Online Matching with High Probability

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Randomized Algorithms and Concentration

The Power of Randomized Algorithms

Many problemss in computer science can be solved in a more natural, efficient, or better way using randomization. You all know many examples such as:

- Quicksort
- Miller-Rabin primality test
- Hashing
- Polynomial identity testing
- Perfect matching on parallel machines
- etc...

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- Fewer know: $\mathbb{P}[C > c_0 \cdot n \log n] < \frac{1}{n}$ for some c_0.
- But did you know:

$$\mathbb{P}[|C/\mathbb{E}[C] - 1| > \epsilon] < n^{-2\epsilon(\ln \ln n - \ln(1/\epsilon) + O(\ln \ln \ln n))}$$

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However, outside of a few areas (e.g. graph coloring), concentration results are relatively rare because of boosting:

- Want good performance? Simply run the algorithm $O(\log n)$ many times.
- Want good runtime? Simply run the algorithm $O(\log n)$ many times in parallel.

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Should be more results like this!

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See "Concentration of Measure for the Analysis of Randomized Algorithms" by Dubhashi and Panconesi

Online Bipartite Matching

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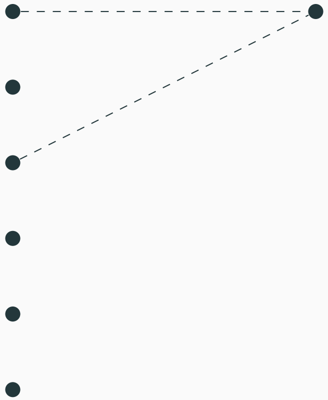

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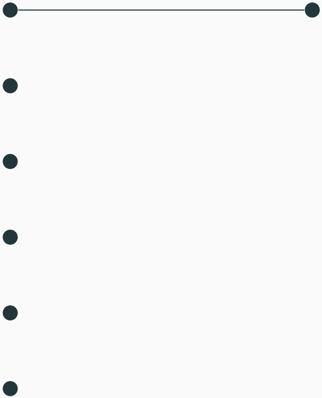

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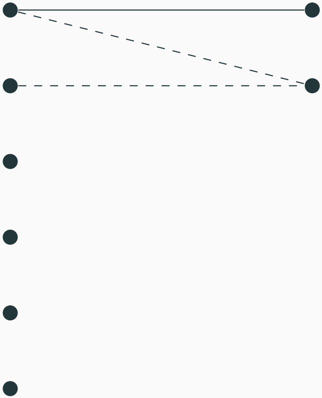

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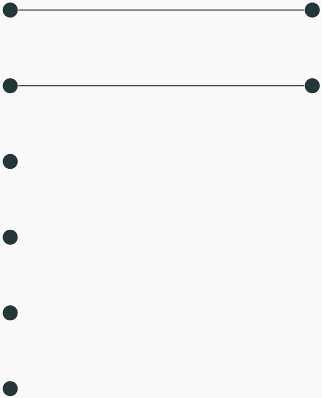

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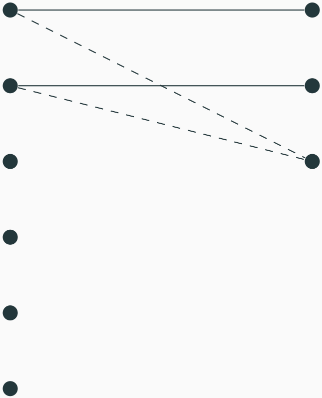

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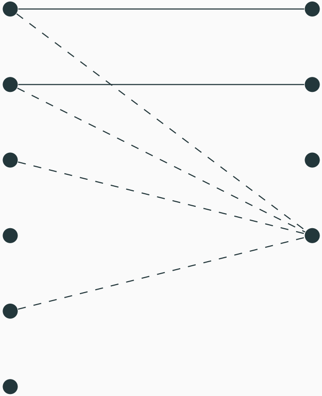

