not satiated so increasing their budget increases utility.

Thus $u_i x_i > u_i x_{i'}$, i.e. envy! □

Key Idea 1: Almost Equal Budgets From Almost Envy-Freeness

Lemma

Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq \epsilon \max\{\alpha_i, \alpha_{i'}\}$.

Key Idea 1: Almost Equal Budgets From Almost Envy-Freeness

Lemma

Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq \epsilon \max\{\alpha_i, \alpha_{i'}\}$.

Proof Sketch. Substantially higher budget still implies envy since utilities are close.

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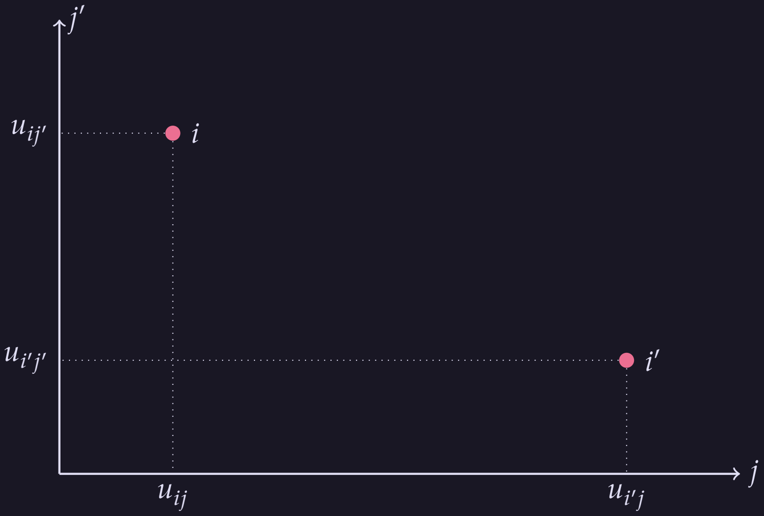
Non-satiation is replaced by dependence on $\max\{\alpha_i, \alpha_{i'}\}$. □

Key Idea 2: Interpolation



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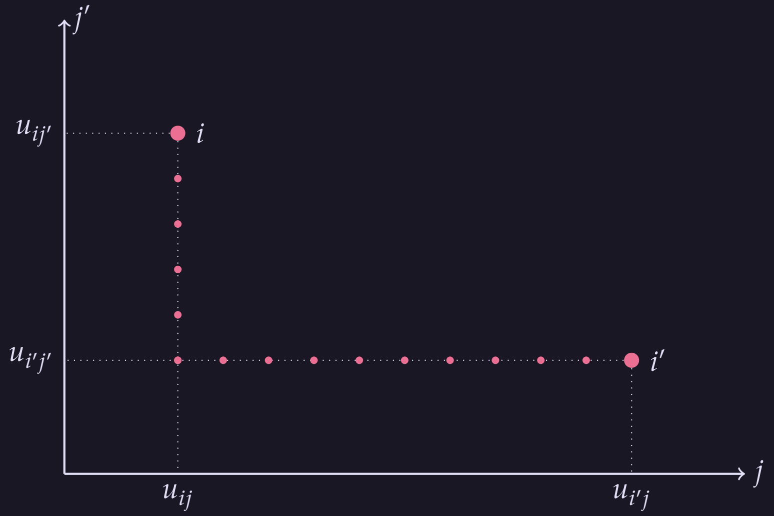
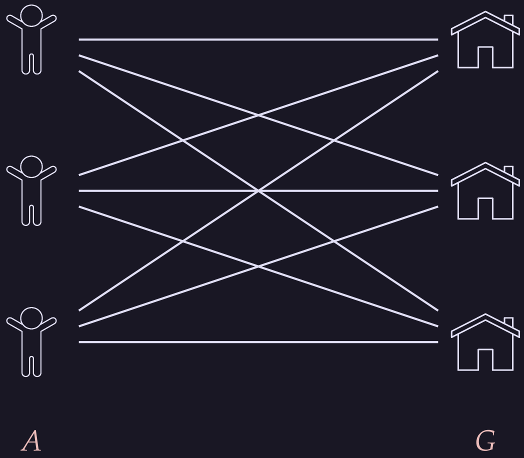


