Dichotomies in Equilibrium Computation, and Complementary Pivot Algorithms for a New Class of Non-Separable Utility Functions*

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Abstract

After more than a decade of work in TCS on the computability of market equilibria, complementary pivot algorithms have emerged as the best hope of obtaining practical algorithms. So far they have been used for markets under separable, piecewise-linear concave (SPLC) utility functions [30] and SPLC production sets [31]. Can his approach extend to non-separable utility functions and production sets? A major impediment is *rationality*, i.e., if all parameters are set to rational numbers, there should be a rational equilibrium.

Recently, [42] introduced classes of non-separable utility functions and production sets, called *Leontief-free*, which are applicable when goods are substitutes. For markets with these utility functions and production sets, and satisfying mild sufficiency conditions, we obtain the following results:

- Proof of rationality.
- Complementary pivot algorithms based on a suitable adaptation of Lemke's classic algorithm.
- A strongly polynomial bound on the running time of our algorithms if the number of goods is a constant, despite the fact that the set of solutions is disconnected.
- Experimental verification, which confirms that our algorithms are practical.
- Proof of PPAD-completeness.

Next we give a proof of membership in FIXP for markets under piecewise-linear concave (PLC) utility functions and PLC production sets by capturing equilibria as fixed points of a continuous function via a nonlinear complementarity problem (NCP) formulation.

Finally we provide, for the first time, dichotomies for equilibrium computation problems, both Nash and market; in particular, the results stated above play a central role in arriving at the dichotomies for exchange markets and for markets with production. We note that in the past, dichotomies have played a key role in bringing clarity to the complexity of decision and counting problems.

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Contents

1 Introduction

Market equilibrium is an inherently algorithmic notion: this should be obvious from the fact that Walras, who while defining this notion in 1874 [61], also gave a mechanism for arriving at an equilibrium, namely the tatonnement process (see Section 1.5 for a brief history of work since then). In 1975, Eaves [25] gave a complementary pivot algorithm for the linear case of the Arrow-Debreu market model. Although this approach was far superior than previous ones (see Section 1.3 for details), it was not extended to more general utility functions until two years ago, when [30] extended it to separable, piecewise-linear concave (SPLC) utility functions and [31] extended it to SPLC production sets.

A major impediment to extension to non-separable utility functions was that a necessary condition for this approach is *rationality*, i.e., if all parameters are set to rational numbers, there should be a rational equilibrium (obviously, this condition is satisfied by SPLC utilities and SPLC production sets). In 1976, Mas-Colell gave an example using a non-separable utility function which has only irrational equilibria (mentioned in [26]). Additionally, the difficulty of finding algorithms for Arrow-Debreu markets under non-separable utility functions was well known. The only positive results we are aware of are: For Fisher's market model, which is a subcase of the Arrow-Debreu model, under constant elasticity of substitution (CES) utility functions [16], and for differentiable, concave utility functions, but in a non-standard model which allows perfect price discrimination [59]. For Arrow-Debreu market model under CES utility functions for $\rho \geq -1$ [14].

Our first result is a complementary pivot algorithm for a class of non-separable utility functions and production sets, called *Leontief-free (LF)*, defined recently in [42] (see Section 1.1). We first prove rationality for this class – this does not contradict Mas-Colell's example, since it used Leontief utility functions which do not lie in this class. Experiments confirm that our algorithms are practical and, since they are path-following algorithms, they yield proofs of membership of these problems in the class PPAD, defined by Papadimitriou in [**?**]. Additionally, we also establish PPAD-hardness for these problems. In case the number of goods is a constant, we establish strongly polynomial bounds on the running time of our algorithms, despite the fact that the set of solutions is disconnected. For problems whose solutions lie in a continuous domain (i.e., the convex combination of two solutions does qualify for being a solution, but may not be one), it is well known that polynomial time algorithms exploit convexity of the set of solutions in a critical manner and very few such algorithms are known for problems in which the solution set is not convex; we are only aware of [1, 30, 31].

In economics, it is customary to assume that utility functions are concave, and production sets are convex. Since we are in a finite precision model of computation, we will assume that utility functions are piecewise-linear and concave (PLC) and production sets are polyhedral; we call it PLC production since the boundary of polyhedral production set can be defined by a PLC correspondence. Clearly by making the pieces fine enough, the approximation to the original utilities and production sets can be made as good as needed.

Our second result concerns the class FIXP [28], which captures the complexity of computing an equilibrium for k-player Nash, henceforth denoted k-Nash, for $k \geq 3$ [28]. We prove membership in FIXP for a very general class of markets, namely markets under PLC utility functions (which include SPLC as well as non-separable PLC functions) and PLC production sets. These proofs involve capturing equilibria as fixed points of a continuous function via a nonlinear complementarity problem (NCP) formulation. We note that at present very few problems have been shown to be in FIXP and we believe this technique, using an NCP formulation, will find use in the future.

In the endeavor, over the last half century, to classify natural computational problems by their complexity, dichotomies have played a key role in bringing much clarity; these dichotomies characterize how the complexity of a certain problem changes as a certain parameter is changed. Perhaps the most well known of these dichotomies is Schaefer's theorem, which gives a complete characterization of when a restriction of SAT, defined via relations over the Boolean domain, is in P and when it is NP-complete. Following this result, a lot of work was done on dichotomies for decision problems, e.g., see [7, 18], and for counting problems, see the extensive survey [9]; in the latter case the dichotomy is between P and #P-complete.

Our third result provides, for the first time, dichotomies for equilibrium computation problems. We start by observing that the results already known on Nash equilibrium lead to a dichotomy that respects three different criteria, computation complexity being one of them, see Table 1. In a nutshell, this dichotomy establishes a qualitative difference between 2-Nash and k-Nash for $k \geq 3$. The two results stated above, together with other results, lead to analogous dichotomies for market equilibrium. Table 4 gives a dichotomy for exchange markets, establishing a qualitative difference between LF utility functions and piecewise-linear concave (PLC) utility functions. Table 5 gives it for markets with production, but with utility functions being the simplest possible, i.e., linear; it draws a sharp contrast between LF production sets and PLC production sets. Interestingly enough, the same three criteria apply to both these dichotomies as well.

After a decade of intense work on equilibrium computation, at this point two facts are self-evident: First, equilibrium problems have their own character[1] which is quite distinct from that of decision, optimization or counting problems. Second, equilibrium computation has grown into a full-fledged area within the theory of algorithms and computational complexity.

1.1 Leontief-free utility functions and Leontief-free production sets

A utility function over a set of divisible goods is said to be *separable* if it is the sum of utilities of individual goods, and *non-separable* otherwise. If the utilities of individual goods are PLC and the joint utility is separable, then we have a *separable, PLC (SPLC) utility function.* An analogous notion for production was given in [31], namely, the *production of a firm is separable, piecewise-linear concave (SPLC)* if the firm produces a single finished good from any one of a set of raw goods, with the production of the finished good from each raw good being given by a PLC function, and the total production of the finished good being additive over all raw goods. For example, suppose a firm produces bread from either wheat or corn, each given by a PLC function. If the total quantity of bread produced from both wheat and corn is additive[2], then the firm's production function is SPLC.

The irrational example of Mas-Colell, mentioned above, used *Leontief utilities*, which are non-separable, and are applicable when goods are complements. A typical example is bread and butter, assuming that the agent wants these goods in a certain proportion and derives no utility from only bread or only butter. Next consider an agent who has the option of eating bread or bagels at breakfast. Assume she has PLC utilities for each and eventually gets satiated from each. Clearly, her joint utility for consuming a combination of bread and bagels should not be additive, since both satiate her desire for the same type of food at breakfast, and it should be sub-additive. Thus, a non-separable utility function is called for. However, the kind of non-separability that needs to be formalized here is quite different from that captured by Leontief utilities, since in this case, goods

[1] E.g., an early observation of [41] was that NP-hardness could not be used for establishing intractability of computing a Nash equilibrium: since this is a total problem, a proof of NP-hardness would be tantamount to showing NP = co-NP, a result considered highly unlikely.

[2] Clearly, it would be more realistic to assume that the total production is sub-additive, since the same machinery and labor are presumably being used for producing bread from wheat and from corn. This extension is achieved below via the notion of Leontief-free production.

are substitutes and not complements.

[42] introduced the notion of *Leontief-free (LF) utility functions* for this purpose, see Section 4 for a formal definition. SPLC utilities are a subclass of LF utilities, and LF utilities are a subclass of submodular utilities (shown in [42]), see Figure 1. [42] also introduced the notion of *Leontief-free production* to model a firm that uses a set of raw goods, that are substitutes, to produce a set of finished goods, that are also substitutes, e.g., a firm that uses full-fat milk and low-fat milk as raw goods to produce the finished goods yogurt and ice cream. Given the PLC production function of each finished good from each raw good, Leontief-free production sets help model the sub-additivities that set in when the firm uses several raw goods to produce several finished goods. Once again, SPLC production is a subclass of LF production. A natural application of our complementary pivot algorithms for these notions is for pricing a new good, since it will be competing with its substitutes.

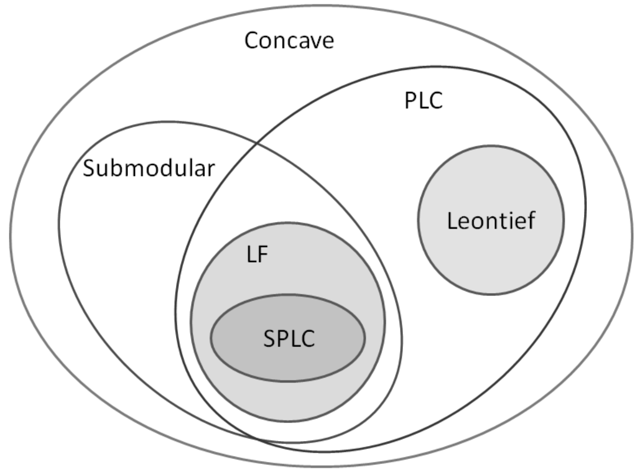

Figure 1:

The name Leontief-free was chosen to indicate that Leontief-type constraints, i.e., desiring two or more goods in fixed proportions, are disallowed in these utility functions and production sets, in fact as shown in [42] adding even one such constraint to a Leontief-free utility function or the production set can lead to irrationality.

1.2 The classes PPAD and FIXP

The two complexity classes PPAD, defined by Papdimitriou [47], and FIXP, defined by Etessami and Yannakakis [28], have played an important role in this theory, e.g., they capture the complexity of 2-Nash and k-Nash, for $k \geq 3$, respectively. These classes appear to be quite disparate – whereas solutions to problems in the former are rational numbers, those to the latter are algebraic numbers, as observed in [58]. And whereas the former is contained in function classes NP $\cap$ co-NP, the latter lies somewhere between P and PSPACE, and is likely to be closer to the harder end of PSPACE [62].

Informally, PPAD is the class of problems that allow for "path-following algorithms" and for this reason, PPAD has an intimate connection with complementary pivot algorithms: obtaining such an algorithm for a problem gives, together with Todd's result [56], membership of the problem in PPAD. Furthermore, the Lemke-Howson algorithm provided a key motivation for the definition of this class. On the other hand, a problem is in FIXP if its solutions are in one-to-one correspondence with the fixed points of a function which is defined using the operations of $+$, $*$, $/$, max, and an arbitrary number of rational constants.

The only results showing membership in FIXP or proving FIXP-hardness for market equilibrium questions we are aware of are: [28] prove that the problem of computing an equilibrium in an Arrow-Debreu market is FIXP-complete provided the excess demand is an algebraic function of the prices and this model is a simplified version of the standard model in that individual utility functions are not given, only the aggregate excess demand function is given. [12] show that an Arrow-Debreu market under CES utility functions is in FIXP provided the elasticity parameter for each agent is a rational number $\rho_i < 1$ and is given in unary. [62] show that an Arrow-Debreu market under Leontief utility functions is in FIXP; observe that in the latter cases as well, excess demand is an algebraic function of the prices. No markets with production have been shown to be in FIXP, and whereas several standard market models are expected to be FIXP-hard, see [58], none are shown FIXP-hard yet.

For markets under PLC utility functions, considered in this paper, optimal bundles of buyers are not unique. Therefore, excess demand will not be a function, it will be a correspondence – this is a new difficulty we need to overcome. For markets with production, the amount of each good available is not a constant, which leads to another difficulty to be overcome. As stated above, these markets may not have rational equilibria and so don't admit an LCP. Instead, we give an *nonlinear complementarity problem (NCP)* whose solutions are in one-to-one correspondence with market equilibria. We then design a continuous function F over a convex, compact domain which is computable by a FIXP circuit, and we show that the fixed points of F are in one-to-one correspondence with the solutions of the NCP, and hence market equilibria. We believe this technique for proving membership in FIXP using an NCP formulation will find use in the future.

1.3 Complementary pivot algorithms

An algorithm that walks on the one-skeleton of a polyhedron to find a solution, which is necessarily at a vertex of the polyhedron, is called a *pivoting-based algorithm*. The classic example of such an algorithm is the simplex algorithm of Dantzig [19] for linear programming. An algorithm which additionally is attempting to satisfy certain complementarity conditions is called a *complementary pivot algorithm*, classic examples being the Lemke-Howson algorithm [37] for 2-Nash and Eaves' algorithm [26] for the linear case of the Arrow-Debreu market model; the latter is based on Lemke's algorithm [36] (see Appendix A for a brief description).

The common feature of these three algorithms is that they run fast on randomly chosen examples (established in [55, 48, 30], respectively) even though they take exponential time in the worst case (established in [35] and [48] for the first two algorithms and left as an open problem in [30] for the third); the worst case examples are artificially contrived to make the algorithm perform poorly. These algorithms also tend to yield deep structural properties of the underlying problem, e.g., strong duality; index, degree and stability for 2-Nash equilibria [51]; and oddness of number of equilibria [30], respectively.

Our complementary pivot algorithms for computing equilibria for an Arrow-Debreu market under Leontief-free utilities and Leontief-free production sets are based on Lemke's algorithm [36]. It turns out that the LCP (linear complementarity problem) that captures the set of equilibria of our market is in a non-standard form – it has variables which do not participate in complementarity conditions. As a result, Lemke's algorithm is not directly applicable: if such a variable becomes zero, the algorithm requires that its complementarity condition be relaxed, but there is none! Let us call such variables *abnormal* and the rest *normal*.

We get around this problem by first observing that our non-standard LCP has additional structure: with each abnormal variable we can associate a set of normal variables. Second, we make the following modification to the basic Lemke algorithm. We show that the algorithm can be executed in such a way that whenever an abnormal variable becomes zero at a vertex, a double label is created corresponding to a normal variable (see Section 9 for an explanation of these terms). We then move out of this vertex by relaxing this double label.

One deficiency of Lemke's algorithm is that it is not guaranteed to terminate with a solution – this requires an additional argument. We prove termination by showing that the polyhedron associated with our augmented LCP does not have any secondary rays (see the Appendix A for a detailed explanation), if the market satisfies a mild sufficiency condition (see Section 10 for details). If the number of goods is a constant then we show how to partition the polyhedron corresponding to the augmented LCP into polynomially many regions so that each region has at most two vertices that are solutions of the augmented LCP. As a consequence, the path traced by the algorithm on the one-skeleton of this polyhedron is only polynomially long, hence showing that our algorithms are strongly polynomial for this cases, in addition to being practical. We note that [22] had given a polynomial time algorithm for general PLC utility functions, provided the number of goods is a constant. However, their algorithm does an exhaustive search over polynomially many configurations and is therefore not practical. Thus our algorithm answers their question of obtaining a "systematic way of finding equilibrium instead of the brute-force way."