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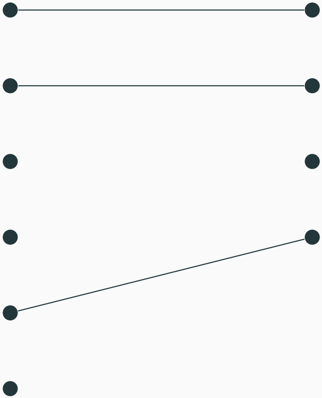

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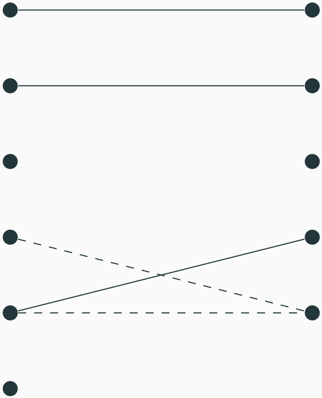

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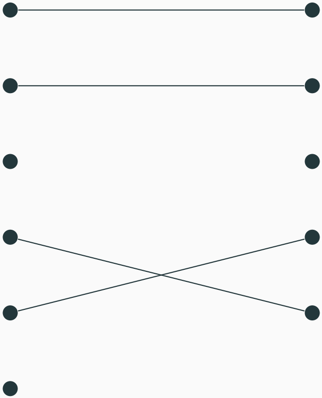

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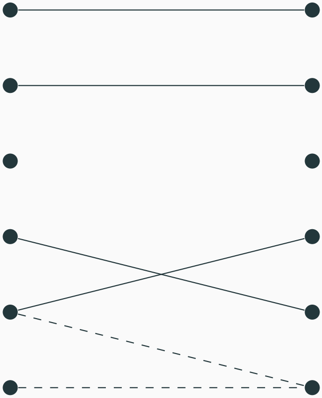

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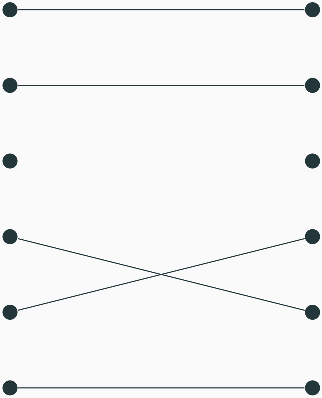

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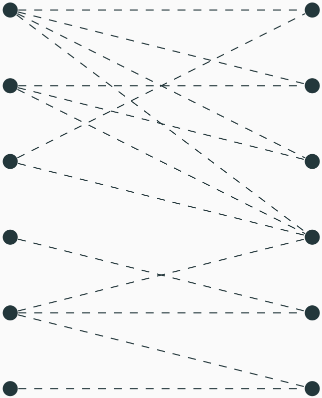

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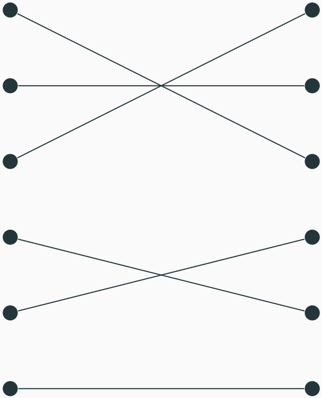

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The goal is to maximize the competitive ratio, i.e.

$$\frac{|M_{\text{online}}|}{\text{OPT}_{\text{offline}}}.$$

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- $(1 - 1/e)$-competitive in expectation is best possible for randomized algorithms.

