Multiplicative Auction Algorithm for Approximate Maximum Weight Bipartite Matching[1]

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Theory Seminar, February 10, 2023

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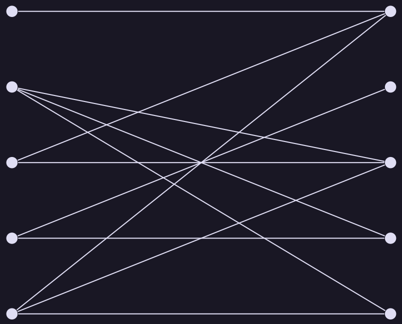
[1] by Da Wei Zheng and Monika Henzinger, to appear in IPCO 2023

Maximum Weight Bipartite Matching



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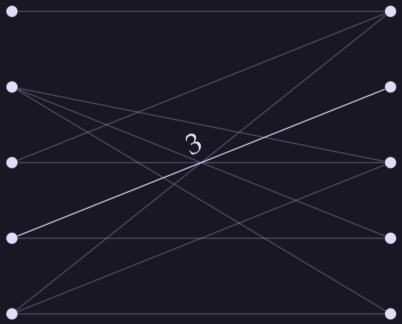

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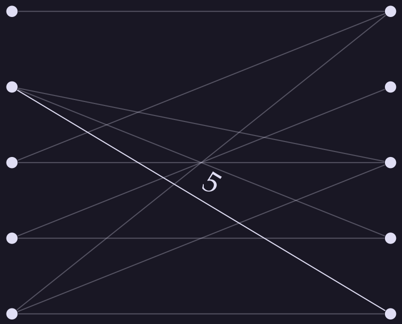

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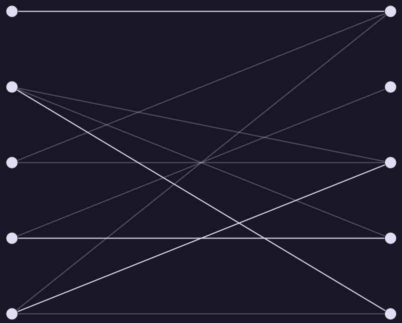

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Algorithms for Matchings and Flows

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  - Chen, Kyng, Liu, Peng, Gutenberg, Sachdeva 2022 $O(m^{1+o(1)})$ for Max Flow

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Zheng and Henzinger achieve:

- $(1-\epsilon)$-apx in $O(m\epsilon^{-1}\log(\epsilon^{-1}))$ for Weighted Bipartite Maximum Matching with a much simpler algorithm
- Algorithm is based on multiplicative weights but beats traditional ϵ^{-2} barrier
- Dynamic edge deletions and one-sided vertex insertions in $O(\epsilon^{-1}\log(\epsilon^{-1}))$ time per edge (amortized)

Recall the primal and dual LPs for Bipartite Maximum Weight Matching on $(G \cup B, E)$.

Primal and Dual LP

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$$\begin{aligned} \max \quad & \sum_{e \in E} w_e x_e \\ \text{s.t.} \quad & x(\delta(j)) \leq 1 \ \forall j \in G, \\ & x(\delta(i)) \leq 1 \ \forall i \in B, \\ & x \geq 0. \end{aligned}$$

$$\begin{aligned} \min \quad & \sum_{j \in G} p_j + \sum_{i \in B} u_i \\ \text{s.t.} \quad & p_j + u_i \geq w_{ij} \forall \{j, i\} \in E, \\ & p \geq 0, \\ & q \geq 0. \end{aligned}$$

Complementary Slackness

Lemma (Complementary Slackness)

Let x be a matching and p, u dual variables such that:

- *If $p_j > 0$, then j is matched.*
- *If $u_i > 0$, then i is matched.*
- *If i is matched to j then $w_{ij} = p_j + u_i$.*
- *For all $\{i, j\} \in E$, $p_j + u_i \geq w_{ij}$.*

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Proof. Complementary slackness.

Approximate Complementary Slackness

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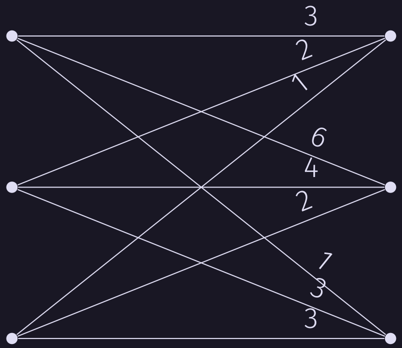
- *For all $\{i, j\} \in E$, $p_j + u_i \geq (1 - \epsilon) w_{ij}$.*

Then x is a $(1 - \epsilon)$-approximate maximum weight matching.