Key Idea 3: Expand the Instance $(k = 4)$



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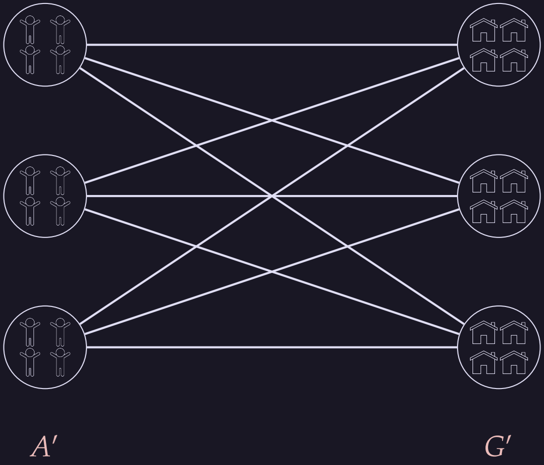

Expanding Works Out

Lemma

If j and j' are goods of the same type, then $p_j = p_{j'}$.

Lemma

If i and i' are agents of the same type, then $b_i = b_{i'}$.

Note: technically need non-satiation - next slide!

Goal: Approximate HZ

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- each good j satisfies $\sum_{i \in A} x_{ij} \in [1 - \epsilon, 1]$,
- no agent overspends, i.e. $p \cdot x_i \leq 1$,
- each agent i gets an almost optimal bundle, i.e.

$$u_i \cdot x_i \geq \max \left\{ u_i \cdot y \;\middle|\; \sum_{j \in G} y_j = 1, p \cdot y \leq 1 \right\} - \epsilon.$$

Add k/n awesome goods with utility 2 for all agents.

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No agent gets 0.6 of any awesome good.

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Proof. Lets say i gets 0.6 of an awesome good. Let $i' \in A$. Then $u_{i'} \cdot x_i \geq 1.2$. So to avoid envy, i' must get 0.2 of an awesome good. Not enough awesome goods for that! □

Corollary

For all $i \in A$, $u_i \cdot x_i \leq 1.6$.

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Lemma

Rescale so that the largest budget is 1. Then, for any i, *we have* $\alpha_i \leq 5n^2$.

Corollary

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Lemma

Rescale so that the largest budget is 1. Then, for any i, we have $\alpha_i \leq 5n^2$.

Corollary

Let $i, i' \in A$ be such that utilities agree up to one good where they differ by at most ϵ. Then $|b_i - b_{i'}| \leq 5n^2\epsilon$.

Question

How many interpolating agents are there between any two normal agents?

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Answer: Up to $\frac{n}{\epsilon}$.

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Answer: Up to $\frac{n}{\epsilon}$.

So $|b_i - b_{i'}| \leq 5n^3$. Completely useless! ☹

Structure of Optimal Bundles

Optimal bundles at budgets t for i are:

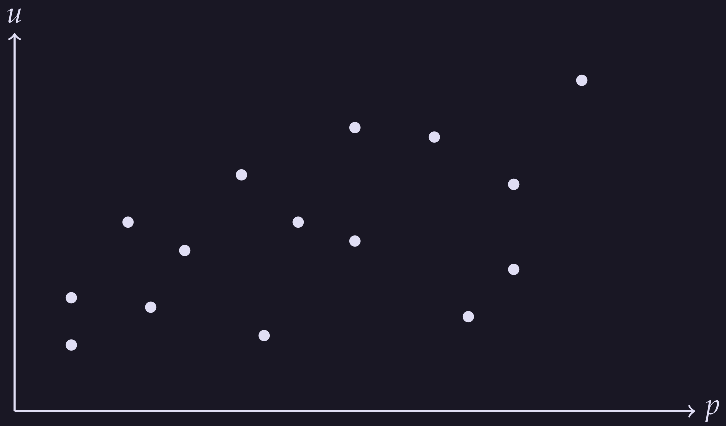
$$\begin{aligned} \max \quad & u_i \cdot x_i \\ \text{s.t.} \quad & \sum_{j \in G} x_{ij} \leq 1, \\ & p \cdot x_i \leq t, \\ & x_i \geq 0. \end{aligned}$$