1.4 Dichotomies and summary of results

We will assume throughout this paper that all numbers given in an instance are rational. Table 1 gives the dichotomy for Nash equilibrium computation. The rationality of 2-Nash was first established as a corollary of the Lemke-Howson algorithm [37], and the first 3-Nash game having only irrational equilibria was given by Nash [46].

Table 1:

	2-Nash	**k-Nash, $k \geq 3$**
Nature of solution	Rational [37]	Algebraic; irrational example [46]
Complexity	PPAD-complete [47, 20, 11]	FIXP-complete [28]
Practical algorithms	Lemke-Howson [37]	?

Recent results have yielded analogous dichotomies for market equilibrium computation and are presented in Tables 2 and 3, for consumption and production, respectively. These results include the complexity results of [10, 60], establishing PPAD-completeness of computing equilibria for Arrow-Debreu markets under SPLC utilities, the new complementary pivot algorithms [30] and [31], and a proof of membership of PLC markets in FIXP, which is established in the current paper. Note that in the tables, results of the current paper have been indicated as $\mathcal{CP}$.

We note that the separable vs. non-separable dichotomy is a very natural one and has arisen in other situations before, e.g., for the min-cost flow problem where the objective is a convex function of flows through individual edges, a polynomial time algorithms have been known for a while if the convex function is additively separable over edges [43, 32][3]. Hence there was every reason to believe that we had arrived at as good an understanding of the complexity of computing market equilibria (via as convincing a dichotomy), as we had for Nash equilibrium. However, that turned out not to be the case, as described below.

Table 2:

	SPLC utilities	**PLC utilities**
Nature of solution	Rational [22, 60]	Algebraic [22]; irrational example [26]
Complexity	PPAD-complete [10, 60]	FIXP: $\mathcal{CP}$ (Theorem 3.6); FIXP-hardness?
Practical algorithms	GMSV [30] (based on Lemke [36])	?

[3] A slight extension to non-separable convex functions was later given by [44].

Table 3:

	SPLC production	PLC production
Nature of solution	Rational [31]	Algebraic: $\mathcal{CP}$ (Theorem 3.9) irrational example [31]
Complexity	PPAD-complete [31]	FIXP: $\mathcal{CP}$ (Theorem 3.17); FIXP-hardness?
Practical algorithms	GV [31] (based on Lemke [36])	?

Using the notions of Leontief-free utilities and production sets, we extend the dichotomies given in Tables 2 and 3 to those in Tables 4 and 5, respectively. Proofs of rationality in both cases came as a surprise, considering the non-separability involved. In both cases, we assume that the market satisfies mild sufficiency conditions (see Section 8.1 for details), and we derive a linear complementarity problem (LCP) formulation whose solutions are in one-to-one correspondence with the set of equilibria. As a corollary, we get a proof of rationality for both these markets. Note that an LCP with rational data always has a rational solution (similar to an LP).

For Tables 3 and 5, which give dichotomies for production, we also need to specify the class of utility functions of agents. For this, we have used the following convention. For "negative" results, such as PPAD-hardness or irrational example, we assume the most restricted utilities, i.e., linear in both tables. For "positive" results, such as containment in PPAD or rationality of equilibria, we assume the most general utilities, i.e., SPLC in Table 3 and Leontief-free in Table 5.

Table 4:

	Leontief-free utilities	PLC utilities
Nature of solution	Rational: $\mathcal{CP}$ (Theorem 7.7)	Algebraic [22]; irrational example [26]
Complexity	PPAD-complete: $\mathcal{CP}$ (Theorem 10.12)	In FIXP: $\mathcal{CP}$ (Theorem 3.6); FIXP-hardness?
Practical algorithms	$\mathcal{CP}$ (based on Lemke [36])	?

Table 5:

	Leontief-free production	PLC production
Nature of solution	Rational: $\mathcal{CP}$ (Theorem 8.13)	Algebraic: $\mathcal{CP}$ (Theorem 3.9; irrational example [31]
Complexity	PPAD-complete: $\mathcal{CP}$ (Theorem 10.12)	In FIXP: $\mathcal{CP}$ (Theorem 3.17) FIXP-hardness?
Practical algorithms	$\mathcal{CP}$ (modification of Lemke's algorithm)	?

1.5 A brief history of work on computability of market equilibria

The introduction of the tatonnement process, by Walras [61], was followed by decades of concerted effort within mathematical economics for proving that it converges to an equilibrium. However, in the 1960s, serious issues were found: Scarf gave an example [50] on which the tatonnement process cycles and the Sonnenschien-Debreu-Mantel theorem [54, 21, 38] showed that assumptions on individual demand functions do not constrain aggregate demand function, implying that the task was hopeless.

At this point, interest switched to centralized algorithms, rather than distributed mechanisms, since equilibrium computation is important, e.g., for policy analysis, especially taxation policy, see [52, 24]. Impressive approaches were given by Scarf [49] and Smale [53]; however, these algorithms were quite slow and moreover they suffered from numerical instability issues. Despite these shortcomings, these algorithms were used. See [16] for other works in economics, including the discovery of some remarkable convex programs that capture equilibrium allocations and prices for specific market models, including the famous Eisenberg-Gale program [27].

With applications to markets on the Internet in the backdrop, around twelve years ago researchers in TCS started bringing to bear tools from the modern theory of algorithms and computational complexity to this question. After the linear utilities case was successfully tackled [23, 33], the next general case was SPLC utilities. However, when this long-standing open question was settled in the negative – the problem was shown PPAD-complete [10, 13, 60] – there seemed little point in proceeding to more general utility functions. In this situation, complementary pivot algorithms have brought new hope as far as practical algorithms are concerned. The limits of this approach are currently unclear and need to be understood thoroughly. Another important question is to find ways of dealing with irrationality by extending this approach, e.g., via a suitable way of approximation.

Notations. We mostly follow: capital letters denote matrices of constants, like W; bold lower case letters denote vector of variables, like $\boldsymbol{x}, \boldsymbol{y}$; Greek letters are used for dual variables, and calligraphic capital letters denote sets like $\mathcal{A}, \mathcal{G}$. Indices i, j, k and f refer to agent i, good j, segment k, and firm f respectively. Similarly, $\sum_i, \sum_j, \sum_k$, and $\sum_f$ refer to summation over all agents, all goods, all segments, and all firms respectively. Appendix B summarizes the notation used in this paper for a quick reference.

2 The Arrow-Debreu Market Model

The Arrow-Debreu market model [3] consists of a set $\mathcal{G}$ of divisible goods, a set $\mathcal{A}$ of agents and a set $\mathcal{F}$ of firms. Let n denote the number of goods in the market.

The production capabilities of a firm is defined by a set of production schedules. If a firm can produce a bundle $\boldsymbol{x}^p$ of goods using bundle $\boldsymbol{x}^r$ as raw material, then such a production schedule defines a production possibility vector (PPV) $(\boldsymbol{x}^p - \boldsymbol{x}^r)$. The set of PPVs of a firm determines its production capabilities. Let $\mathcal{S}^f \in \mathbb{R}^n$ denote the PPV set of firm f. Following are the standard and natural assumptions on $\mathcal{S}^f$ (see [3]).

1. Set $\mathcal{S}^f$ is closed and convex, and contains the origin.

2. The set of produced goods and raw goods of a firm are disjoint. Define $\mathcal{R}^f \stackrel{\text{def}}{=} \{j \in \mathcal{G} \mid v_j < 0,\ \boldsymbol{v} \in \mathcal{S}^f\}$ to be the set of raw goods and $\mathcal{P}^f \stackrel{\text{def}}{=} \{j \in \mathcal{G} \mid v_j > 0,\ \boldsymbol{v} \in \mathcal{S}^f\}$ to be the set of produced goods, then $\mathcal{R}^f \cap \mathcal{P}^f = \emptyset$.

3. *Downward close* - Adding to raw material does not decrease the production, i.e., if $\boldsymbol{v} \in \mathcal{S}^f$, and $\boldsymbol{w} \leq \boldsymbol{v}$, while $w_j \geq 0, \forall j \in \mathcal{P}^f$ then $\boldsymbol{w} \in \mathcal{S}^f$.

4. *No production out of nothing* - $\{\oplus_{f \in \mathcal{F}} \mathcal{S}^f\} \cap \mathbb{R}^n_+ = \mathbf{0}$.

The goal of a firm is to produce as per a profit maximizing (optimal) schedule. Firms are owned by agents: Θ^i_f is the profit share of agent i in firm f such that $\forall f \in \mathcal{F},\ \sum_{i\in\mathcal{A}} \Theta^i_f = 1$.

Each agent i comes with an initial endowment of goods; W^i_j is amount of good j with agent i. The preference of an agent i over bundles of goods is captured by a non-negative, non-decreasing and concave utility function $U_i : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$. Non-decreasingness is due to free disposal property, and concavity captures the law of diminishing marginal returns. Each agent wants to buy a (optimal) bundle of goods that maximizes her utility to the extent allowed by her earned money – from initial endowment and profit shares in the firms. Without loss of generality, we assume that total initial endowment of every good is 1, i.e, $\sum_{i\in\mathcal{A}} W^i_j = 1, \forall j \in \mathcal{G}$[4].

Given prices of goods, if there is an assignment of optimal production schedule to each firm and optimal affordable bundle to each agent so that there is neither deficiency nor surplus of any good, then such prices are called *market clearing* or *market equilibrium* prices. The market equilibrium problem is to find such prices when they exist. In a celebrated result, Arrow and Debreu [3] proved that market equilibrium always exists under some mild conditions, however the proof is non-constructive and uses heavy machinery of Kakutani fixed point theorem.

A well studied restriction of Arrow-Debreu model is *exchange economy*, i.e., markets without production firms.

3 Membership in FIXP

In this section, we show that equilibrium computation problem in markets with PLC utility functions and PLC production functions is in FIXP [28].

We first obtain a characterization of market equilibrium in terms of the solutions of a *non-linear* complementarity problem[5] (NCP) formulation and then design a continuous function F over a convex and compact domain, computable by a FIXP circuit, i.e., algebraic circuit with $\{max, min, +, -, *, /\}$ operators and rational constants. Further we show that assuming the weakest known sufficiency conditions for the existence of market equilibrium given by Arrow and Debreu [3][6], fixed points of F are in one-to-one correspondence with the solutions of NCP, and hence are related to market equilibria.

Etessami and Yannakakis [28] showed membership in FIXP for exchange markets (markets without production) with explicit algebraic demand function, however this approach does not work for markets with PLC utilities. A major difficulty is that the demand of an agent (or firm) is not an explicit algebraic function of given prices; it is not even unique. The same difficulty was experienced by [60] in proving membership of exchange markets with SPLC utilities in PPAD, and they resort to the characterization of PPAD (given in [28]) as a class of exact fixed-point computation problems for polynomial time computable piecewise-linear Brouwer functions. No such characterization for FIXP is known. Further we also consider markets with production firms, which has its own difficulties, like handling market clearing conditions becomes non-trivial due to indefinite quantities of goods in the market.

We develop a novel technique for proving membership in FIXP (PPAD) from NCP (LCP), which may be of independent interest. To keep things simple, first we show our result for the exchange markets with PLC utilities, and then extend it to also include PLC production.

[4]This is like redefining the unit of goods by appropriately scaling utility and production parameters.
[5]see [17, 45] for the definition of nonlinear complementarity problem.
[6]We note that Maxfield [39] sufficiency conditions based on economy graph are not suitable for PLC markets.

3.1 Exchange economy

The piecewise-linear and concave (PLC) utility function, of agent i, $u_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ can be described as

$$u_i(x^i) = \min_k \{\sum_j U^i_{jk} x^i_j + T^i_k\},$$

where U^i_{jk}'s and T^i_k's are given non-negative rational numbers. Given prices $\boldsymbol{p}$, agent i's optimal bundle is a solution of the following linear program (LP):

$$\begin{aligned} &\max u_i \\ u_i \le &\sum_j U^i_{jk} x^i_j + T^i_k, \ \forall k \\ \sum_j x^i_j p_j \le &\sum_j W^i_j p_j \\ x^i_j \ge &\ 0, \ \forall j \end{aligned} \tag{1}$$

Let γ^i_k and λ_i be the non-negative dual variables of constraints in the above LP. From the optimality conditions, we get the following linear constraints and complementarity conditions. Note that the constraints are linear assuming prices are given. All variables introduced will have a non-negativity constraint; for the sake of brevity, we will not write them explicitly.

$$\begin{aligned} \forall j: \ \sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j \quad &\text{and} \quad x^i_j(\sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j) = 0 \\ \forall k: \ u_i \le \sum_j U^i_{jk} x^i_j + T^i_k \quad &\text{and} \quad \gamma^i_k(u_i - \sum_j U^i_{jk} x^i_j - T^i_k) = 0 \\ \sum_j x^i_j p_j \le \sum_j W^i_j p_j \quad &\text{and} \quad \lambda_i(\sum_j x^i_j p_j - \sum_j W^i_j p_j) = 0 \\ \sum_k \gamma^i_k = 1 & \end{aligned} \tag{2}$$

From strong duality, (1) and (2) are equivalent. Further by simple algebra, these conditions also give

$$u_i = \lambda_i \sum_j W^i_j p_j + \sum_k \gamma^i_k T^i_k. \tag{3}$$

Hence u_i is a redundant variable and can be eliminated using the above expression, however for clarity we keep it as a placeholder variable for the above expression. We get the above constraints for each agent i and all together, they capture the optimal bundle and budget constraints of every agent. At market equilibrium, we also need market clearing of each good, which is essentially, $\sum_i x^i_j \le 1, \ \forall j$. By putting these together and now treating price $\boldsymbol{p}$ as variables, we get the nonlinear complementarity problem (NCP) formulation as shown in Table 6. Since equilibrium prices are scale invariant in Arrow-Debreu market, we have put $\sum_j p_j = 1$ as well.

The next lemma follows from the above analysis.

Lemma 3.1 *If $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$ is a solution of E-NCP, then $(\boldsymbol{p}, \boldsymbol{x})$ is a market equilibrium. Further if $(\boldsymbol{p}, \boldsymbol{x})$ is a market equilibrium, then $\exists(\boldsymbol{\lambda}, \boldsymbol{\gamma})$ such that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$ is a solution of E-NCP.*

Sufficiency Conditions. Market equilibrium may not exist, and it is NP-complete to decide whether there exists an equilibrium even in markets with SPLC utility functions [60]. Arrow-Debreu

Table 6: E-NCP

$$\begin{array}{rcr}
\forall(i,j):\ \sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j & \text{and} & x^i_j(\sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j) = 0 \\
\forall(i,k):\ u_i \le \sum_j U^i_{jk}x^i_j + T^i_k & \text{and} & \gamma^i_k(u_i - \sum_j U^i_{jk}x^i_j - T^i_k) = 0 \\
\forall i:\ \sum_j x^i_j p_j \le \sum_j W^i_j p_j & \text{and} & \lambda_i(\sum_j x^i_j p_j - \sum_j W^i_j p_j) = 0 \\
\forall j:\ \sum_i x^i_j \le 1 & \text{and} & p_j(\sum_j x^i_j - 1) = 0 \\
\forall i:\ \sum_k \gamma^i_k = 1 & \text{and} & u_i = \lambda_i \sum_j W^i_j p_j + \sum_k \gamma^i_k T^i_k \\
\sum_j p_j = 1 & &
\end{array}$$

[3] showed that market equilibrium exists under the following sufficiency conditions: $W > 0$ and each agent is non-satiated. In case of PLC utilities, non-satiation condition implies that for every k, there exists a j such that $U^i_{jk} > 0$.