Additive Auction Example



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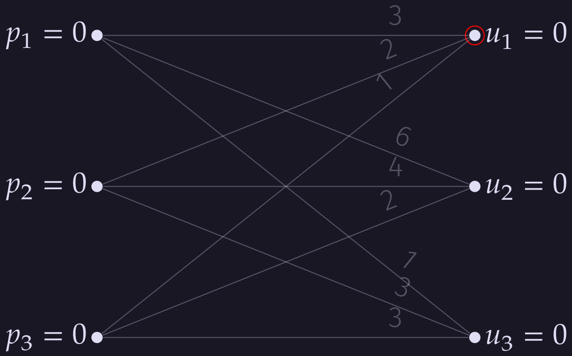

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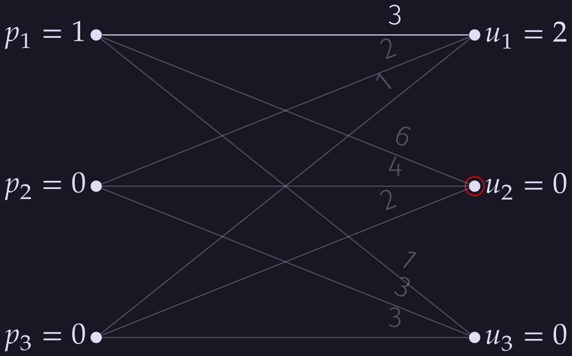

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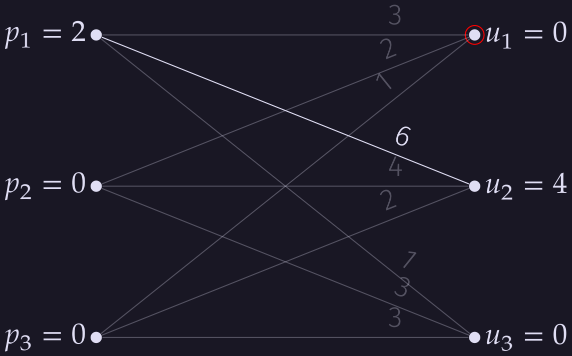

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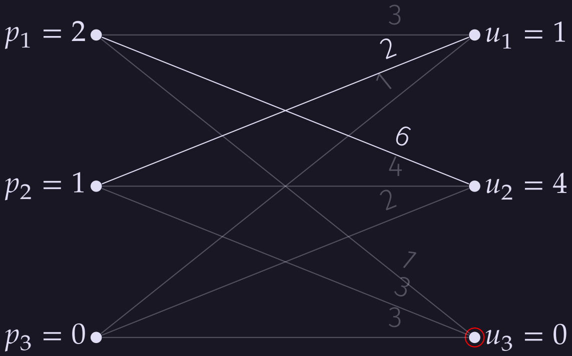

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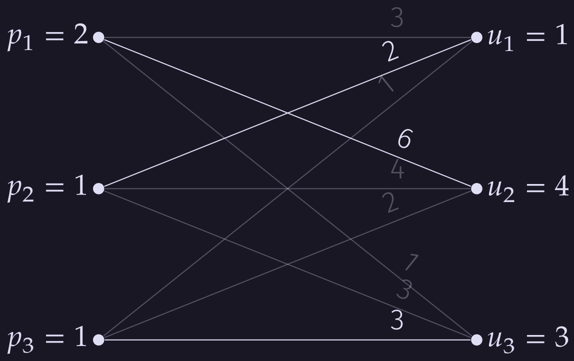

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- If $p_j > 0$, then j is matched.
- If $u_i > 0$, then i is matched.
- If $\{i, j\} \in M$, then $p_j + u_i = w_{ij}$.
- If for some i, we have $p_j + u_i \geq (1 - \epsilon)w_{ij}$ for all j at the time that i was matched, then this continues to hold until the match is destroyed.

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- Ensure that prices rise fast enough to get a good runtime.

Simplifying Assumptions

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- $\frac{w_{\max}}{w_{\min}} \leq \frac{n}{\epsilon}$
- Each w_{ij} is of the form $(1+\epsilon)^{l_{ij}}$ for some $0 \leq l \leq log_{1+\epsilon}(n/\epsilon)$.

Multiplicative Auction

Algorithm 1: Multiplicative Auction

Create a list of pairs Q.

For each $\{i, j\} \in E$, add triples (t, i, j), $(t + 1, i, j)$, ..., (l_{ij}, i, j) to Q where t is maximal such that $(1 + \epsilon)^{l_{ij} - t} > \frac{1}{\epsilon}$.

Sort Q in non-increasing order using bucket sort.

For each i, let $Q_i = \{(k, j) \mid (k, i, j) \in Q\}$.