Ranking with High Probability

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RANKING Example

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0.5

0.4

0.9

0.7

0.3

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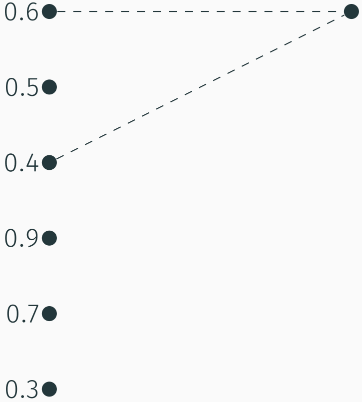

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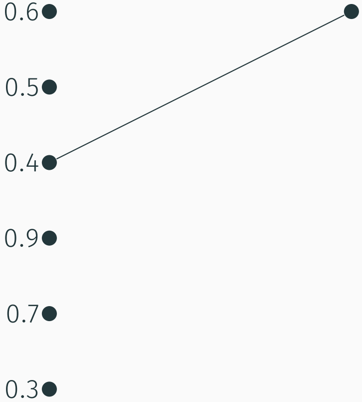

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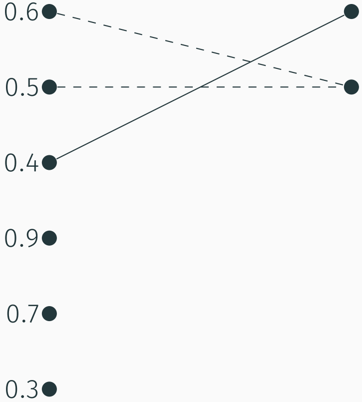

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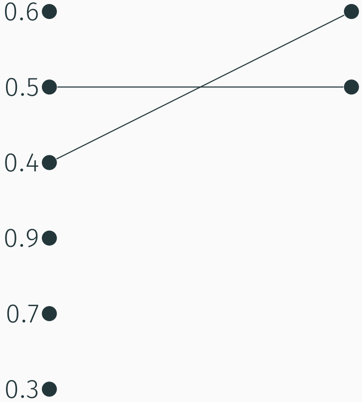

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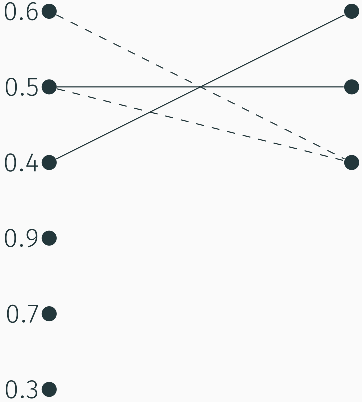

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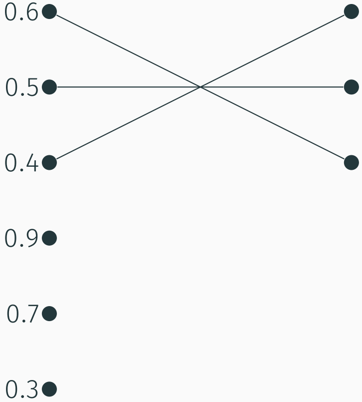

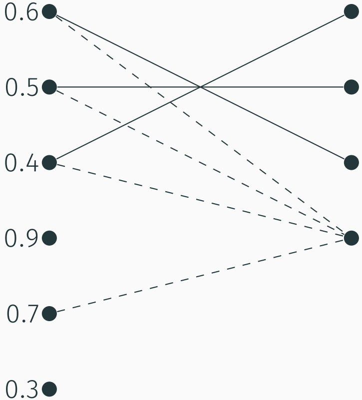
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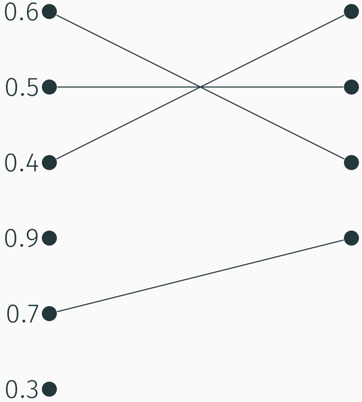