Next we define a continuous function $F : D \to D$, where D is convex and compact and show that fixed points of F are in one-to-one correspondence with the solutions of E-NCP, and hence are related to market equilibria using Lemma 3.1. Since F is continuous on a convex and compact D, there exists a fixed point. Clearly, for such a theorem, we need to assume sufficiency conditions.

To define D, first we obtain upper bounds on all variables at equilibrium. Let $x_{max} \overset{\text{def}}{=} 1.1$, $W_{min} \overset{\text{def}}{=} \min_{(i,j)} W^i_j$, $U_{max} \overset{\text{def}}{=} \max_{(i,j,k)} U^i_{jk}$, $T_{max} \overset{\text{def}}{=} \max_{(i,k)} T^i_k$, and $\lambda_{max} \overset{\text{def}}{=} 2n^{(U_{max}+T_{max})}/_{W_{min}}$. Note that $W_{min} > 0$ under sufficiency conditions. Since the total quantity of every good is 1, $0 \le x^i_j < x_{max}$ at equilibrium. Using (3), we get $\lambda_i \le n^{(U_{max}+T_{max})}/_{W_{min}} < \lambda_{max}$ at equilibrium.

Let $D \overset{\text{def}}{=} \{(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\gamma}, \boldsymbol{\lambda}) \in \mathbb{R}^N_+ \mid \sum_j p_j = 1;\ x^i_j \le x_{max};\ \sum_k \gamma^i_k = 1;\ \lambda_i \le \lambda_{max}\}$, where N is the total number of variables, and let $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\lambda}}) \overset{\text{def}}{=} F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{\gamma}, \boldsymbol{\lambda})$ as given in Table 7.

The following claim is straightforward using Lemma 3.1 and we omit its proof.

Claim 3.2 *Every market equilibrium gives a fixed point of F.*

Next assuming sufficiency conditions for the existence of market equilibrium, we show that every fixed point of F gives a market equilibrium. Table 8 gives all the conditions that might lead to a fixed point of F based on update rule in Table 7.

Reading Table 8. Consider (3.1), which says that if $x^i_j = 0$, then for this input to be a fixed point, it must be the case that $\sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j$, otherwise $\overline{x}^i_j \ne x^i_j$. Similarly, suppose $\sum_k U^i_{jk}\gamma^i_k < \lambda_i p_j$, then it must be the case that $x^i_j = 0$. Next consider (1.2), which says that if $p_j > 0$ and $\sum_i x^i_j > 1$ for some j, then for this input to be fixed point, it must be the case that whenever $p_j > 0$ we have $\sum_i x^i_j > 1$ and whenever $p_j = 0$, we have $\sum_i x^i_j \le 1$, otherwise $\overline{\boldsymbol{p}} \ne \boldsymbol{p}$.

Table 7: FIXP Circuit for Exchange Economy

$$\overline{p}_j = \frac{p_j + \max\{\sum_i x^i_j - 1, 0\}}{\sum_j (p_j + \max\{\sum_i x^i_j - 1, 0\})}$$

$$\overline{\gamma}^i_k = \frac{\gamma^i_k + \max\{u_i - \sum_j U^i_{jk} x^i_j - T^i_k, 0\}}{\sum_k (\gamma^i_k + \max\{u_i - \sum_j U^i_{jk} x^i_j - T^i_k, 0\})}$$

$$\overline{x}^i_j = \min\left\{\max\left\{x^i_j + \sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j, 0\right\}, x_{max}\right\}$$

$$\overline{\lambda}_i = \min\left\{\max\left\{\lambda_i + \sum_j x^i_j p_j - \sum_j W^i_j p_j, 0\right\}, \lambda_{max}\right\}$$

Table 8: Conditions for a Fixed Point Based on Update Rule in Table 7

$\overline{\boldsymbol{p}} = \boldsymbol{p}$	case 1:	$\sum_i x^i_j \leq 1$	(1.1)
	case 2:	If $p_j = 0$, then $\sum_i x^i_j \leq 1$ If $p_j > 0$, then $\sum_i x^i_j > 1$	(1.2)
$\overline{\boldsymbol{\gamma}}^i = \boldsymbol{\gamma}^i$	case 1:	$u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k$	(2.1)
	case 2:	If $\gamma^i_k = 0$, then $u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k$ If $\gamma^i_k > 0$, then $u_i > \sum_j U^i_{jk} x^i_j + T^i_k$	(2.2)
$\overline{x}^i_j = x^i_j$	$x^i_j = 0$	$\sum_k U^i_{jk}\gamma^i_k \leq \lambda_i p_j$	(3.1)
	$0 < x^i_j < x_{max}$	$\sum_k U^i_{jk}\gamma^i_k = \lambda_i p_j$	(3.2)
	$x^i_j = x_{max}$	$\sum_k U^i_{jk}\gamma^i_k \geq \lambda_i p_j$	(3.3)
$\overline{\lambda_i} = \lambda_i$	$\lambda_i = 0$	$\sum_j x^i_j p_j \leq \sum_j W^i_j p_j$	(4.1)
	$0 < \lambda_i < \lambda_{max}$	$\sum_j x^i_j p_j = \sum_j W^i_j p_j$	(4.2)
	$\lambda_i = \lambda_{max}$	$\sum_j x^i_j p_j \geq \sum_j W^i_j p_j$	(4.3)

Next we show that none of the conditions in shaded rows, namely (1.2), (2.2), (3.3) and (4.3), are satisfied at fixed points of F, which implies that each fixed point of F gives a solution of E-NCP, and hence market equilibrium.

Claim 3.3 *At every fixed point of F, $0 < \lambda_i < \lambda_{max}, \forall i$.*

Proof : First suppose that $\lambda_i = \lambda_{max}$ for some i at a fixed point. It implies that for every good

j such that $p_j \geq \frac{W_{min}}{2n}$, we have $x^i_j = 0$ (from (3.1)). Hence, $\sum_j x^i_j p_j < W_{min}$, which contradicts $\sum_j x^i_j p_j \geq \sum_j W^i_j p_j$ (from (4.3)). Hence $\lambda_i < \lambda_{max}, \forall i$ at a fixed point.

Next suppose that $\lambda_i = 0$ for some i at a fixed point. It implies that for every $\gamma^i_k > 0$ and $U^i_{jk} > 0$, we have $x^i_j = x_{max}$ (from (3.3)). Note that here we use the *sufficiency condition* that for every k there exists a j such that $U^i_{jk} > 0$. Further $p_j > 0$ for such goods and $\sum_i x^i_j > 1$ for all goods whose $p_j > 0$ (from (1.2)). By this, we get $\sum_{i,j} x^i_j p_j > 1 = \sum_{i,j} W^i_j p_j$. This further implies that $\exists i'$ such that $\sum_j x^{i'}_j p_j > \sum_j W^{i'}_j p_j$ and $\lambda_{i'} = \lambda_{max}$, which is a contradiction. □

Claim 3.4 *At every fixed point of F, $\sum_i x^i_j \leq 1, \forall j$.*

Proof : Suppose $\exists j$ such that $\sum_i x^i_j > 1$. It implies that $p_j > 0$ (from (1.2)). This further implies that whenever $p_j > 0$, we have $\sum_i x^i_j > 1$. Hence, we have $\sum_{ij} x^i_j p_j > 1 = \sum_{i,j} W^i_j p_j$. By this, we get that $\exists i'$ such that $\sum_j x^{i'}_j p_j > W^{i'}_j p_j$ and hence $\lambda_{i'} = \lambda_{max}$ (from (4.3)), contradicting Claim 3.3. □

Note that Claim 3.4 implies that $x^i_j < x_{max}, \forall (i,j)$ at every fixed point of F.

Claim 3.5 *At every fixed point of F, $u_i \leq \sum_j U^i_{jk} x^i_j + T^i_k, \forall (i,k)$.*

Proof : Note that u_i is a placeholder variable for $\lambda_i \sum_j W^i_j p_j + \sum_k \gamma^i_k T^i_k$. Suppose $\exists (i,k)$ such that $u_i > \sum_j U^i_{jk} x^i_j + T^i_k$, then we have $\forall (i,k)$, $\gamma^i_k > 0 \Rightarrow u_i > \sum_j U^i_{jk} x^i_j + T^i_k$ (from (2.2)). This implies that $\sum_k u_i \gamma^i_k > \sum_{j,k} U^i_{jk} x^i_j \gamma^i_k + \sum_k T^i_k \gamma^i_k$. From Claims 3.4 and 3.3, we have $\sum_k U^i_{jk} \gamma^i_k x^i_j = \lambda_i p_j x^i_j, \forall (i,j)$ and $\sum_j x^i_j p_j \lambda_i = \sum_j W^i_j p_j \lambda_i, \forall i$. Putting these together, we get that $u_i > \sum_j W^i_j p_j \lambda_i + \sum_k T^i_k \gamma^i_k$, which is a contradiction. □

Claims 3.3, 3.4, 3.5 imply that none of the conditions (1.2), (2.2), (3.3), (4.3) are satisfied at fixed points of F. Therefore, we get the following theorem.

Theorem 3.6 *Assuming sufficient conditions of the existence of market equilibrium, every fixed point of F gives a solution of E-NCP and hence a market equilibrium. Further, F can be computed by a FIXP-circuit and hence market equilibrium computation problem for PLC utilities is in FIXP.*

Remark 3.7 *This technique can be used to obtain Linear-FIXP (equivalent to PPAD) circuit for markets with SPLC utilities using the linear complementary problem (LCP) formulation given in [30], thereby giving alternate proof of membership in PPAD for such markets. However, the same approach for proving membership in Linear-FIXP does not seem to work for 2-Nash using its LCP formulation.*

3.2 Markets with production

Recall from Section 2 that each firm has a production technology to produce a set of goods from a set of *different* raw goods. The PLC production technology of firm f can be described as

$$\sum_j D^f_{jk} x^{f,p}_j \leq \sum_j C^f_{jk} x^{f,r}_j + T^f_k, \ \forall k,$$

where D^f_{jk}'s, C^f_{jk}'s and T^f_k's are given non-negative rational numbers, and $x^{f,p}_j$ and $x^{f,r}_j$ denote the amount of good j produced and used respectively. In the above expression, the first summation is on goods j which can be produced by firm f, and the second summation is on goods j which can be used as a raw material. These two sets of goods are disjoint as described in Section 2, however for simplicity we do not introduce more symbols and taking summation over all goods. Further, variables $x_j^{f,p}$ and $x_j^{f,r}$ are respectively defined only for those goods j which can be produced and used by firm f.

Given prices $\boldsymbol{p}$, firm f's profit maximizing plan is a solution of the following linear program (LP):

$$
\begin{aligned}
\max \sum_j p_j x_j^{f,p} - \sum_j p_j x_j^{f,r} \\
\sum_j D_{jk}^f x_j^{f,p} \leq \sum_j C_{jk}^f x_j^{f,r} + T_k^f, \ \forall k \\
x_j^{f,p} \geq 0, x_j^{f,r} \geq 0
\end{aligned}
\tag{4}
$$

Let δ_k^f be the non-negative dual variable of constraints in the above LP. From the optimality conditions, we get the following linear constraints and complementarity conditions. Note that the constraints are linear assuming prices are given. All variables introduced will have a non-negativity constraint; for the sake of brevity, we will not write them explicitly.

$$
\begin{aligned}
\forall k: \ \sum_j D_{jk}^f x_j^{f,p} \leq \sum_j C_{jk}^f x_j^{f,r} + T_k^f \quad \text{and} \qquad & \delta_k^f(\sum_j D_{jk}^f x_j^{f,p} - \sum_j C_{jk}^f x_j^{f,r} + T_k^f) = 0 \\
\forall j: \ p_j \leq \sum_k D_{jk}^f \delta_k^f \quad \text{and} \qquad & x_j^{f,p}(p_j - \sum_k D_{jk}^f \delta_k^f) = 0 \\
\forall j: \ \sum_k C_{jk}^f \delta_k^f \leq p_j \quad \text{and} \qquad & x_j^{f,r}(\sum_k C_{jk}^f \delta_k^f - p_j) = 0
\end{aligned}
\tag{5}
$$

From strong duality, (4) and (5) are equivalent. Let ϕ^f captures the profit of firm f, i.e., $\phi^f = \sum_j p_j x_j^{f,p} - \sum_j p_j x_j^{f,r}$. Further by simple algebra, these conditions also give

$$\phi^f = \sum_k \delta_k^f T_k^f. \tag{6}$$

We get the above constraints for each firm k and all together, they capture the optimal production plan of every firm. Next we need to add constraints capturing optimal bundle of agents and market clearing. For this, we only need to modify market clearing constraints in (6) appropriately and we get the nonlinear complementarity problem (NCP) formulation AD-NCP for market equilibrium as shown in Table 9.

The next lemma and theorem follow from the construction.

Lemma 3.8 *If $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{x}^p, \boldsymbol{x}^r, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ is a solution of AD-NCP, then $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{x}^p, \boldsymbol{x}^r)$ is a market equilibrium. Further, if $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{x}^p, \boldsymbol{x}^r)$ is a market equilibrium, then $\exists(\boldsymbol{\lambda}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ such that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{x}^p, \boldsymbol{x}^r, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ is a solution of AD-NCP.*

Theorem 3.9 *Equilibrium prices are algebraic in markets with PLC utilities and PLC production.*

Sufficiency Conditions. For markets with production, Arrow-Debreu [3] gave the following sufficiency conditions for the existence of equilibrium: $W > 0$, each agent is non-satiated, no

Table 9: AD-NCP

$$\begin{aligned}
\forall(f,k):\ \sum_j D^f_{jk} x^{f,p}_j \le \sum_j C^f_{jk} x^{f,r}_j + T^f_k \quad &\text{and} \quad \delta^f_k\Big(\sum_j D^f_{jk} x^{f,p}_j - \sum_j C^f_{jk} x^{f,r}_j - T^f_k\Big) = 0\\
\forall(f,j):\ p_j \le \sum_k D^f_{jk}\delta^f_k \quad &\text{and} \quad x^{f,p}_j\Big(p_j - \sum_k D^f_{jk}\delta^f_k\Big) = 0\\
\forall(f,j):\ \sum_k C^f_{jk}\delta^f_k \le p_j \quad &\text{and} \quad x^{f,r}_j\Big(\sum_k C^f_{jk}\delta^f_k - p_j\Big) = 0\\
\forall(i,j):\ \sum_k U^i_{jk}\gamma^i_k \le \lambda_i p_j \quad &\text{and} \quad x^i_j\Big(\sum_l U^i_{jk}\gamma^i_k - \lambda_i p_j\Big) = 0\\
\forall(i,k):\ u_i \le \sum_j U^i_{jk} x^i_j + T^i_k \quad &\text{and} \quad \gamma^i_k\Big(u_i - \sum_j U^i_{jk} x^i_j - T^i_k\Big) = 0\\
\forall i:\ \sum_j x^i_j p_j \le \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f \quad &\text{and} \quad \lambda_i\Big(\sum_j x^i_j p_j - \sum_j W^i_j p_j - \sum_f \Theta^i_f \phi^f\Big) = 0\\
\forall j:\ \sum_i x^i_j + \sum_f x^{f,r}_j \le 1 + \sum_f x^{f,p}_j \quad &\text{and} \quad p_j\Big(\sum_i x^i_j + \sum_f x^{f,r}_j - 1 - \sum_f x^{f,p}_j\Big) = 0\\
\forall i:\ \sum_k \gamma^i_k = 1 \quad &\text{and} \quad u_i = \lambda_i\Big(\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f\Big) + \sum_k \gamma^i_k T^i_k\\
\forall f:\ \phi^f = \sum_k \delta^f_k T^f_k &\\
\sum_j p_j = 1 &
\end{aligned}$$

production out of nothing and no vacuous production. In case of PLC production, the last two conditions mean that the following linear constraints define a bounded polyhedron.

$$\begin{aligned}
\forall(f,k):\ & \sum_j D^f_{jk} x^{f,p}_j \le \sum_j C^f_{jk} x^{f,r}_j + T^f_k\\
\forall j:\ & \sum_f x^{f,r}_j \le 1 + \sum_f x^{f,p}_j\\
\forall(f,j):\ & x^{f,p}_j \ge 0;\quad x^{f,r}_j \ge 0
\end{aligned} \tag{7}$$

where first is production constraint and second is supply constraint. Let x^* be the maximum possible value of a variable over these constraints. Note that the bit length of x^* is polynomial in the size of input and can be computed in polynomial time.