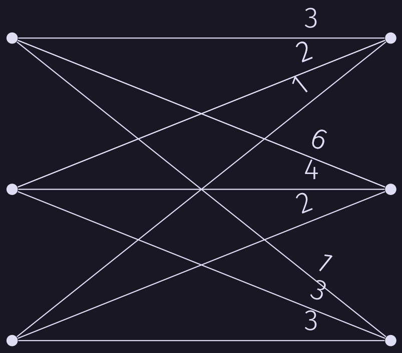
Call $\textsc{Match}(i)$ on unmatched i until the matching stabilizes.

MATCH(i)

Algorithm 2: MATCH(i)

```
while Q_i is not empty do
    Pop top element (k, j) from Q.
    u_ij := w_ij − p_j
    if u_ij ≥ (1 + ε)^k then
        Match i to j (unmatching previous partner).
        p_j ← p_j + ε u_ij
```

Multiplicative Auction Example $\epsilon = 1/3$



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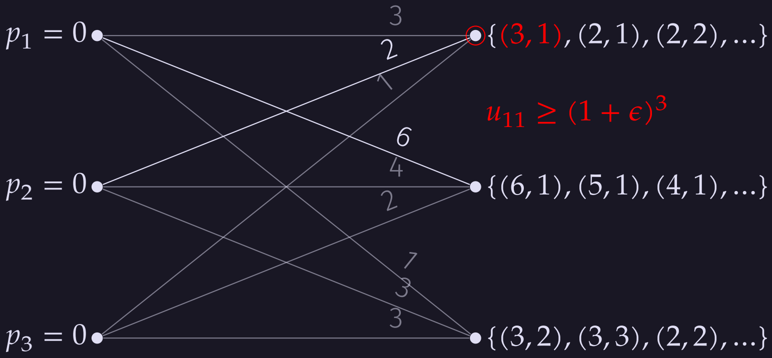

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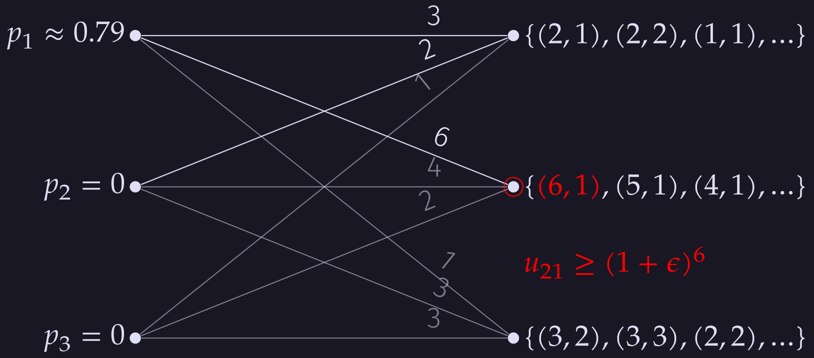

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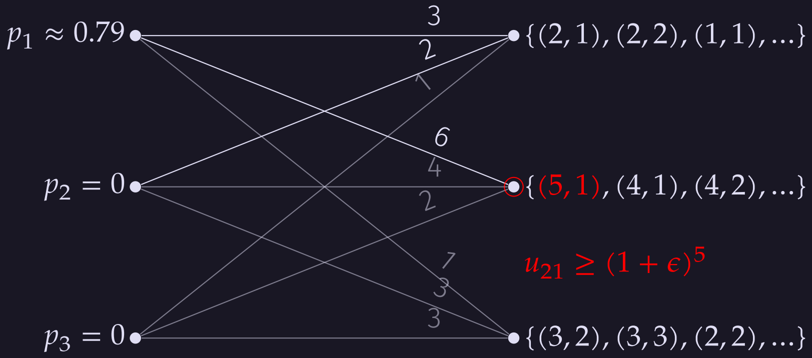

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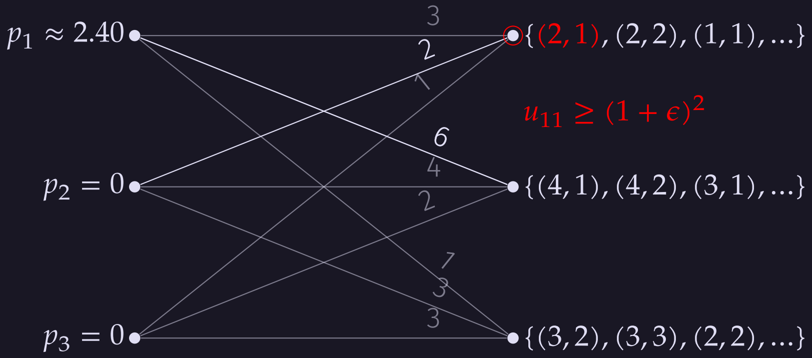