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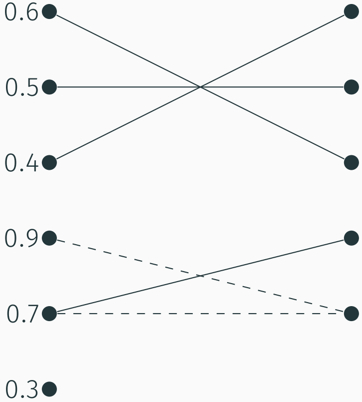

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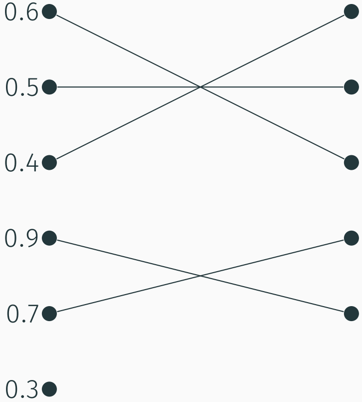

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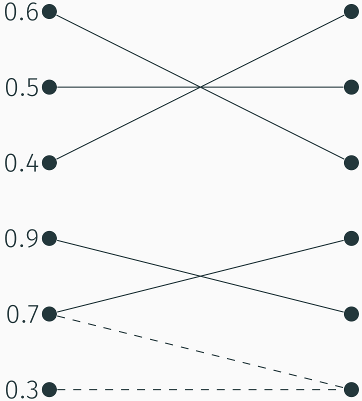

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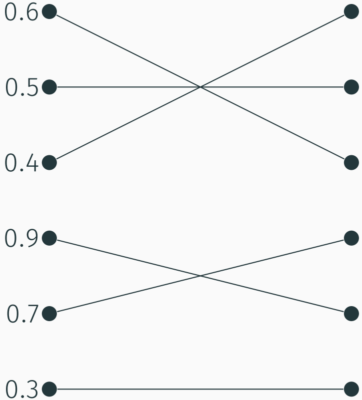

Main Theorem

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Consider an instance (S, B, E) of the Biparite Online Matching Problem which admits a matching of size n. Then for any $\alpha > 0$ and any arrival order,

$$\mathbb{P}\left[|M| < \left(1 - \frac{1}{e} - \alpha\right) n\right] < e^{-2\alpha^2 n}$$

where M is the random variable denoting the matching generated by Ranking.

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where M is the random variable denoting the matching generated by RANKING.

For now assume that n is also the number of offline / online vertex (i.e. there is a perfect matching).

McDiarmid's Inequality

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Lemma (McDiarmid 1989)

Let $c_1, \ldots, c_n \in \mathbb{R}_+$ and consider some function $f : [0,1]^n \to \mathbb{R}$ satisfying

$$|f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_n)| \leq c_i$$

for all $x \in [0,1]^n$, $i \in [n]$ and $x_i' \in [0,1]$. Moreover let Δ^n be the uniform distribution on $[0,1]^n$. Then for all $t > 0$, we have

$$\mathbb{P}_{x \sim \Delta^n}[f(x) < \mathbb{E}_{y \sim \Delta^n}[f(y)] - t] < e^{-\frac{2t^2}{\sum_{i=1}^n c_i^2}}.$$

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2. Prove the necessary bounded differences property of f.

Lemma (Bounded Differences)

Let $x \in [0,1]^S$, $j^\star \in S$ and $\theta \in [0,1]$ be arbitrary. Define x'_j to be θ if $j = j^\star$ and x_j otherwise. Then $|f(x) - f(x')| \leq 1$.

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3. Apply McDiarmid's inequality with $t = \alpha n$ and $c_i = 1$.

Bounded Differences

Bounded differences follows directly from the following:

Lemma

Let $j \in S$, then we can define the graph G_{-j} which contains all vertices of G except for j. For some fixed values of $x \in [0,1]^S$, we let M be the matching produced by Ranking *in G and let M_{-j} be the matching produced by* Ranking *in G_{-j}. Then $|M_{-j}| \leq |M| \leq |M_{-j}| + 1$.*

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Proof. Live. $\square$

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- The proof is elegant and uses a nice result from probability theory with an equally nice structural lemma about matchings.
- This should be just as well-known as $\mathbb{E}[|M|] \geq (1 - 1/e)n$!