Next we define a continuous function $F : D \to D$, where D is convex and compact and show that fixed points of F are in one-to-one correspondence with market equilibrium. Since F is continuous on a convex and compact D, there exists a fixed point. Clearly, for such a theorem, we need to assume sufficiency conditions.

To define D, first we obtain upper bounds on all variables at equilibrium. Let $x^p_{max} \stackrel{\text{def}}{=} x^*+1$ and $x_{max} \stackrel{\text{def}}{=} 2lx^p_{max}+2$, where l is total number of firms and x^* is as discussed above in sufficiency condition. Next define min and max of every input C, D, T, U, W like $C_{min} \stackrel{\text{def}}{=} \min_{(f,j,k)}\{C^f_{jk} \mid C^f_{jk} > 0\}$

and $C_{max} \stackrel{\text{def}}{=} \max_{(f,j,k)} C^f_{jk}$. Let $x^r_{max} \stackrel{\text{def}}{=} x_{max} + (nx^p_{max} D_{max}/C_{min})$, $\delta_{max} \stackrel{\text{def}}{=} \max\{1/C_{min}, 1/D_{min}\}+1$, and $\lambda_{max} \stackrel{\text{def}}{=} 4nx_{max}(U_{max}+T_{max})/W_{min}$.

Clearly, $x^i_j < x_{max}$, $x^{f,p}_j < x^p_{max}$ and $x^{f,r}_j < x^r_{max}$ at equilibrium. Using $u_i = \lambda_i(\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f) + \sum_k \gamma^i_k T^i_k$, we get $\lambda_i \leq nx_{max}(U_{max}+T_{max})/W_{min} < \lambda_{max}$ at equilibrium. Using $\sum_k C^f_{jk}\delta^f_k \leq p_j$, we get an upper bound on δ^f_k at equilibrium as: $\delta^f_k \leq 1/C_{min} < \delta_{max}$.

Let $D \stackrel{\text{def}}{=} \{(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{x}^p, \boldsymbol{x}^r, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\lambda}) \in \mathbb{R}^N_+ \mid \sum_j p_j = 1;\ x^i_j \leq x_{max};\ x^{f,p}_j \leq x^p_{max};\ x^{f,r}_j \leq x^r_{max};\ \sum_k \gamma^i_k = 1;\ \delta^f_k \leq \delta_{max};\ \lambda_i \leq \lambda_{max}\}$, where N is the total number of variables, and let $(\overline{\boldsymbol{p}}, \overline{\boldsymbol{x}}, \overline{\boldsymbol{x}}^p, \overline{\boldsymbol{x}}^r, \overline{\boldsymbol{\gamma}}, \overline{\boldsymbol{\delta}}, \overline{\boldsymbol{\lambda}}) \stackrel{\text{def}}{=} F(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{x}^p, \boldsymbol{x}^r, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\lambda})$ as given in Table 10.

Table 10: FIXP Circuit for Markets with Production

$$\overline{p}_j = \frac{p_j + \max\{\sum_i x^i_j + \sum_f x^{f,r}_j - 1 - \sum_f x^{f,p}_j, 0\}}{\sum_j (p_j + \max\{\sum_i x^i_j + \sum_f x^{f,r}_j - 1 - \sum_f x^{f,p}_j, 0\})}$$

$$\overline{\gamma}^i_k = \frac{\gamma^i_k + \max\{u_i - \sum_j U^i_{jk} x^i_j - T^i_k, 0\}}{\sum_k \gamma^i_k + \max\{u_i - \sum_j U^i_{jk} x^i_j - T^i_k, 0\}}$$

$$\overline{\delta}^f_k = \min\left\{\max\left\{\delta^f_k + \sum_j D^f_{jk} x^{f,p}_j - \sum_j C^f_{jk} x^{f,r}_j - T^f_k, 0\right\}, \delta_{max}\right\}$$

$$\overline{x}^i_j = \min\left\{\max\left\{x^i_j + \sum_k U^i_{jk}\gamma^i_k - \lambda_i p_j, 0\right\}, x_{max}\right\}$$

$$\overline{x}^{f,p}_j = \min\left\{\max\left\{x^{f,p}_j + p_j - \sum_k D^f_{jk}\gamma^i_k, 0\right\}, x^p_{max}\right\}$$

$$\overline{x}^{f,r}_j = \min\left\{\max\left\{x^{f,r}_j + \sum_k C^f_{jk}\delta^f_k - p_j, 0\right\}, x^r_{max}\right\}$$

$$\overline{\lambda}_i = \min\left\{\max\left\{\lambda_i + \sum_j x^i_j p_j - \sum_j W^i_j p_j, 0\right\}, \lambda_{max}\right\}$$

The following claim is straightforward using Lemma 3.8 and we omit its proof.

Claim 3.10 *Every market equilibrium is a fixed point of F.*

Next assuming sufficiency conditions for the existence of market equilibrium, we show that every fixed point of F is a market equilibrium. Table 11 gives all the conditions that might lead to a fixed point of F based on the update rule in Table 10 (see *reading Table 8* in previous section for how to read this). We show that none of the conditions in shaded rows, namely (1.2), (2.2), (3.3), (4.3), (5.3), (6.3) and (7.3), are satisfied at fixed points of F, which implies that each fixed point of F gives a solution of AD-NCP in Table (9) and hence a market equilibrium.

Claim 3.11 *At every fixed point of F, $\sum_j D^f_{jk} x^{f,p}_j \leq \sum_j C^f_{jk} x^{f,r} + T^f_k, \forall(f, k)$.*

Proof : Suppose $\exists(f,k)$ such that $\sum_j D^f_{jk} x^{f,p}_j > \sum_j C^f_{jk} x^{f,r} + T^f_k$, then we have $\delta^f_k = \delta_{max}$ (from (7.3)). This implies that whenever $D^f_{jk} > 0$, we have $x^{f,p}_j = 0$ (from (6.1)), which contradicts the starting assumption. ◻

Table 11: Conditions for a Fixed Point Based on Update Rule in Table 10

$\overline{\boldsymbol{p}} = \boldsymbol{p}$	case 1:	$\sum_i x_j^i + \sum_f x_j^{f,r} \leq 1 + \sum_f x_j^{f,p}$	(1.1)
	case 2:	$p_j = 0$ and $\sum_i x_j^i + \sum_f x_j^{f,r} \leq 1 + \sum_f x_j^{f,p}$ $p_j > 0$ and $\sum_i x_j^i + \sum_f x_j^{f,r} > 1 + \sum_f x_j^{f,p}$	(1.2)
$\overline{\boldsymbol{\gamma}}^i = \boldsymbol{\gamma}^i$	case 1:	$u_i \leq \sum_j U_{jk}^i x_j^i + T_k^i$	(2.1)
	case 2:	$\gamma_k^i = 0$ and $u_i \leq \sum_j U_{jk}^i x_j^i + T_k^i$ $\gamma_k^i > 0$ and $u_i > \sum_j U_{jk}^i x_j^i + T_k^i$	(2.2)
$\overline{x}_j^i = x_j^i$	$x_j^i = 0$	$\sum_k U_{jk}^i \gamma_k^i \leq \lambda_i p_j$	(3.1)
	$0 < x_j^i < x_{max}$	$\sum_k U_{jk}^i \gamma_k^i = \lambda_i p_j$	(3.2)
	$x_j^i = x_{max}$	$\sum_k U_{jk}^i \gamma_k^i \geq \lambda_i p_j$	(3.3)
$\overline{\lambda_i} = \lambda_i$	$\lambda_i = 0$	$\sum_j x_j^i p_j \leq \sum_j W_j^i p_j + \sum_{f,k} \Theta_f^i \delta_k^f T_k^f$	(4.1)
	$0 < \lambda_i < \lambda_{max}$	$\sum_j x_j^i p_j = \sum_j W_j^i p_j + \sum_{f,k} \Theta_f^i \delta_k^f T_k^f$	(4.2)
	$\lambda_i = \lambda_{max}$	$\sum_j x_j^i p_j \geq \sum_j W_j^i p_j + \sum_{f,k} \Theta_f^i \delta_k^f T_k^f$	(4.3)
$\overline{x}_j^{f,r} = x_j^{f,r}$	$x_j^{f,r} = 0$	$\sum_k C_{jk}^f \delta_k^f \leq p_j$	(5.1)
	$0 < x_j^{f,r} < x_{max}^r$	$\sum_k C_{jk}^f \delta_k^f = p_j$	(5.2)
	$x_j^{f,r} = x_{max}^r$	$\sum_k C_{jk}^f \delta_k^f \geq p_j$	(5.3)
$\overline{x}_j^{f,p} = x_j^{f,p}$	$x_j^{f,p} = 0$	$p_j \leq \sum_k D_{jk}^f \delta_k^f$	(6.1)
	$0 < x_j^{f,p} < x_{max}^p$	$p_j = \sum_k D_{jk}^f \delta_k^f$	(6.2)
	$x_j^{f,p} = x_{max}^p$	$p_j \geq \sum_k D_{jk}^f \delta_k^f$	(6.3)
$\overline{\delta}_k^f = \delta_k^f$	$\delta_k^f = 0$	$\sum_j D_{jk}^f x_j^{f,p} \leq \sum_j C_{jk}^f x_j^{f,r} + T_k^f$	(7.1)
	$0 < \delta_k^f < \delta_{max}$	$\sum_j D_{jk}^f x_j^{f,p} = \sum_j C_{jk}^f x_j^{f,r} + T_k^f$	(7.2)
	$\delta_k^f = \delta_{max}$	$\sum_j D_{jk}^f x_j^{f,p} \geq \sum_j C_{jk}^f x_j^{f,r} + T_k^f$	(7.3)

Claim 3.12 *At every fixed point of F, $\sum_k C_{jk}^f \delta_k^f \leq p_j, \forall(f,j)$.*

Proof : Suppose $\exists(f,j)$ such that $\sum_k C_{jk}^f \delta_k^f > p_j$, then we have $x_j^{f,r} = x_{max}^r$ (from (5.3)).

It implies that whenever $C^f_{jk} > 0$, we have $\delta^f_k = 0$ (from (7.1)), which contradicts the starting assumption. □

Claim 3.13 *If* $\sum_{i,j} x^i_j p_j + \sum_{f,j} x^{f,r}_j p_j > \sum_{i,j} W^i_j p_j + \sum_{f,j} x^{f,p}_j p_j$*, then* $\exists i$ *such that* $\sum_j x^i_j p_j > \sum_j W^i_j p_j + \sum_{f,k} \Theta^i_f T^f_k \delta^f_k$.

Proof : This proof is by contradiction. Suppose we have $\sum_j x^i_j p_j \le \sum_j W^i_j p_j + \sum_{f,k} \Theta^i_f T^f_k \delta^f_k, \forall i$, then summing it over all i and using $\sum_i \Theta^i_f = 1$, we get

$$\sum_{i,j} W^i_j p_j + \sum_{f,k} T^f_k \delta^f_k \ge \sum_{i,j} x^i_j p_j > \sum_{i,j} W^i_j p_j + \sum_{f,j} x^{f,p}_j p_j - \sum_{f,j} x^{f,r}_j p_j \qquad (8)$$

Claims 3.11 and 3.12 imply that $x^{f,r}_j(\sum_k C^f_{jk}\delta^f_k - p_j) = 0, \forall(f,j)$ and $\delta^f_k(\sum_j D^f_{jk}x^{f,p}_j - \sum_j C^f_{jk}x^{f,r}_j - T^f_k) = 0, \forall(f,k)$, and it further implies that $\sum_{f,k} T^f_k \delta^f_k = \sum_{f,j,k} \delta^f_k D^f_{jk} x^{f,p}_j - \sum_{f,j} x^{f,r}_j p_j$. Using this and (8) we get $\sum_{f,j,k} \delta^f_k D^f_{jk} x^{f,p}_j > \sum_{f,j} x^{f,p}_j p_j$, which is a contradiction because (6.1), (6.2) and (6.3) imply that $\sum_{f,j,k} \delta^f_k D^f_{jk} x^{f,p}_j \le \sum_{f,j} x^{f,p}_j p_j$. □

Claim 3.14 *At every fixed point of* F*,*

- $0 < \lambda_i < \lambda_{max}, \forall i$
- $\sum_i x^i_j + \sum_f x^{f,r}_j \le 1 + \sum_f x^{f,p}_j, \forall j$
- $x^i_j < x_{max}, \forall(i,j)$

Proof : First suppose that $\lambda_i = \lambda_{max}$ for some i at a fixed point. It implies that for every good j such that $p_j \ge W_{min}/2nx_{max}$, we have $x^i_j = 0$ (from (3.1)). Hence, $\sum_j x^i_j p_j < W_{min}$, which contradicts (4.3). Hence $0 < \lambda_i < \lambda_{max}, \forall i$ at a fixed point.

Next suppose that $\lambda_i = 0$ for some i at a fixed point. It implies that for every $\gamma^i_k > 0$ and $U^i_{jk} > 0$, we have $x^i_j = x_{max}$ (from (3.3)). Note that here we use the *sufficiency condition* that for every k there exists a j such that $U^i_{jk} > 0$. Since x_{max} is much larger than $1 + \sum_f x^{f,p}_j$, we have $p_j > 0$ for such goods and $\sum_i x^i_j + \sum_f x^{f,r}_j > 1 + \sum_f x^{f,p}_j$ for all goods whose $p_j > 0$ (from (1.2)). By this, we get $\sum_{i,j} x^i_j p_j + \sum_{f,j} x^{f,r}_j p_j > \sum_{i,j} W^i_j p_j + \sum_{f,j} x^{f,p}_j p_j$. Using Claim 3.13, it implies that $\exists i'$ such that $\sum_j x^{i'}_j p_j > \sum_j W^{i'}_j p_j + \sum_{f,k} \Theta^{i'}_f T^f_k \delta^f_k$ and $\lambda_{i'} = \lambda_{max}$ (from (4.3)), which is a contradiction.

Finally suppose that there exists a j such that $\sum_i x^i_j + \sum_f x^{f,r}_j > 1 + \sum_f x^{f,p}_j$, then we have $p_j > 0$ and whenever $p_j > 0$, $\sum_i x^i_j + \sum_f x^{f,r}_j > 1 + \sum_f x^{f,p}_j$ (from (1.2)). It implies that there exists an i such that $\lambda_i = \lambda_{max}$, which is a contradiction. This also implies that $x^i_j < x_{max}, \forall(i,j)$. □

The proof of next claim is similar as in Claim 3.5, hence omitted.

Claim 3.15 *At every fixed point of* F*,* $u_i \le \sum_j U^i_{jk} x^i_j + T^i_k, \forall(i,k)$.

Claim 3.16 *At every fixed point of* F*,* $p_j \le \sum_k D^f_{jk}\delta^f_k, \forall(f,j)$.