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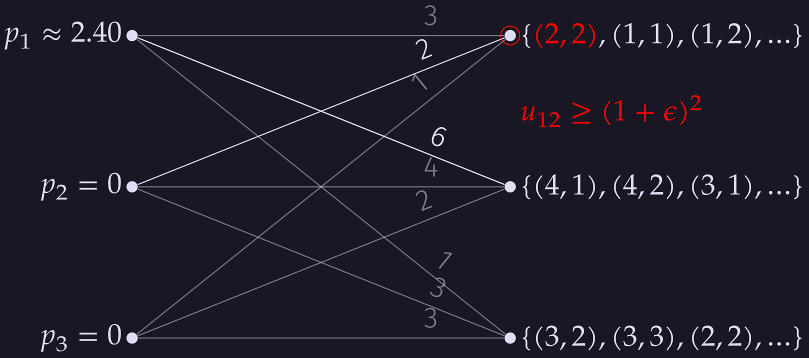

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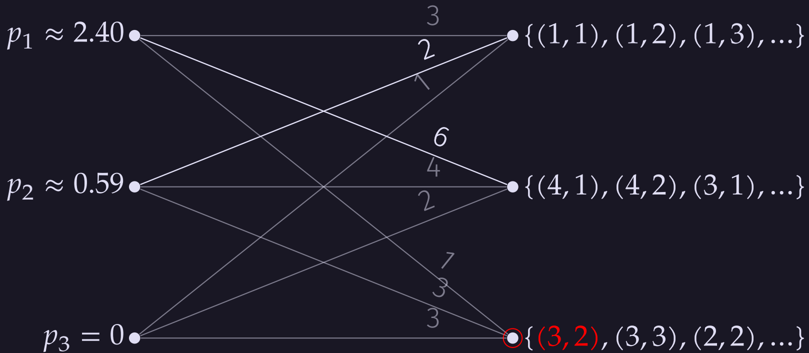

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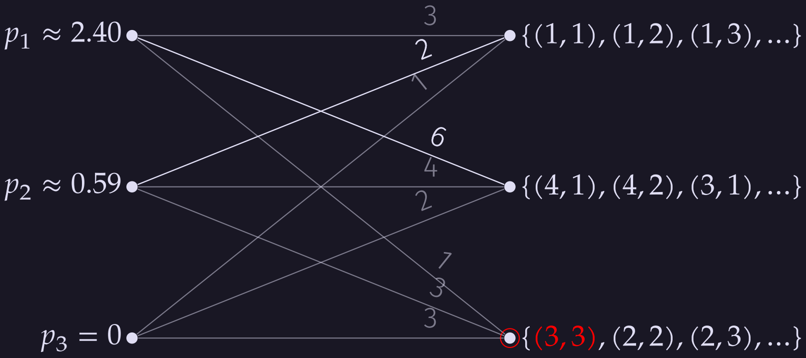

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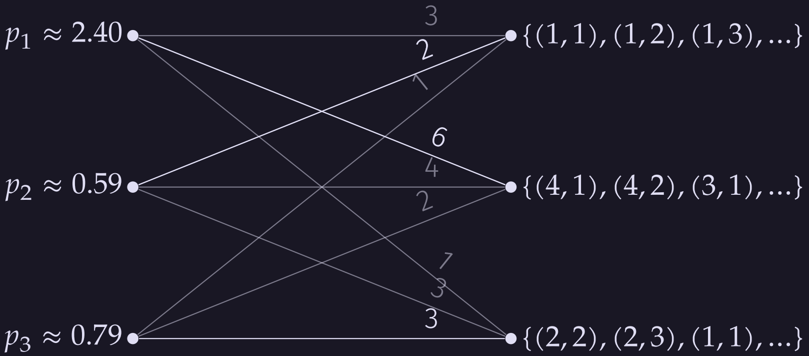

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So, after matching i to j:

$$\begin{aligned} u_i + p_{j'} = u_{ij} + w_{ij'} - u_{ij'} &\geq \frac{1-\epsilon}{1+\epsilon} u_{ij'} + w_{ij'} - u_{ij'} \\ &\geq (1-2\epsilon)u_{ij'} + w_{ij'} - u_{ij'} = w_{ij'} - 2\epsilon u_{ij'} \\ &\geq (1-2\epsilon) w_{ij'}. \end{aligned}$$

Correctness II

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So for every j, we know $u_{ij} < \epsilon w_{ij}$ because we removed (t, j) and $(1+\epsilon)^t < \epsilon(1+\epsilon)^{l_{ij}} = \epsilon w_{ij}$.

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Thus:

$$u_i + p_j = w_{ij} - u_{ij} \geq (1-\epsilon) w_{ij}. \quad \square$$

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Thank your for listening!