Can this be extended to other Online Matching Problems? Yes!

Generalizations

Fully Online Matching

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- Vertices arrive and depart in adverserial order.
- Vertices must be matched after they arrive and before they depart.

$\Rightarrow$ Models e.g. ride-sharing problems and is a direct generalization of Online Bipartite Matching!

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Huang, Kang, Tang, Wu, Zhang 2018: 0.521-competitive in general, 0.567-competitive on bipartite graphs.

Concentration for Fully Online Matching

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Proof. Almost the same as for Online Bipartite Matching! ☐

Vertex-Weighted Matching

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Note. Edge-weighted also exists but is much harder!

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Well known that this still gives $(1 - \frac{1}{e})$-competitive!

Concentration for Vertex-Weighted Matching

Theorem

For any $\alpha > 0$, there exists a variant of RANKING *such that for any instance $G = (S, B, E)$ with weights $w : S \to \mathbb{R}_+$ of the Online Vertex-Weighted Bipartite Matching, any arrival order of B and any matching M^*,*

$$\mathbb{P}\left[w(M) < \left(1 - \frac{1}{e} - \alpha\right) w(M^\star)\right] < e^{-\frac{1}{50}\alpha^4 \frac{w(M^\star)^2}{||w||_2^2}}$$

where M denotes the matching generated by RANKING *and*

$$w(M) := \sum_{\{i,j\} \in M} w_j.$$

Proof Idea

Initial idea, show:

Lemma (Weighted Bounded Differences)

Let $x \in [0,1]^S$, $j^\star \in S$ and $\theta \in [0,1]$ be arbitrary. Define x'_j to be θ if $j = j^\star$ and x_j otherwise. Then $|f(x) - f(x')| \leq w_j$ where f is the weight of the Ranking *output.*

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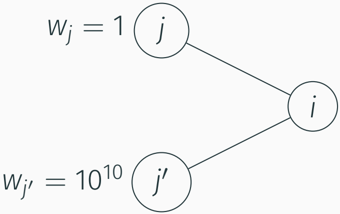
This would give

$$\mathbb{E}\left[w(M) < \left(1 - \frac{1}{e} - \alpha\right) w(M^\star)\right] < e^{-2\frac{w(M^\star)^2}{||w||_2^2}}$$

via weighted McDiarmid.

The Problem

But this does not work!



Consider $x_{j'} > 1 - 10^{-11}$. Then for some values of x_j, i picks j over j' because:

$$w_{j'}(1 - e^{x_{j'}-1}) < w_j(1 - e^{x_j-1}).$$

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This is still $(1 - 1/e - \epsilon)$-competitive!

The Fix II

Now we get:

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1. Run $\frac{\alpha}{2}$-Ranking to be $(1 - 1/e - \alpha/2)$-competitive.

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*Let $x \in [0,1]^S$, $j^\star \in S$ and $\theta \in [0,1]$ be arbitrary. Define x'_j to be θ if $j = j^\star$ and x_j otherwise. Then $|f(x) - f(x')| \leq \frac{2}{\epsilon} w_j$ where f is the weight of the ϵ-*RANKING *output.*

So to get concentration above $(1 - 1/e - \alpha)w(M^\star)$:

1. Run $\frac{\alpha}{2}$-RANKING to be $(1 - 1/e - \alpha/2)$-competitive.
2. Use McDiarmid with $\alpha/2$ to get concentration above $(1 - 1/e - \alpha/2 - \alpha/2)w(M^\star)$. $\square$

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- Open problem: is there a way to get dependence on $M^\star$ instead of $||w||_2^2$?
- Open problem: Can you show that these bounds are tight in some sense?

Thank You!