Structure of Optimal Bundles II

The dual is the key:

$$\begin{aligned} \min \quad & \mu + \rho t \\ \text{s.t.} \quad & \mu + p_j \rho \geq u_{ij}, \\ & \mu, \rho \geq 0. \end{aligned}$$

Geometry of Optimal Bundles



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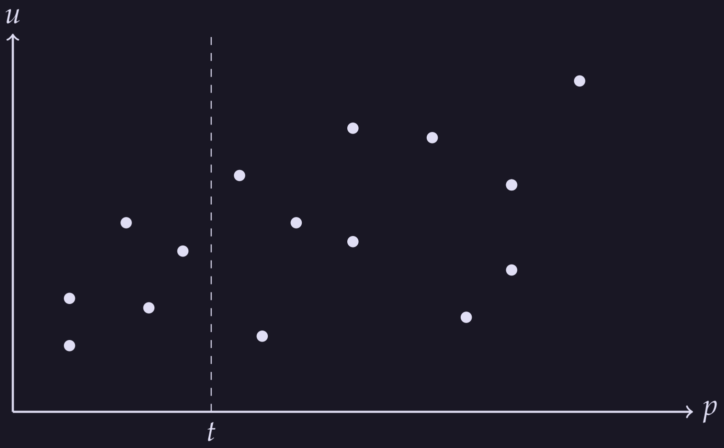

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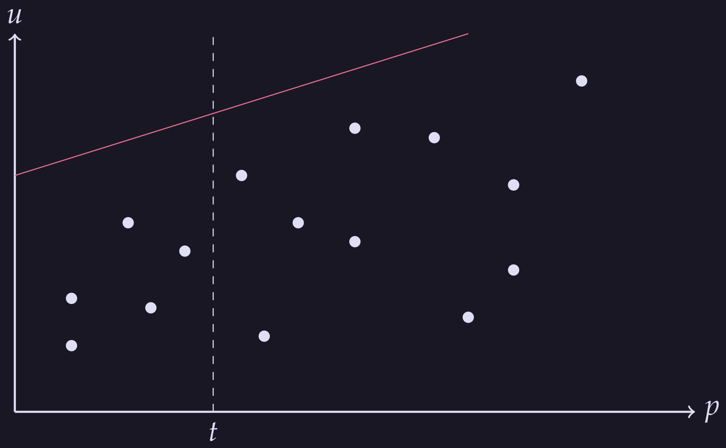

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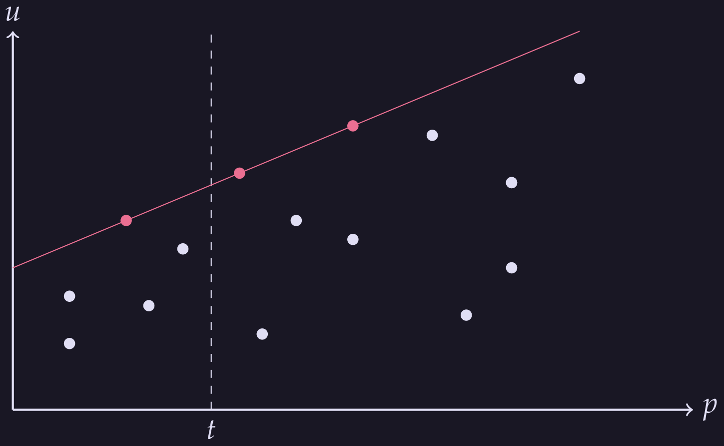