Proof : Suppose there exists a (f, j) such that $p_j > \sum_k D^f_{jk}\delta^f_k$ at a fixed point, then $x^{f,p}_j = x^p_{max}$ (from (6.3)). Claims 3.14 and 3.11 imply that $\sum_i x^i_j + \sum_f x^{f,r}_j \leq 1 + \sum_f x^{f,p}_j, \forall j$ and $\sum_j D^f_{jk} x^{f,p}_j \leq \sum_j C^f_{jk} x^{f,r} + T^f_k, \forall (f, k)$, which leads to a contradiction since $x^{f,p}_j = x^p_{max}$ cannot satisfy these constraints as discussed in the sufficiency conditions. This claim uses the *no production out of nothing* and *no vacuous production* conditions. □

Together Claims 3.11, 3.12, 3.14, 3.15 and 3.16 imply that none of the conditions (1.2), (2.2), (3.3), (4.3), (5.3), (6.3), and (7.3) are satisfied at fixed points of F. Therefore, we get the following theorem.

Theorem 3.17 *Assuming sufficient conditions of the existence of market equilibrium, every fixed point of F gives a solution of AD-NCP and hence a market equilibrium. Further, F can be computed by a FIXP-circuit and hence market equilibrium computation problem for PLC utilities and PLC production is in FIXP.*

Remark 3.18 *This technique can be used to obtain Linear-FIXP (equivalent to PPAD) circuit for markets with SPLC utilities and SPLC production using the linear complementary problem (LCP) formulation given in [31], thereby giving alternate proof of membership in PPAD for such markets.*

4 Leontief-free Utility Functions and Production Sets

We first give a high level description of these notions, which were introduced in [42] for dealing with the situation in which goods are substitutes. Interestingly enough, there was no analogous notion in economics: the only notion dealing with substitutes in economics is *constant elasticity of substitution (CES) utilities*; however, as noted in [42], these utility functions are too restrictive, since the requirement that the agent have constant elasticity of substitution over the whole domain is too stringent and moreover can lead to odd optimal bundles. Furthermore, CES utilities satisfy constant returns to scale and hence do not capture decreasing marginal utilities due to satiation.

Let G be a set of divisible goods, $G = \{1, 2, \ldots, n\}$, and f be a PLC utility function of an agent for these goods, $f : \mathbb{R}^n_+ \rightarrow \mathbb{R}_+$. Recall from the Introduction that f is *separable* if it is the sum of utilities of individual goods, i.e. $f(x) = \sum_{j \in G} f_j(x_j)$, where $f_j : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is the utility function of the agent for good j, $j \in G$. A utility function that is not separable is said to be *non-separable*. Leontief utilities are an important class of non-separable utilities, capturing situations in which goods are complements. Given parameters a_j, $j \in G$, the *Leontief utility* of a bundle is defined to be $f(x) = \min_{j \in G} \left\{ \frac{x_j}{a_j} \right\}$. Clearly, if $a_j = 0$, good j is not desired at all; we will assume that at least two of these parameters are non-zero; if only one is, then this function is linear.

Observe that under a Leontief utility function, every infinitesimal amount of utility derived is obtained from *all goods having non-zero a_j's*, consumed in the desired proportions. In contrast, under a Leontief-free (LF) function, *goods compete* for every infinitesimal amount of utility derived via the mechanism of *segments*. We will first introduce the notion of segments for an separable PLC (SPLC) utility function $f(x) = \sum_{j \in G} f_j(x_j)$; recall that SPLC utilities form a subclass of LF utilities. Each "piece" of f_j, for each good j, defines a segment; each segment has three parameters. Clearly, each piece has two parameters, an upper bound in utility that can be accrued for the good obtained under this piece, which could be ∞, and the rate at which utility is accrued per unit of good obtained. These two parameters and the name of the good, i.e., j, are the three parameters associated with each segment of f_j. In preparation for introducing general LF functions, let us compute the utility derived for bundle x under f as follows. Write an LP that considers all possible allocations of x to the segments and attempts to maximize the sum of the utilities accrued under all segments (the utility accrued under a segment cannot exceed the upper bound specified). Clearly, if f is SPLC, the optimal solution will allocate each good to the segments having the highest rates.

Next, let us turn to a general LF function. The three parameters of each of its segments are:

1. An upper bound on the total utility that can be accrued for goods obtained under this segment, which could be ∞.
2. A non-empty subset of goods.
3. Corresponding to each good specified above, the rate at which utility is accrued per unit of good obtained.

Once again, the utility derived for a bundle x is computed via an analogous LP. Unlike the SPLC case, the solution of the LP has no easy description. In particular, which good(s) of a segment get how much allocation depends on the bundle x and can change drastically from one bundle to another.

A *Leontief-free production set*, which generalizes SPLC production, is also specified via segments. Let disjoint sets S_r and S_f be the set of raw goods and finished goods of a firm. In general, any good in S_f can be produced from any good in S_r; the precise rate of production is specified via the parameters of the segments[7]. Before defining the parameters of segments, let us give the notion of *raw units*. Each of the raw goods is first converted to raw units and then the raw units are converted to finished goods. The three parameters of each of its segments are:

1. An upper bound on the total number of raw units this segment can handle, which could be ∞.
2. Two non-empty subset, T_r and T_f, where $T_r \subseteq S_r$ and $T_f \subseteq S_f$.
3. Corresponding to each good $j \in T_r$ the rate at which raw units are obtained from j, and corresponding to each good $j' \in T_r$ the rate at which j' is obtained from raw units.

For given prices of raw goods and finished goods, a production schedule yielding optimal profit can be obtained by solving an LP which is analogous to that for an LF utility function.

4.1 A min-max relation

As indicated in Section 4, substitutability among goods is intimately connected to satiation, i.e., sub-additivity, in the joint utility obtained from a bundle of goods. In order to show that the notion of Leontief-free utility functions is well-founded, [42] study the extreme cases of satiation, assuming PLC utilities. They note that as far as intra-good satiation is concerned, the extreme cases are linear (in case of no satiation) and a PLC function which goes flat after only one piece (in case of maximal satiation). For inter-good satiation, they define the following notions.

Fix PLC functions $f_j : \mathbb{R}_+ \to \mathbb{R}_+,\ \forall j \leq n$, where f_j represents the utility obtained when only good j is consumed. Let $\sqsubseteq^j$ denote an n-dimensional unit vector with one on j^{th} co-ordinate. The following definition is with respect to f_js.

Definition 4.1 *We say that a PLC function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is* consistent*, if it is Leontief-free, and its restriction to good j is f_j. Formally, $f(x * \sqsubseteq^j) = f_j(x), \forall j \leq n, \forall x \in \mathbb{R}_+$.*

[7]Of course, if the rate turns out to be zero for a pair of goods j, j', then j' cannot be produced from j.

Let $f_{max} : \mathbb{R}^n_+ \to \mathbb{R}$ denote the joint utility for the extreme case when there is no inter-good satiation. Clearly, this function should be simply additive over $f_j(x_j)$, i.e., with no sub-additivity. Hence

$$\forall \boldsymbol{x} \in \mathbb{R}^n_+, \quad f_{max}(\boldsymbol{x}) = \sum_j f_j(x_j),$$

i.e., it is simply the SPLC combination of the utilities for individual goods.

For the other extreme, i.e., maximal inter-good satiation, define

$$f_{min}(\boldsymbol{x}) = \min_{f:\ \text{consistent}} f(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n_+ \tag{9}$$

Definition 4.2 *Given a bundle $\boldsymbol{x}$, we say that utility $t \in \mathbb{R}_+$ is* feasible *if the following holds: There is a division of $[0,\ t]$ into sub intervals say $I_1, \ldots, I_h$, where each interval is closed and $\sum k \leq h|I_k| = t$. Further, there is an assignment of intervals to goods, such that $x_j \leq \sum_{I \in S_j} f_j^{-1}(I), \forall j \leq n$, where S_j is the set of intervals assigned to good j.*

They show that f_{min} satisfies the following min-max relation:

$$\forall \boldsymbol{x} \in \mathbb{R}^n_+ \quad \min_{f:consistent} f(\boldsymbol{x}) = \max_{t: feasible for \boldsymbol{x}} t$$

Further, they show that f_{min} is a Leontief-free function an LP for it can be computed from the f_js in polynomial time. Moreover, predictions made by f_{min} are consistent with the function one would derive for maximal inter-good satiation in natural situations.

The next two sections will introduce the notation we will follow in this paper for specifying LF utility functions and production sets; as in the definition of Leontief functions given above, we will dispense with explicitly specifying subsets of goods corresponding to each segment by allowing rates to be zero.

4.2 Utility functions

A Leontief-free utility function U_i is specified by a set of segments. On segment k the agent derives utility $\sum_j c_j x_j$ from bundle $\boldsymbol{x}$, say up to l units of utility, where c_j's are non-negative. Intuitively, on this segment a unit amount of good j fetches c_j amount of utility. The common limit l on the maximum amount of utility that can be derived on this segment makes it non-separable. We define $U^i_{jk} \stackrel{\text{def}}{=} c_j$ and $L^i_k \stackrel{\text{def}}{=} l$. We note that L^i_k can be infinity.

Let x_{jk} denote the amount of good j consumed by the agent on this segment, then the overall utility $U_i(\boldsymbol{x})$ from a bundle $\boldsymbol{x}$ is calculated by solving the following linear program.

$$\begin{array}{ll} \max: & \sum_k u_k \\ s.t. & \forall k: \ u_k = \sum_j U^i_{jk} x_{jk}; \quad u_k \leq L^i_k \\ & \forall j: \ \sum_k x_{jk} \leq x_j \end{array}$$

Since each U^i_{jk} is non-negative, U_i is non-negative. It is non-decreasing due to maximization, and it is concave because for a convex combination $\boldsymbol{z} = \lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y},\ 0 < \lambda < 1$ of bundles $\boldsymbol{x}$ and $\boldsymbol{y}$, the corresponding convex combination of their optimal is a feasible point in the polyhedron corresponding to $U_i(\boldsymbol{z})$, and hence $U_i(\boldsymbol{z}) \geq \lambda U_i(\boldsymbol{x}) + (1-\lambda)U_i(\boldsymbol{y})$.

In a Leontief-type preference a set of goods are needed in a fixed proportion to derive non-zero utility, i.e., goods are complementary. For e.g., an agent needs one bread and half cube butter to make a sandwich (a unit utility), i.e., $u \leq$ #bread, $u \leq 2 \cdot$ #butter cubes. Such a preference is not allowed in above construction, as every u_k depends on exactly one linear equation on amounts, hence is *Leontief-free.*

4.3 Production sets

Recall from Section 2 that each firm f has a set $\mathcal{S}^f$ of production possibility vectors. We assume that set $\mathcal{S}^f$ is polyhedral and define a subclass of polyhedral production sets called *Leontief-free.* Such a production set is defined by a set of segments. A segment is defined as follows, where x_j^r's and x_j^p's denote the amount of goods used and produced respectively. We treat $\frac{0}{0}$ as 0.

$$R = \sum_{j \in \mathcal{R}^f} c_j x_j^r; \quad \sum_{j \in \mathcal{P}^f} \frac{x_j^p}{d_j} \leq R; \quad R \leq l$$

Essentially the above expression implies that on this segment $1/c_j$ amount of $j \in \mathcal{R}^f$ contributes to one unit of raw material and from this at most $d_{j'}$ amount of $j' \in \mathcal{P}^f$ can be produced.[8] Further, at most l units of raw material can be used to do production at this rate. This representation disallows Leontief-type productions such as making a sandwich from two slices of bread *and* a butter cube, or producing a unit of gasoline *and* a unit of petroleum gas from a unit of petroleum. For the k^{th} segment of firm f, we define $C_{jk}^f \stackrel{\text{def}}{=} c_j$, $D_{jk}^f \stackrel{\text{def}}{=} d_j$ and $L_k^f \stackrel{\text{def}}{=} l$. We note that L_k^f can be infinity. Let x_{jk}^r's and x_{jk}^p's denote the goods used and produced on segment k respectively, then combined production of firm f on all the segments is:

$$\begin{array}{llll} \forall k: & R_k = \sum_{j \in \mathcal{R}^f} C_{jk}^f x_{jk}^r; & \sum_{j \in \mathcal{P}^f} \frac{x_{jk}^p}{D_{jk}^f} \leq R_k; & R_k \leq L_k^f \\ \forall k: & \forall j \in \mathcal{R}^f,\ x_{jk}^r \geq 0; & \forall j \in \mathcal{P}^f,\ x_{jk}^p \geq 0 & \end{array}$$

Let $\mathcal{S}^f$ be the projection of the above set on $-\sum_k x_{jk}^r,\ \forall j \in \mathcal{R}^f$ and $\sum_k x_{jk}^p,\ \forall j \in \mathcal{P}^f$. Clearly, $\mathcal{S}^f$ is a polyhedral set, and by construction it satisfies all the four assumptions stated in Section 2.

We will call this market, with Leontief-free utility and Leontief-free production sets, by *Leontief-free market* and denote it by $\mathcal{M}$.

5 Market Equilibrium Characterization

Given prices, each firm produces as per a profit maximizing (optimal) production plan and each agent buys a utility maximizing (optimal) bundle that is affordable. At equilibrium, demand of each good meets its total supply. In this section we characterize optimal production plan and optimal bundle for Leontief-free markets.

Suppose prices of goods are given, and p_j denotes the price of good j. Since, agents earn from their shares in the profits of firms too, let ϕ^f denote the maximum profit of firm f at prices $\boldsymbol{p}$, which we will calculate later. First we characterize optimal bundles. Define the *bang-per-buck of agent i from good j on segment (i,k) relative to prices $\boldsymbol{p}$* as $\text{bpb}_{jk}^i \stackrel{\text{def}}{=} \frac{U_{jk}^i}{p_j}$.

The value bpb_{jk}^i represents the utility derived by agent i per unit of money while obtaining good j on segment (i,k). Since the budget of agent i is fixed at given prices, clearly she will prefer to obtain goods with higher bang-per-buck. Using this intuition next we state conditions for optimal bundles. Define bang-per-buck of a segment (i,k) to be $\text{bpb}_k^i \stackrel{\text{def}}{=} \max_j \text{bpb}_{jk}^i$. Let $\boldsymbol{x}$ be a bundle of goods for agent i, where x_{jk} is the amount of good j she obtains on segment (i,k).

[8] The reason behind putting d_js in the denominator is to not allow production of good j when d_j is zero.

B_0. Feasibility: $\forall k, \forall j : x_{jk} \geq 0$; $\forall k : \sum_j U^i_{jk} x_{jk} \leq L^i_k$, and $\sum_{j,k} x_{jk} p_j \leq \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f$.

B_1. An agent, if obtains goods on a segment, obtains only those yielding maximum bang-per-buck. Formally, if $x_{jk} > 0$ then $\text{bpb}^i_{jk} = \text{bpb}^i_k$.

B_2. Goods are obtained on segment (i, k) only if all the segments with bang-per-buck higher than bpb^i_k are bought fully. Formally, if $x_{jk} > 0$ and $\text{bpb}^i_{k'} > \text{bpb}^i_k$ then $\sum_j U^i_{jk'} x_{jk'} = L^i_{k'}$.

Thus, an optimal bundle of agent i can be computed as follows: sort her segments by decreasing bang per buck bpb^i_k and partition the segments by equality, i.e., each equivalence class will consist of all segments having equal bang-per-buck. Let the classes be: $\mathcal{Q}_1, \mathcal{Q}_2, \ldots$. At prices $\boldsymbol{p}$, segments in $\mathcal{Q}_l$ make i equally happy. She starts buying partitions in order, until all her money ($\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f$) is exhausted. Suppose she exhausts all her money at k_i^{th} partition. The segments in partitions 1 to k_{i-1} are called *forced*, those in partition k_i are called *flexible* and those in partitions k_{i+1} and higher are called *undesired*.