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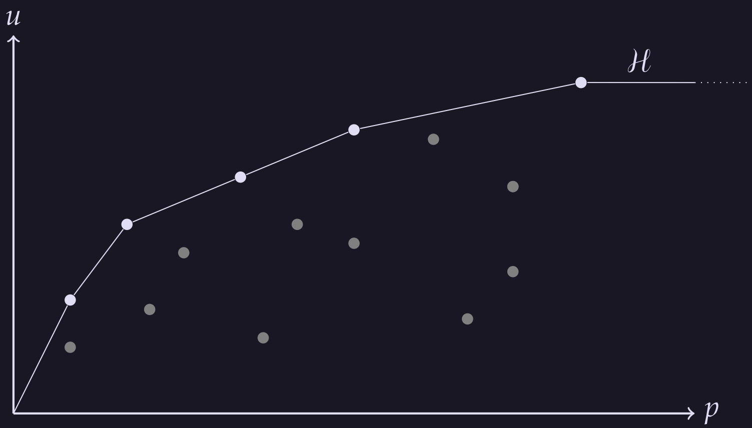

Optimal Bundle Function

Definition (Optimal Bundle Function)

For $i \in A$ and $t \geq 0$ define:

$$\theta_i(t) := \{j \in G \mid j \text{ can be in optimum bundle at budget } t\}$$

Lemma

Let $i, i' \in A$ be such that $\theta_i = \theta_{i'}$, then $b_i = b_{i'}$.

Definition (Optimal Bundle Function)

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Lemma

Let $i, i' \in A$ be such that $\theta_i = \theta_{i'}$, then $b_i = b_{i'}$.

Proof Sketch. Assume otherwise and wlog. $b_i > b_{i'}$. Can use $\theta_i = \theta_{i'}$ to show that x_i is optimum bundle for i' at budget $b_{i'}$. Causes envy due to non-satiation! □

Key Idea 5: θ_i Rarely Changes

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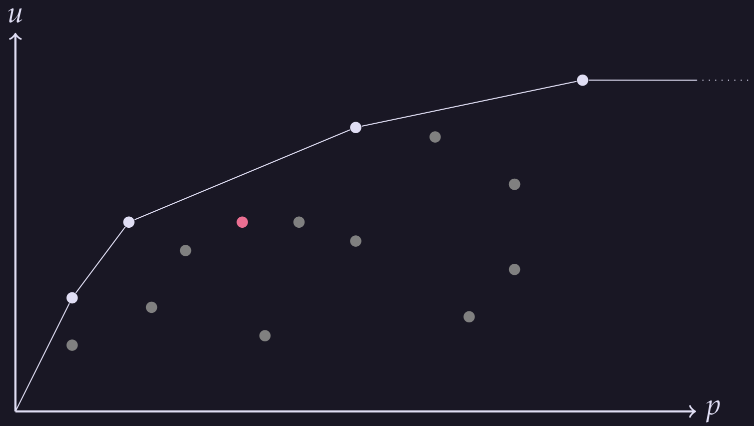

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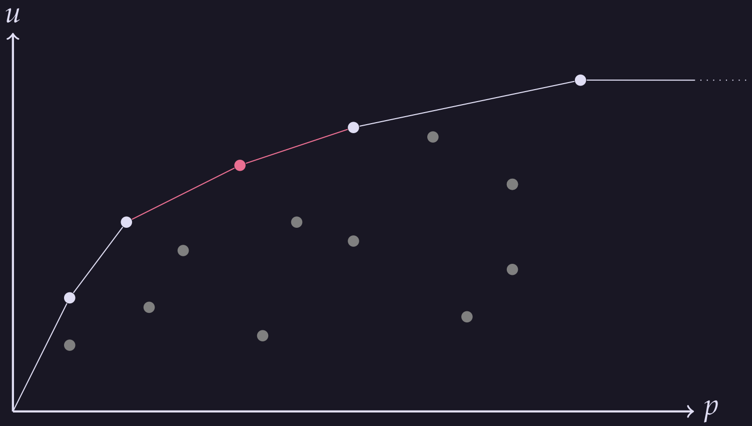