Lemma 5.1 *At prices $\boldsymbol{p}$, bundle $\boldsymbol{x}'$ is an optimal bundle for agent i iff it satisfies $\boldsymbol{B_0}$, $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$.*

Proof : It is easy to see that an optimal bundle of agent i is a solution of the following optimization problem.

$$\begin{array}{lll} \max & : & \sum_k \sum_j U^i_{jk} x_{jk} \\ \text{subject to} & & \forall k : \sum_j U^i_{jk} x_{jk} \leq L^i_k \\ & & \sum_{j,k} x_{jk} p_j \leq \sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f \\ & & \forall (j,k) : x_{jk} \geq 0 \end{array} \tag{10}$$

Note that, given prices the above formulation is a linear program (LP). Therefore, feasibility and Karush-Kuhn-Tucker (KKT) conditions completely characterize its solutions [5], Hence bundle $\boldsymbol{x}'$ has to satisfy them. Further, feasibility and $\boldsymbol{B_0}$ are equivalent. Next we show that KKT conditions are equivalent to $\boldsymbol{B_1}$ and $\boldsymbol{B_2}$. Let β_k, and δ be the non-negative dual variables of the first and second inequalities of (10) respectively. The KKT conditions of this LP are:

$$\begin{array}{lllll} \forall (j,k) : & \frac{\delta}{1-\beta_k} \geq \frac{U^i_{jk}}{p_j} & \text{and} & x_{jk} > 0 \ \Rightarrow & \frac{\delta}{1-\beta_k} = \frac{U^i_{jk}}{p_j} \\ \forall k : & \beta_k > 0 & & \Rightarrow & \sum_j U^i_{jk} x_{jk} = L^i_k \end{array}$$

Note that, if goods are obtain on (i, k) then $\text{bpb}^i_k = \delta/1-\beta_k$ else $\text{bpb}^i_k \leq \delta/1-\beta_k$. Therefore, $\boldsymbol{B_1}$ and the first KKT condition are equivalent. Suppose goods are obtained on (i, k) and $\text{bpb}^i_{k'} > \text{bpb}^i_k$, then it must be the case that $\beta_{k'} > \beta_k \geq 0$. Hence, the second KKT condition and $\boldsymbol{B_2}$ are also equivalent. $\square$

Next we characterize optimal production plans for firms. Recall that on segment (f, k), $1/C^f_{jk}$ units of good $j \in \mathcal{R}^f$ is considered as a unit of raw material and can be used to produce $D^f_{j'k}$ amount of good $j' \in \mathcal{P}^f$. Therefore, p_j/C^f_{jk} is the cost per unit raw material when good j is used, and $D^f_{j'k} p_{j'}$ is the revenue earned by producing good j' from a unit raw material. We define cost-per-unit (cpu), revenue-per-unit (rpu) and profit-per-unit (ppu) of firm f on segment (f, k) to be

$$\text{cpu}^f_k \stackrel{\text{def}}{=} \min_{j \in \mathcal{R}^f} \frac{p_j}{C^f_{jk}}, \quad \text{rpu}^f_k \stackrel{\text{def}}{=} \max_{j \in \mathcal{P}^f} D^f_{jk} p_j, \quad \text{and} \quad \text{ppu}^f_k \stackrel{\text{def}}{=} \text{rpu}^f_k - \text{cpu}^f_k \quad \text{respectively.}$$

Let $(\boldsymbol{x}^r, \boldsymbol{x}^p)$ be the bundles of goods used and produced by firm f, where x^r_{jk} and x^p_{jk} are the amount of good j used and produced on segment k respectively. Consider the following conditions for optimality.

$\boldsymbol{P_0}$. Feasibility. $\forall k : \sum_{j\in\mathcal{P}^f} \frac{x^p_{jk}}{D^f_{jk}} \leq \sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk} \leq L^f_k$, and $\forall k : x^p_{jk} \geq 0, \forall j \in \mathcal{P}^f;\ x^r_{jk} \geq 0, \forall j \in \mathcal{R}^f$.

$\boldsymbol{P_1}$. A firm, if it produces on a segment, uses the least cost raw goods and produces maximum revenue fetching goods. Formally, if $x^r_{jk} > 0$ then ${p_j}/{C^f_{jk}} = \text{cpu}^f_k$, and if $x^p_{jk} > 0$ then $D^f_{jk}p_j = \text{rpu}^f_k$.

$\boldsymbol{P_2}$. No production on the loss making segments. Formally, if $\text{ppu}^f_k < 0$ then $x^r_{jk} = 0, x^p_{jk} = 0, \forall j$.

$\boldsymbol{P_3}$. Segments with positive profit are utilized fully. Formally, if $\text{ppu}^f_k > 0$ then $\sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk} = L^f_k$.

$\boldsymbol{P_4}$. If the price of produced good is positive then production should match the raw material used, i.e., $\sum_{j\in\mathcal{P}^f} \frac{x^p_{jk}}{D^f_{jk}} = \sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk}$.

Intuitively an optimal production plan for firm f can be obtained as follows. If $\text{ppu}^f_k > 0$ then segment (f, k) should be utilized to its maximum production limit. If $\text{ppu}^f_k < 0$ then no production on (f, k) at all, and if $\text{ppu}^f_k = 0$ then it doesn't matter how much of segment (f, k) is utilized. The segments with strictly positive profit will be called *forced*, zero profit segments will be called *flexible* and negative profit segments will be called *undesired*. The total profit of firm f is $\phi^f \stackrel{\text{def}}{=} \sum_{k,\text{ppu}^f_k\geq 0} L^f_k\text{ppu}^f_k$.

Lemma 5.2 *At prices* $\boldsymbol{p}$, $(\boldsymbol{x}^{r'}, \boldsymbol{x}^{p'})$ *correspond to an optimal production of firm* f *iff it satisfies* $\boldsymbol{P_0}$-$\boldsymbol{P_4}$.

Proof : An optimal production plan of firm f at prices $\boldsymbol{p}$ is a solution of the following LP.

$$\begin{array}{lll} \max & : & \sum_{j\in\mathcal{P}^f} p_j \sum_k x^p_{jk} - \sum_{j\in\mathcal{R}^f} p_j \sum_k x^r_{jk} \\ \text{subject to} & \forall k: & \sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk} \leq L^f_k \\ & \forall k: & \sum_{j\in\mathcal{P}^f} \frac{x^p_{jk}}{D^f_{jk}} \leq \sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk} \\ & \forall k: & \forall j \in \mathcal{P}^f,\ x^p_{jk} \geq 0; \quad \forall j \in \mathcal{R}^f, x^r_{jk} \geq 0 \end{array} \tag{11}$$

Since feasibility and KKT conditions characterize solutions of an LP, plan $(\boldsymbol{x}^{r'}, \boldsymbol{x}^{p'})$ has to satisfy them. Clearly, condition $\boldsymbol{P_0}$ is equivalent to feasibility in (11). We show that the KKT conditions are equivalent to conditions $\boldsymbol{P_1}$-$\boldsymbol{P_4}$. Let α_k and π_k be the non-negative dual variables of first and second inequalities of LP (11) respectively. The KKT conditions are:

$$\begin{array}{llll} \forall k, \forall j \in \mathcal{R}^f: & \pi_k - \alpha_k \leq \frac{p_j}{C^f_{jk}} & \text{and} & x^r_{jk} > 0 \Rightarrow \pi_k - \alpha_k = \frac{p_j}{C^f_{jk}} \\ \forall k, \forall j \in \mathcal{P}^f: & \pi_k \geq D^f_{jk}p_j & \text{and} & x^p_{jk} > 0 \Rightarrow \pi_k = D^f_{jk}p_j \\ \forall k: & \alpha_k > 0 \Rightarrow \sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk} = L^f_k & & \\ \forall k: & \pi_k > 0 \Rightarrow \sum_{j\in\mathcal{P}^f} \frac{x^p_{jk}}{D^f_{jk}} = \sum_{j\in\mathcal{R}^f} C^f_{jk}x^r_{jk} & & \end{array}$$

First two KKT conditions are equivalent to condition $\boldsymbol{P_1}$. Thus for segments with production $\text{cpu}_k^f = \pi_k - \alpha_k$, $\text{rpu}_k^f = \pi_k$ and $\text{ppu}_k^f = \alpha_k$. On segment (f,k) if price of a good produced is positive, then clearly $\pi_k > 0$ and hence the fourth KKT condition and $\boldsymbol{P_4}$ are equivalent. If $\text{ppu}_k^f < 0$ then either the first or the second KKT condition hold with strict inequality, and therefore $\boldsymbol{P_2}$ is satisfied. Given that there is production on segment k, it is easy to see that $\text{ppu}_k^f = \alpha_k$. The third KKT condition is equivalent to $\boldsymbol{P_3}$. □

We note that if $\boldsymbol{p}$ is an equilibrium price vector then so is $\alpha \cdot \boldsymbol{p}$, for any $\alpha > 0$, with optimal bundle and optimal production plans unchanged - *scale invariant*.

6 Leontiefness and Irrationality

In this section we describe examples of markets with exactly one Leontief type utility or production segment and only irrational equilibrium prices. First we construct an exchange market where all utility functions are linear except one.

Example 6.1 *Consider a market with two goods and three agents. The endowments of agents are $w^1 = w^2 = w^3 = (1,1)$. Utility function of every agent has only one segment, and there is no limit on the segments. Let x_j^i be the amount of good j obtained by agent i on its segment. For agent 1 it is $U_1 = x_1^1$, and for agent 2 it is $U_2 = x_2^2$; both linear. For agent 3 it is Leontief, namely $U_3 \le x_1^3/2$ and $U_3 \le x_2^3$.*

Since, equilibrium prices are scale invariant, set $p_1 = 1$. The fact that both the inequality of U_3 should hold with equality at equilibrium, together with market clearing conditions, give us $p_2^2 + 2p_2 - 2 = 0$. The only non-negative solution is $p_2 = \sqrt{3} - 1$. Equilibrium allocations are $x_1^1 = \sqrt{3}, x_2^2 = \sqrt{3}/(\sqrt{3}-1)$ and $x_1^3 = 2x_2^3 = \sqrt{3}/(\sqrt{3}+1)$.

Next two examples are of markets with firms. Utility functions of all the agents are linear in both. The first has one Leontief type constraint on raw goods, and the second has it on the produced goods.

Example 6.2 *Consider a market with three goods, three agents and one firm. Endowments of agents are $w^1 = w^2 = w^3 = (1,1,0)$. Each utility function has one linear segment; $U_1 = x_1^1$, $U_2 = x_2^2$ and $U_3 = x_3^3$. The firm is owned by agent 3, i.e., $\Theta_1^3 = 1$. It has exactly one production segment without any upper limit on the raw material used. i.e., $L_1 = \infty$, and needs two units of good 1 and a unit of good 2 to produce a unit of good 3. Let x_j^rs and x_j^ps be the amount of goods used and produce by the firm on its only segment, then the conditions are $2 \cdot x_3^p \le x_1^r$, and $x_3^p \le x_2^r$.*

Again set $p_1 = 1$. Due to the demand of the third agent, the firm has to produce at equilibrium. Further, firm makes zero profit or else it will want to produce infinite. Hence, we have $p_3 = 2 + p_2$. The market clearing conditions give $p_2^2 + 2p_2 - 2 = 0$. Thus the only equilibrium prices of this market are $p_1 = 1, p_2 = \sqrt{3} - 1$ and $p_3 = (1+\sqrt{3})/2$. At equilibrium the allocation and production variables are: $x_1^1 = \sqrt{3}$, $x_2^2 = \sqrt{3}/(\sqrt{3}-1)$ and $x_3^3 = x_3^p = x_1^r/2 = x_2^r = \sqrt{3}/(\sqrt{3}+1)$.

Example 6.3 *Again the goods and agents are as in Example 6.2, and there is one firm with only one production segment without any upper limit. But now the firm can produce a unit of good 3 and two units of good 2 from a unit of good 1, i.e., $x_3^p \le x_1^r$ and $x_2^p \le 2 \cdot x_1^r$.*

Set $p_1 = 1$. At equilibrium the firm will produce with zero profit. Hence, $p_1 = p_3 + 2p_2$. Market clearing conditions give $2p_2^2 - 6p_2 + 1 = 0$. By solving this, the only equilibrium prices we get are $p_1 = 1, p_2 = (3-\sqrt{7})/2$ and $p_3 = \sqrt{7} - 2$, and allocation and production variables are $x_1^1 = 5-\sqrt{7}/2, x_2^2 = (5-\sqrt{7})/(3-\sqrt{7})$ and $x_3^3 = x_3^p = x_2^p/2 = x_1^r = (5-\sqrt{7})/2(\sqrt{7}-2)$.

7 LCP Formulation for Leontief-free Exchange Market

In order to convey the main ideas without introducing too much complication, we first derive linear complementarity problem (LCP)[9] formulation for exchange markets with Leontief-free utilities. In case of exchange markets, it needs to capture two main aspects: optimal bundle to each agent, and market clearing. It is relatively easy to ensure market clearing so we do that first.

We define variable p_j to denote the price of good j. We define variable q^i_{jk} to denote money spent by agent i to obtain good j on segment k, instead of the amount variable x^i_{jk} to avoid quadratic expression in agent's market clearing condition, namely $\sum_{j,k} x^i_{jk} p_j$ is the money spent by agent i. All variables have non-negativity constraints; for the sake of brevity, we will not write them explicitly. As is the case with Garg et al. [30] and Eaves [26] the task of market clearing does not need complementarity – just non-negativity suffices. However, with every constraint we also include corresponding complementarity condition to obtain an LCP in standard form.

$$\forall j \in \mathcal{G}: \quad \sum_{i,k} q^i_{jk} \le p_j \quad \text{and} \quad p_j(\sum_{i,k} q^i_{jk} - p_j) = 0 \tag{12}$$

$$\forall i \in \mathcal{A}: \quad \sum_j W^i_j p_j \le \sum_{j,k} q^i_{jk} \quad \text{and} \quad \lambda_i(\sum_j W^i_j p_j - \sum_{j,k} q^i_{jk}) = 0 \tag{13}$$

We will refer to the constraints as follows: The equation number refers to the inequality and the equation number with a prime refers to the complementarity condition, i.e., (12) refers to the inequality and (12') refers to the complementarity condition.

Lemma 7.1 *The set of constraints given in (12) and (13) hold if and only if the market clears.*

Proof : Adding the constraints in (12) over all goods and those in (13) over all agents we get

$$\sum_{i,j,k} q^i_{jk} \le \sum_j p_j \quad \text{and} \quad \sum_{i,j} W^i_j p_j \le \sum_{i,j,k} q^i_{jk},$$

respectively. Since $\sum_{i,j} W^i_j p_j = \sum_j p_j$, both these inequalities are equalities. Finally, by non-negativity, all the constraints in (12) and (13) must hold with equality, which proves the lemma. □

Since, utilities are non-separable, ensuring optimal bundle is more tricky. For optimal bundle of agent i, we derive the constraints through KKT conditions of LP (10) restated below. Here x^i_{jk} is the amount of good j agent i obtains on segment k.

$$\forall (j,k): \quad \frac{\delta}{1-\beta_k} \ge \frac{U^i_{jk}}{p_j} \quad \text{and} \quad x^i_{jk} > 0 \;\Rightarrow\; \frac{\delta}{1-\beta_k} = \frac{U^i_{jk}}{p_j}$$

$$\forall k: \quad \beta_k > 0 \quad \Rightarrow \quad \sum_j U^i_{jk} x^i_{jk} = L^i_k$$

To ensure $\boldsymbol{B_0}$, the total utility on segment (i,k) should not be more than L^f_k. We use the above conditions to derive an equivalent upper bound on the total money spent of segment (i,k).

$$\forall k: \; \sum_j U^i_{jk} x^i_{jk} \le L^i_k \;\Rightarrow\; \sum_j \frac{U^i_{jk}}{p_j}(x^i_{jk} p_j) \le L^i_k \;\Rightarrow\; \frac{\delta}{1-\beta^i_k} \sum_j q^i_{jk} \le L^i_k \;\Rightarrow\; \sum_j q^i_{jk} \le L^i_k \left(\frac{1}{\delta} - \frac{\beta_k}{\delta}\right)$$

[9] Refer to Appendix A for detailed description of LCP.