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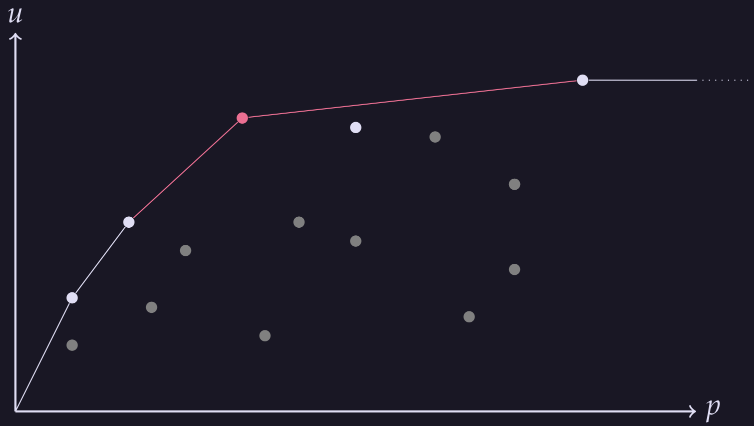

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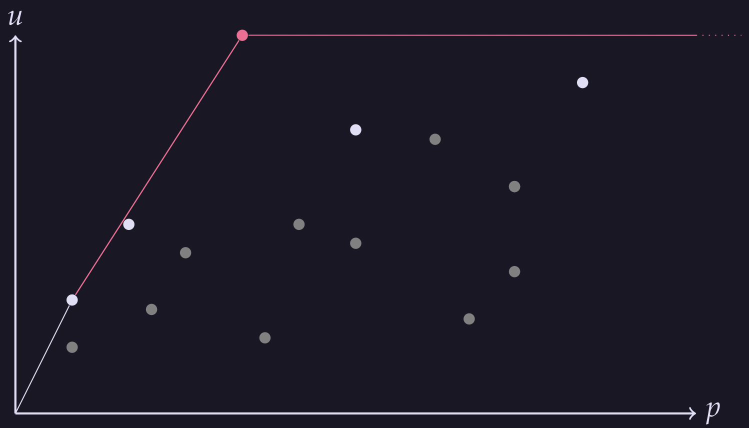

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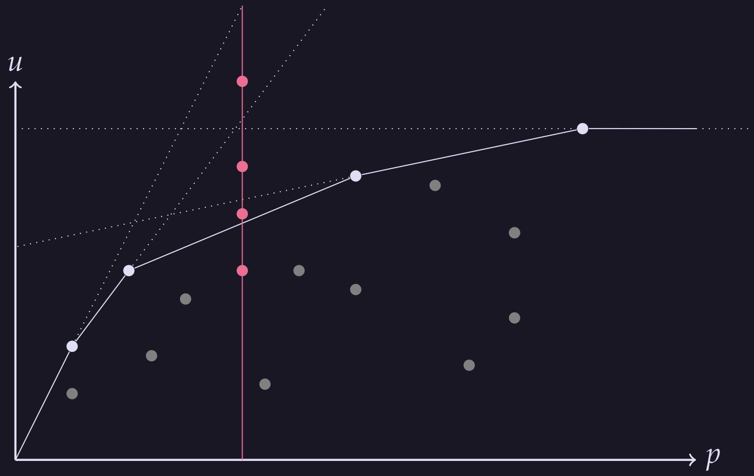

Lemma

Let $i_1, \dots, i_m$ be a set of agents such that all agents agree on all utilities except for possibly one type of good. Then $|\{\theta_{i_1}, \dots, \theta_{i_m}\}| \leq 2n + 1$.

Lemma

Let $i_1, \ldots, i_m$ *be a set of agents such that all agents agree on all utilities except for possibly one type of good. Then* $|\{\theta_{i_1}, \ldots, \theta_{i_m}\}| \leq 2n + 1$.

Lemma

Let $i, i' \in A$, *then* $|b_i - b_{i'}| \leq 5\epsilon n^4$.

Lemma

Let $i_1, \ldots, i_m$ be a set of agents such that all agents agree on all utilities except for possibly one type of good. Then $|\{\theta_{i_1}, \ldots, \theta_{i_m}\}| \leq 2n + 1$.

Lemma

Let $i, i' \in A$, then $|b_i - b_{i'}| \leq 5\epsilon n^4$.

Proof. Between two agents, at most $2n^2$ changes can happen. Each contributes at most $5\epsilon n^2$. $\square$

Bringing It Together II

Theorem

If $\epsilon \le \frac{1}{5n^5}$ *and* $k = \frac{n^3}{\epsilon}$, *then* (x, p) *is a* $\frac{3}{n}$*-approximate HZ equilibrium in the original instance.*

Theorem

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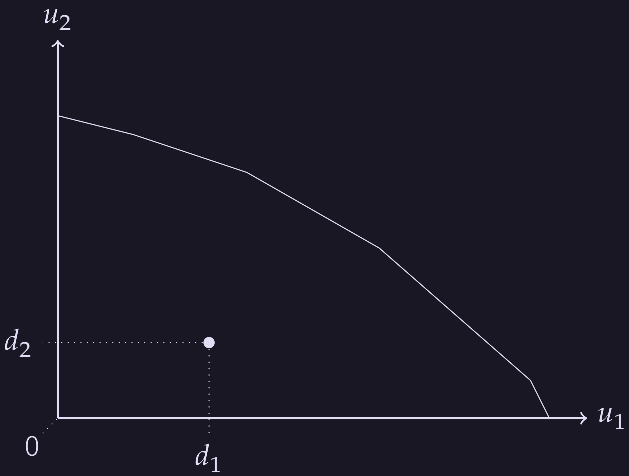
Theorem

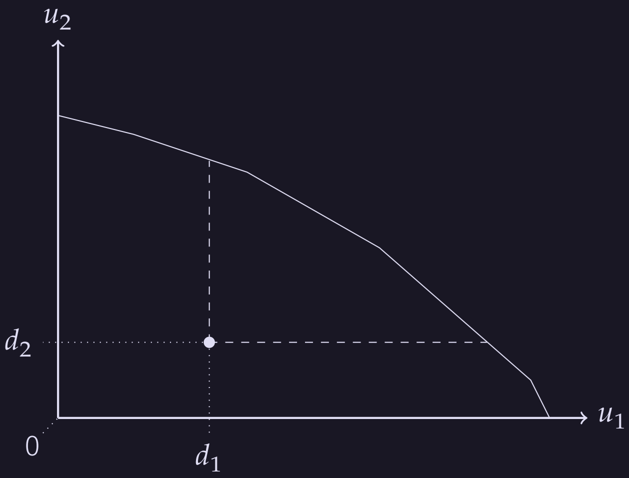
The problem of finding an EF+PO allocation in one-sided cardinal-utility matching market is PPAD-complete.