To get linear constraints, we replace ${}^1/_\delta$ with λ_i and ${}^{\beta_k}/_\delta$ with γ_k^i. Intuitively $1/\lambda_i$ captures the bang-per-buck of flexible segments for agent i, and γ_k^i's are supplements for her forced segments. The next two conditions follows from the above KKT two conditions respectively.

$$\forall(i,k), \forall j: \quad U_{jk}^i(\lambda_i - \gamma_k^i) - p_j \leq 0 \quad \text{and} \quad q_{jk}^i(U_{jk}^i(\lambda_i - \gamma_k^i) - p_j) = 0 \tag{14}$$

$$\forall(i,k): \quad \textstyle\sum_j q_{jk}^i \leq L_k^i(\lambda_i - \gamma_k^i) \quad \text{and} \quad \gamma_k^i(\sum_j q_{jk}^i - L_k^i(\lambda_i - \gamma_k^i)) = 0 \tag{15}$$

Remark 7.2 *Without using KKT conditions, equations (14) and (15) are not clear at all, as the trick of adding supplement variables in prices from [30] does not extend to Leontief-free utilities.*

Lemma 7.3 *The set of constraints given in (14) and (15) hold if every agent gets an optimal bundle.*

Proof : Consider an equilibrium of $\mathcal{M}$. Substitute for the variables p_j, q_{jk}^i, λ_i in the manner described above. Substitute for the variables γ_k^i as follows: if segment (i,k) is flexible or undesirable, set it to zero, and if it is forced, set it so that the following equality is satisfied

$$\frac{1}{\lambda_i - \gamma_k^i} = \max_j \frac{U_{jk}^i}{p_j}.$$

Clearly, all the γ_k^is satisfy non-negativity. Now, it is easy to verify that in each of the three cases – that the segment (i,k) is forced, flexible or undesirable – the constraints (14) and (14') are satisfied. For upper bound on the utility, we need to ensure that $\sum_j U_{jk}^i x_{jk}^i \leq L_k^i$. Which is equivalent to $\sum_j {}^{U_{jk}^i}/_{p_j}(x_{jk}^i p_j) \leq L_k^i$. From (14') we have,

$$x_{jk}^i > 0 \Rightarrow q_{jk}^i > 0 \Rightarrow \tfrac{1}{\lambda_i - \gamma_k^i} = \tfrac{U_{jk}^i}{p_j}. \text{ This gives } \tfrac{\sum_j q_{jk}^i}{\lambda_i - \gamma_k^i} \leq L_k^i$$

Thus, constraints (15) and (15') are also satisfied. □

Let the constraints (12) through (15) and (12') through (15'), together with non-negativity on all variables, define our **LCP**. The following lemma is a direct consequence of Lemmas 7.3 and 7.1.

Lemma 7.4 *Every market equilibrium of $\mathcal{M}$ is a solution of* **LCP**.

The polyhedron defined by the linear inequalities of an LCP is called its feasible region. Note that all inequalities of **LCP** are homogeneous, forming a cone with origin being its vertex as the feasible region. Lemke's scheme when applied to **LCP** will terminate at origin, which is not an equilibrium. Further, **LCP** may admit more non-equilibrium solutions where prices of a set of goods is set to zero.

We overcome all these shortcomings as follows. First we ensure that price of every good is non-zero at equilibrium. Define desire(j) to be roughly the lower bound on total amount that agents may demand if price of good j is set to zero, *i.e.,*

$$\text{desire}(j) = \sum_{(i,k):U_{jk}^i>0} \frac{L_k^i}{\max_d U_{dk}^i}$$

Lemma 7.5 *Equilibrium prices of an exchange market $\mathcal{M}$ are strictly positive if* desire$(j) > 1$, $\forall j$.

Proof : Suppose, some goods have zero price at an equilibrium. Let $\mathcal{Z} \subseteq \mathcal{G}$ be the set of such goods. If goods of $\mathcal{Z}$ fetches non-zero utility on a segment, then it is a forced segment with infinite bang-per-buck, and only zero priced goods are obtained it. Let $\mathcal{S}$ be the set of such segments.

$$\begin{aligned}\sum_{(i,k)\in\mathcal{S}, j\in\mathcal{Z}} x^i_{jk} &= \sum_{j\in\mathcal{Z};(i,k)\in\mathcal{S}:U^i_{jk}>0} \frac{U^i_{jk}x^i_{jk}}{U^i_{jk}} \\ &\geq \sum_{(i,k)\in\mathcal{S}} \frac{1}{\max_j U^i_{jk}} \sum_{j\in\mathcal{Z}} U^i_{jk}x^i_{jk} = \sum_{(i,k)\in\mathcal{S}} \frac{L^i_k}{\max_j U^i_{jk}} \\ &\geq \textstyle\sum_{j\in\mathcal{Z}} \text{desire}(j) > \sum_{j\in\mathcal{Z}} 1\end{aligned}$$

A contradiction to market clearing condition. □

We assume that in market $\mathcal{M}$, desire$(j) > 1$, for each good j. This ensures that equilibrium prices are strictly positive (Lemma 7.5). Since, equilibrium prices are scale invariant, we will lower bound the price variables by one in **LCP** to get a non-homogeneous LCP as follows: Replace p_j with $p'_j + 1$ and $p'_j \geq 0$. The resulting LCP, call it **NH-LCP**, is as follows.

$$\forall(i,k), \forall j: \quad U^i_{jk}(\lambda_i - \gamma^i_k) - p'_j \leq 1 \quad \text{and} \quad q^i_{jk}(U^i_{jk}(\lambda_i - \gamma^i_k) - p'_j - 1) = 0 \tag{16}$$

$$\forall(i,k): \quad \textstyle\sum_j q^i_{jk} \leq L^i_k(\lambda_i - \gamma^i_k) \quad \text{and} \quad \gamma^i_k(\sum_j q^i_{jk} - L^i_k(\lambda_i - \gamma^i_k)) = 0 \tag{17}$$

$$\forall j \in \mathcal{G}: \quad \textstyle\sum_{i,k} q^i_{jk} - p'_j \leq 1 \quad \text{and} \quad p'_j(\sum_{i,k} q^i_{jk} - (p'_j + 1)) = 0 \tag{18}$$

$$\forall i \in \mathcal{A}: \quad \textstyle\sum_j W^i_j p'_j - \sum_{j,k} q^i_{jk} \leq -\sum_j W^i_j \quad \text{and} \quad \lambda_i(\sum_j W^i_j(p'_j + 1) - \sum_{j,k} q^i_{jk}) = 0 \tag{19}$$

Next theorem is a direct consequence of Lemmas 7.1, 7.4, 7.5 and 7.3

Theorem 7.6 *The solutions of* **NH-LCP** *capture exactly the equilibria of Leontief-free exchange market $\mathcal{M}$ up to scaling, assuming that for every good j the* desire(j) *is at least* 1*.*

Since every LCP has a solution which forms a vertex of its polyhedron (see Appendix A), we get the following using as a corollary of Theorem 7.6.

Theorem 7.7 *Leontief-free exchange market $\mathcal{M}$ with all parameters rational that has an equilibrium admits equilibrium prices which are polynomial sized rational numbers, assuming that for every good j* desire(j) *is at least* 1*.*

Since **NH-LCP** is in the standard form, Lemke's scheme[10] can be directly applied to it. The procedure may end on a *secondary ray*, as market equilibrium may not always exist [60]. Assuming weakest known sufficiency conditions [39] we can show that there are no secondary rays in the polyhedron corresponding to **NH-LCP**. However, we omit this proof. It subsumes in the no-secondary ray proof for the complete LCP of Leontief-free market in Section 10.1. Next, we proceed to a more interesting problem of capturing production as complementarity.

[10]Refer to Appendix A for the description of Lemke's scheme

8 LCP for Arrow-Debreu Market with Leontief-free Production

At Arrow-Debreu market equilibrium firms produce as per optimal production plans, agents obtain optimal bundles and market clears. We will derive LCP conditions to ensure optimal production plan to each firm. The conditions (14) and (15) for optimal bundle remain as they are, however market clearing conditions have to be modified to take into account the produced and used goods by firms.

As done for optimal bundle, LCP constraints for optimal production plans can be derived through KKT conditions of LP (11). However it is mathematically involved and unintuitive. Here we describe a more (intuitive) direct LCP construction, using $\boldsymbol{P_0}$-$\boldsymbol{P_4}$ of Section 5.

Remark 8.1 *Amount variables are more suitable to capture optimal production. However we finally need to merge it with the formulation of optimal bundle and market clearing to get a complete LCP. Hence, we have to use the money (value) variables here too, which makes the task of capturing conditions $\boldsymbol{P_0}$ and $\boldsymbol{P_4}$ very tricky.*

Define variable $q^{f,r}_{jk}$ to denote the money firm f spends to buy raw good j on segment k ($\equiv p_j x^{f,r}_{jk}$), and variable $q^{f,p}_{jk}$ to denote the revenue firm f earns by producing good j on segment k ($\equiv p_j x^{f,p}_{jk}$). To ensure optimal production for firm f, we need to capture forced, flexible and undesired segments correctly. On segment (f,k) raw good $j \in \mathcal{R}^f$ costs p_j/C^f_{jk} per unit raw material, and we will capture the least such cost in variable τ^f_k. Condition $\boldsymbol{P_1}$ implies,

$$\forall (f,k), \forall j \in \mathcal{R}^f : \quad \tau^f_k - \frac{p_j}{C^f_{jk}} \leq 0 \quad \text{and} \quad q^{f,r}_{jk}(\tau^f_k - \frac{p_j}{C^f_{jk}}) = 0 \tag{20}$$

As per conditions $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$, production happens only on segments with non-negative profit and that too only of maximum revenue fetching goods. On segment (f,k) revenue from good $j \in \mathcal{P}^f$ is $D^f_{jk} p_j$ per unit raw material used. Then the profit per unit raw material is $\max_{j \in \mathcal{P}^f} D^f_{jk} p_j - \tau^f_k$. We will capture this value, if non-negative, in variable δ^f_k as follows.

$$\forall (f,k), \forall j \in \mathcal{P}^f : \quad D^f_{jk} p_j - \tau^f_k - \delta^f_k \leq 0 \quad \text{and} \quad q^{f,p}_{jk}\left(D^f_{jk} p_j - \tau^f_k - \delta^f_k\right) = 0 \tag{21}$$

For feasibility condition $\boldsymbol{P_0}$, non-negativity on $\boldsymbol{q}^r$ and $\boldsymbol{q}^p$ will ensure non-negativity on $\boldsymbol{x}^r$ and $\boldsymbol{x}^p$. Ensuring the upper bound of L^f_k on $\sum_{j \in \mathcal{R}^f} C^f_{jk} x^{f,r}_{jk}$ is tricky, because direct substitution of amount variables with money variables yields multi-variate polynomials. Using (20') we get a linear constraints in money variables instead, as follows.

$$\begin{aligned} \sum_{j \in \mathcal{R}^f} C^f_{jk} x^{f,r}_{jk} \leq L^f_k \quad &\Rightarrow \quad \sum_{j \in \mathcal{R}^f} \frac{C^f_{jk}}{p_j} p_j x^{f,r}_{jk} \leq L^f_k \quad (\text{Here } p_j x^{f,r}_{jk} = q^{f,r}_{jk}) \\ &\Rightarrow \quad \frac{1}{\tau^f_k} \sum_{j \in \mathcal{R}^f} q^{f,r}_{jk} \leq L^f_k \quad \Rightarrow \quad \sum_{j \in \mathcal{R}^f} q^{f,r}_{jk} \leq L^f_k \tau^f_k \end{aligned}$$

Condition $\boldsymbol{P_3}$ needs that segments with positive profit are utilized completely (forced), and zero profit segments can be utilized partially (flexible). As δ^f_k captures the profit for these two types of segments, we get

$$\forall (f,k) : \quad \sum_{j \in \mathcal{R}^f} q^{f,r}_{jk} - L^f_k \tau^f_k \leq 0 \quad \text{and} \quad \delta^f_k (\sum_{j \in \mathcal{R}^f} q^{f,r}_{jk} - L^f_k \tau^f_k) = 0 \tag{22}$$

Finally, for $\boldsymbol{P_4}$, that the amount produced depends on the amount of raw material used, we observe that total revenue is total cost plus profit.

$$\forall (f,k) : \quad \sum_{j \in \mathcal{P}^f} q^{f,p}_{jk} = \sum_{j \in \mathcal{R}^f} q^{f,r}_{jk} + L^f_k \delta^f_k \tag{23}$$

Lemma 8.2 *The constraints (20) - (23) hold if every firm produces as per an optimal production plan.*

Proof : Consider an equilibrium of $\mathcal{M}$. Substitute p_j, $q^{f,r}_{jk}$ and $q^{f,p}_{jk}$ in the manner described above. Set τ^f_k to $\min_{j\in\mathcal{R}^f} {p_j}/{C^f_{jk}}$ and δ^f_k to $\max_{j\in\mathcal{P}^f} D^f_{jk}p_j - \tau^f_k$ if non-negative, else set it to zero. Note that due to Lemma 5.2 the optimal production plans satisfy $\boldsymbol{P_0}$-$\boldsymbol{P_4}$.

Since, τ^f_k captures cpu^f_k and δ^f_k captures ppu^f_k if non-negative, conditions $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$ ensure that (20), (21), (20') and (21') are satisfied.

The way (22) is derived, clearly it has to be satisfied as every production adheres to feasibility ($\boldsymbol{P_0}$) Further (22') follows due to $\boldsymbol{P_3}$. Finally we observe that due to $\boldsymbol{P_4}$, if profit is zero on a segment, then its revenue is same as its cost, and if the profit is positive, then revenue is cost plus profit, which is exactly what (23) captures. $\square$

To take production into account the market clearing conditions (12) and (13) are modified as follows.

$$\begin{aligned}\forall j \in \mathcal{G}: \quad & \sum_{i,k} q^i_{jk} + \sum_{(f,k),j\in\mathcal{R}^f} q^{f,r}_{jk} \le p_j + \sum_{(f,k),j\in\mathcal{P}^f} q^{f,p}_{jk} \\ & \text{and} \quad p_j\left(\textstyle\sum_{i,k} q^i_{jk} + \sum_{(f,k),j\in\mathcal{R}^f} q^{f,r}_{jk} - p_j + \sum_{(f,k),j\in\mathcal{P}^f} q^{f,p}_{jk}\right) = 0\end{aligned} \tag{24}$$

Suppose, ϕ^f denotes the profit $\sum_k L^f_k\delta^f_k$ of firm f.

$$\forall i \in \mathcal{A}: \quad \textstyle\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f \le \sum_{j,k} q^i_{jk} \quad \text{and} \quad \lambda_i(\sum_j W^i_j p_j + \sum_f \Theta^i_f\phi^f - \sum_{j,k} q^i_{jk}) = 0 \tag{25}$$

Lemma 8.3 *The set of constraints given in (24) and (25) hold if and only if the market clears.*

Proof : Similar to Lemma 7.1 the proof follows using $\sum_i W^i_j = 1, \forall j$, and $L^f_k\delta^f_k = \sum_{j\in\mathcal{P}^f} q^{f,p}_{jk} - \sum_{j\in\mathcal{R}^f} q^{f,r}_{jk}$, $\forall (f,k)$ from (23). $\square$

Define **ADLCP** to be the LCP defined by constraints (14), (15), and (20) through (25), together with their complementarity conditions and non-negativity on all the variables. The next lemma follows from Lemmas 7.3, 8.2 and 8.3.