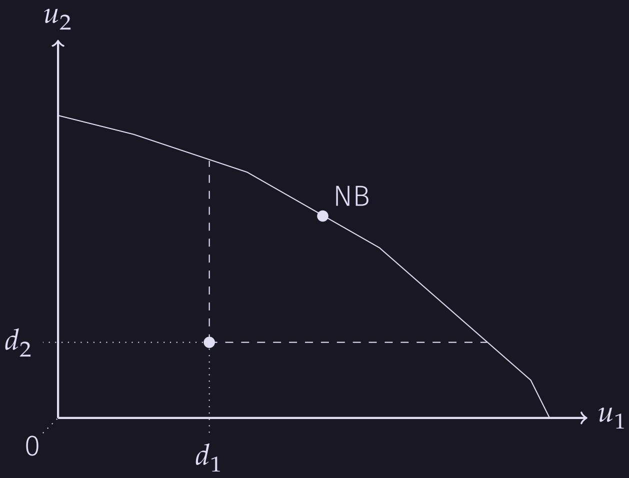
Nash Bargaining

Nash Bargaining Point



Nash Bargaining Point




u_2
NB
d_2
u_1
0
d_1

Theorem (Nash 1950)

Let U, set of utility vectors, be convex. Then

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    - *invariance under affine transformations,*
    - *independence of irrelevant alternatives.*

Theorem (Nash 1950)

Let U, set of utility vectors, be convex. Then

1. *There is a unique point satisfying certain axioms:*
   - *Pareto-optimality,*
   - *symmetry,*
   - *invariance under affine transformations,*
   - *independence of irrelevant alternatives.*
2. *It is the maximizer of $\prod_{i \in A}(u_i - d_i)$ for $u \in U$.*

Nash Bargaining Convex Program

Hosseini, Vazirani 2021: Let's use this for matching markets!

$$\begin{aligned} \max_{x} \quad & \sum_{i \in A} \log(u_i(x)) \\ \text{s.t.} \quad & \sum_{i \in A} x_{ij} \leq 1 \quad \forall j \in G, \\ & \sum_{j \in A} x_{ij} \leq 1 \quad \forall i \in A, \\ & x \geq 0. \end{aligned}$$

Theorem (Tröbst, Vazirani 2024)

If x is a Nash bargaining solution, then x is 2-envy-free.

Definition (Approximate Envy-Freeness)

An allocation x is α-envy-free if $u_i \cdot x_i \geq \frac{1}{\alpha} u_i \cdot x_{i'}$ for all $i, i' \in A$.

Envy-Freeness of Nash Bargaining II

Proof. Assume otherwise, i.e. there are $i, i' \in A$ with $u_i \cdot x_{i'} \geq (2 + \epsilon) u_i \cdot x_i$.

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Proof. Assume otherwise, i.e. there are $i, i' \in A$ with $u_i \cdot x_{i'} \geq (2+\epsilon) u_i \cdot x_i$. Now exchange a δ fraction of x_i and $x_{i'}$.

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Agent i: $u_i \cdot x_i \to (1-\delta) u_i \cdot x_i + \delta(2+\epsilon) u_i \cdot x_i$.

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Product of utilities changes by factor

$$(1-\delta+\delta(2+\epsilon))(1-\delta).$$

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Product of utilities changes by factor

$$(1 - \delta + \delta(2+\epsilon))(1-\delta).$$

Positive derivative at $\delta = 0$, so x was not optimal! $\square$

Theorem (Tröbst, Vazirani 2024)

If x is within $(1+\epsilon)$ of an optimum Nash bargaining point, then x is $(2+3\sqrt{\epsilon})$-envy-free.

Envy-Freeness of Nash Bargaining III

Theorem (Tröbst, Vazirani 2024)

If x is within $(1+\epsilon)$ of an optimum Nash bargaining point, then x is $(2+3\sqrt{\epsilon})$-envy-free.

Theorem (Panageas, Tröbst, Vazirani 2021)

A $(1+\epsilon)$-approximate Nash bargaining point can be found in polynomial time (and efficient in practice).

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This mostly resolves the question of EF+PO allocations in one-sided cardinal-utility matching markets.

- Can we beat 2-EF + PO?
- Can we get 1-EF + α-PO?
- What about two-sided markets?

Thank your for listening!