Lemma 8.4 *Every market equilibrium of $\mathcal{M}$ is a solution of* **ADLCP***.*

As is the case with **LCP**, **ADLCP** too is homogeneous and suffers from the similar shortcomings like conical feasible region with only one vertex, namely origin, and non-equilibrium solutions. Again we can resolve these by constructing a non-homogeneous LCP, where all the market equilibria are preserved up to scaling.

8.1 Conditions for positive equilibrium prices

One of the assumptions on production sets $\mathcal{S}^f$'s was that firms together can not produce something out of nothing. In case of Leontief-free production, on segment (f,k) a unit of $j \in \mathcal{R}^f$ can produce $C^f_{jk}D^f_{j'k}$ units of $j' \in \mathcal{P}^f$. Construct a directed graph $G_{\mathcal{F}}(\mathcal{M})$ where goods are nodes, and there is an edge from j to j' with weight $\max_{(f,k),j\in\mathcal{R}^f,j'\in\mathcal{P}^f} C^f_{jk}D^f_{j'k}$, if it is non-zero. Then *no production out of nothing* condition on production (see Section 2) is equivalent to,

Definition 8.5 (No production out of nothing (restated)) *Market $\mathcal{M}$ satisfies* no production out of nothing, *if weights of edges in every cycle of $G_{\mathcal{F}}(\mathcal{M})$ multiply to strictly less than one.*

The above condition ensures that along any production cycle net amount of some good strictly decreases, while all other goods remain the same. This implies that the net production of every good is finite at any feasible production, as initial endowment of every good is unit. Therefore, the amount of goods available to agents is always finite. Now if demand of a good is infinite at any given prices, then it can not be equilibrium prices. Using this fact we define a condition to ensure positive equilibrium prices.

Definition 8.6 (Non-satiation) *Agent i is said to be* non-satiated *for good j if there exists a segment (i,k) with $U^i_{jk} > 0$ and L^i_k infinity.*

Lemma 8.7 *Equilibrium prices of a Leontief-free market $\mathcal{M}$ are strictly positive if every good has a non-satiated agent in $\mathcal{M}$.*

Proof : Suppose, prices of a set $\mathcal{Z} \subseteq \mathcal{G}$ of goods is zero at an equilibrium. On a segment (i,k) if $U^i_{jk} > 0, j \in \mathcal{Z}$ then bpb^i_{jk} is infinity, implying that (i,k) is forced and only goods of $\mathcal{Z}$ are bought on it. Since every good has a non-satiated agent, demand of at least one of zero priced good is infinity. The *no production out of nothing* condition disallows infinite production of any good, contradicting the market clearing condition. □

Henceforth we assume that every good in $\mathcal{M}$ has a non-satiated agent.[11]

8.2 Non-homogeneous LCP

Now we can lower bound the price variables by a positive number, as equilibrium prices are positive (Lemma 8.7) and scale invariant. However, we also want that negative rhs appears only in agent side market clearing condition (25). This is needed to ensure that all the equilibrium conditions except market clearing are satisfied on the path followed by the algorithm, which is crucial to prove *no secondary rays* and in turn convergence of the algorithm.

Suppose, we lower bound p_j by a positive rational number E_j. We do this by replacing p_j with $p'_j + E_j$ in **ADLCP**. Then to keep rhs of (21) non-negative we replace τ^f_k with $\tau'^f_k + E^f_k$ where $E^f_k \geq \max_{j\in\mathcal{P}^f} D^f_{jk}E_j$. Further, to keep the rhs of (20) non-negative $E^f_k \leq \min_{j\in\mathcal{R}^f} E_j/C^f_{jk}$. In all we have $\max_{j\in\mathcal{P}^f} D^f_{jk}E_j \leq \min_{j\in\mathcal{R}^f} E_j/C^f_{jk}$, i.e., no positive profit on (f,k) at prices $\boldsymbol{E}$. We solve the following to compute such a vector $\boldsymbol{E}$,

$$\begin{array}{lll} \forall (f,k),\ \forall j \in \mathcal{R}^f,\ \forall j' \in \mathcal{P}^f : & D^f_{j'k}E_{j'} \leq \frac{E_j}{C^f_{jk}} \\ \forall j : & E_j \geq 1 \end{array} \tag{26}$$

Since the first condition of (26) is homogeneous, setting all E_js to zeros is a solution, hence the second condition. We denote the polyhedron of (26) by $\mathcal{E}$.

Lemma 8.8 *Polyhedron $\mathcal{E}$ is non-empty and has a non-empty interior.*

[11] This assumption ensures infinite desire(j) for every good j. One can derive a better bound using no production out of nothing condition, and then non-satiation condition will not be needed. But the bound is involved and that adds to unnecessary complications later. To keep the arguments simple and intuitive, we stick to the non-satiation condition.

Proof : Taking logarithms, the first condition of (26) transforms to $\log(E_{j'}) - \log(E_j) \leq -\log(C^f_{jk}D^f_{j'k})$. and the second condition to $\log(E_j) \geq 0$. Rename $\log(E_j)$ by e_j; this gives a system analogues to $Ax \leq b$. By Farkas' lemma this does not have a solution if and only if there is a $y \geq 0,\ y^TA = 0,\ y^Tb = -1$ [4]. It is easy to check that for our system of equations, existence of such a y implies a cycle of weight at least zero in the graph between firms, where the weight on edge from j to j' is $\log(C^f_{jk}D^f_{j'k})$. This contradicts *no production out of nothing* assumption. Further, this condition also implies that there is no cycle with $\log(C^f_{jk}D^f_{j'k})$'s adding to zero, hence $\mathcal{E}$ has a non-empty interior. □

Take a vector $\boldsymbol{E}$ from $\mathcal{E}$ (Lemma 8.8) and define $E^f_k \stackrel{\text{def}}{=} \min_j E_j/C^f_{jk}$ and replace τ^f_k with $\tau'^f_k + E^f_k$. The resulting LCP, call it **NH-ADLCP**, is as follows. There are non-negativity constraints on all the variables, however for brevity we omit them.

$$\forall(i,k), \forall j: \quad U^i_{jk}(\lambda_i - \gamma^i_k) - p'_j \leq E_j \quad \text{and} \quad q^i_{jk}(U^i_{jk}(\lambda_i - \gamma^i_k) - p'_j - E_j) = 0 \tag{27}$$

$$\forall(i,k): \quad \textstyle\sum_j q^i_{jk} \leq L^i_k(\lambda_i - \gamma^i_k) \quad \text{and} \quad \gamma^i_k(\sum_j q^i_{jk} - L^i_k(\lambda_i - \gamma^i_k)) = 0 \tag{28}$$

$$\forall(f,k), \forall j \in \mathcal{R}^f: \quad \tau'^f_k - \frac{p'_j}{C^f_{jk}} \leq \frac{E_j}{C^f_{jk}} - E^f_k \quad \text{and} \quad q^{f,r}_{jk}(\tau'^f_k + E^f_k - \frac{p'_j + E_j}{C^f_{jk}}) = 0 \tag{29}$$

$$\forall(f,k), \forall j \in \mathcal{P}^f: \ D^f_{jk}p'_j - \tau'^f_k - \delta^f_k \leq E^f_k - D^f_{jk}E_j \ \text{ and } \ q^{f,p}_{jk}\left(D^f_{jk}(p'_j + E_j) - \tau'^f_k - E^f_k - \delta^f_k\right) = 0 \tag{30}$$

$$\forall(f,k): \quad \textstyle\sum_{j\in\mathcal{R}^f} q^{f,r}_{jk} - L^f_k\tau'^f_k \leq L^f_kE^f_k \quad \text{and} \quad \delta^f_k(\sum_{j\in\mathcal{R}^f} q^{f,r}_{jk} - L^f_k\tau'^f_k - L^f_kE^f_k) = 0 \tag{31}$$

$$\forall(f,k): \quad \textstyle\sum_{j\in\mathcal{P}^f} q^{f,p}_{jk} = \sum_{j\in\mathcal{R}^f} q^{f,r}_{jk} + L^f_k\delta^f_k \tag{32}$$

$$\begin{aligned} \forall j \in \mathcal{G}: \quad & \textstyle\sum_{i,k} q^i_{jk} + \sum_{(f,k),j\in\mathcal{R}^f} q^{f,r}_{jk} - p'_j - \sum_{(f,k),j\in\mathcal{P}^f} q^{f,p}_{jk} \leq E_j \\ & \text{and} \quad p'_j\left(\textstyle\sum_{i,k} q^i_{jk} + \sum_{(f,k),j\in\mathcal{R}^f} q^{f,r}_{jk} - (p'_j + E_j) - \sum_{(f,k),j\in\mathcal{P}^f} q^{f,p}_{jk}\right) = 0 \end{aligned} \tag{33}$$

$$\begin{aligned} \forall i \in \mathcal{A}: \quad & \textstyle\sum_j W^i_jp'_j + \sum_f \Theta^i_f\phi^f - \sum_{j,k} q^i_{jk} \leq -\sum_j W^i_jE_j \\ & \text{and} \quad \lambda_i\left(\textstyle\sum_j W^i_j(p'_j + E_j) + \sum_f \Theta^i_f\phi^f - \sum_{j,k} q^i_{jk}\right) = 0 \end{aligned} \tag{34}$$

Here ϕ^f is a place holder for $\sum_k L^f_k\delta^f_k$; profit of firm f. Since $\boldsymbol{E}$ is a solution of (26), it is easy to see that rhs of all the above inequalities except (34) are non-negative.

Remark 8.9 *To avoid degeneracies to set in, zeros in the rhs should be avoided. To achieve this for (30), $\boldsymbol{E}$ is taken from the interior of $\mathcal{E}$ (Lemma 8.8). Note that rhs of (28) will still remain zero. We can fix this by replacing λ_i with $\lambda'_i + E^i$, where $E^i \stackrel{\text{def}}{=} \min_{j,k} E_j/(U^i_{jk}+1)$ so that rhs of (27) remain positive.*

Lemma 8.10 *At a solution of* **NH-ADLCP** *every agent receives an optimal bundle.*

Proof : Given a solution of **NH-LCP**, let $p_j = p'_j + 1$ and $x^i_{jk} = {q^i_{jk}}/{p_j}$. We will show that allocation $\boldsymbol{x}$ assigns an optimal bundle to every agent at prices $\boldsymbol{p}$. It suffices to show that for each i, bundle $\boldsymbol{x}^i$ satisfies $\boldsymbol{B_0}$-$\boldsymbol{B_2}$ at prices $\boldsymbol{p}$ (Lemma 5.1).

We have $\lambda_i > 0$ for each i or else due to (16') all the q^i_{jk}'s of agent i will be zero, contradicting market clearing condition (19) of her. From (16) and (16') we have

$$\forall (i,k), \forall j : \frac{1}{\lambda_i - \gamma^i_k} \geq \frac{U^i_{jk}}{p_j} \quad \text{and} \quad q^i_{jk} > 0 \Rightarrow \frac{1}{\lambda_i - \gamma^i_k} = \frac{U^i_{jk}}{p_j}$$

This ensures that, on a segment only maximum bang-per-buck goods are obtained, if at all, thereby satisfying $\boldsymbol{B_1}$. If goods are obtained on segment (i,k), then its bang-per-buck is captured by $\text{bpb}^i_k = {1}/{(\lambda_i - \gamma^i_k)}$. This with (17) also ensures $\boldsymbol{B_0}$ as follows.

$$\forall (i,k) : \sum_j q^i_{jk} \leq L^i_k(\lambda_i - \gamma^i_k) \Rightarrow \sum_j \frac{U^i_{jk}}{p_j} q^i_{jk} \leq L^i_k \Rightarrow \sum_j U^i_{jk} x^i_{jk} \leq L^i_k$$

Suppose for an agent i, goods are obtained on segments both (i,k) and (i,k'), and $\text{bpb}^i_k > \text{bpb}^i_{k'}$. In that case, as desired (i,k) is a forced segment as $\gamma^i_k > \gamma^i_{k'} \geq 0$ and (17'). Further, if goods are not obtained at all on segment (i,k''), then (17') ensures that $\gamma^i_{k''}$ is zero, and then using (16) we have

$$\forall j : \frac{U^i_{jk''}}{p_j} \leq \frac{1}{\lambda_i} \leq \frac{1}{\lambda_i - \gamma^i_{k'}} = \text{bpb}^i_{k'}$$

Thus, $\boldsymbol{x}^i$ satisfies $\boldsymbol{B_2}$. □

Lemma 8.11 *At a solution of* **NH-ADLCP** *every firm operates at an optimal production plan.*

Proof : At a solution of **NH-ADLCP**, the price of good j is $p_j = p'_j + E_j$. Let $x^{f,r}_{jk} = {q^{f,r}_{jk}}/{p_j}$ and $x^{f,p}_{jk} = {q^{f,p}_{jk}}/{p_j}$ be the amount of used and produced goods on respective production segments. These are well defined since p_j's are positive. Let $\tau^f_k = \tau'^f_k + E^f_k$, which is upper bounded by $\min_j {p_j}/{C^f_{jk}}$ (29); minimum cost per unit raw material. Due to Lemma 5.2, it suffices to show that $(\boldsymbol{x}^{f,r}, \boldsymbol{x}^{f,p})$ satisfies $\boldsymbol{P_0}$-$\boldsymbol{P_4}$ at prices $\boldsymbol{p}$. Clearly, non-negativity constraints of $\boldsymbol{P_0}$ are satisfied.

First we observe that on a segment, if at all, (29) and (29') allows to spend money on only least cost raw good and (30) and (30') allows production of only maximum revenue fetching goods hence $\boldsymbol{P_1}$ follows. For segment (f,k) variable δ^f_k is non-zero only if some of $q^{f,r}_{jk}$'s are non-zero too (31'). Therefore, the equality in (32) ensures that there is production if and only if raw material is used.

Suppose (f,k) is undesired at prices $\boldsymbol{p}$ (negative profit). If τ^f_k captures the minimum cost, then (30) holds with strict inequality, and (30') does not allow any production. If it does not capture the minimum cost then due to (29') no raw material can be purchased on this segment. In all no production on segment (f,k), and thus $\boldsymbol{P_2}$ follows.

If the segment is forced or flexible (non-negative profit), then

$$0 \leq \max_{j\in\mathcal{P}^f} D^f_{jk} p_j - \min_{j \in \mathcal{R}^f} \tfrac{p_j}{C^f_{jk}} \leq D^f_{jk} p_j - \tau^f_k, \ \exists j \in \mathcal{P}^f \quad \text{(using (29))}$$

If τ^f_k is strictly less than the minimum cost, then last inequality above is also strict, and to satisfy (30) δ^f_k has to be positive. This is not possible since all $q^{f,r}_{jk}$'s are zero due to (29'). So τ^f_k is exactly the minimum cost per unit for segment (f,k), and then δ^f_k is at least the profit per unit (30). The δ^f_k can not be strictly more than profit, otherwise (30') will not allow any production and violate (31'). To ensure upper limit of L^f_k on the units of raw material used of $\boldsymbol{P_0}$, consider (31).

$$\sum_{j\in\mathcal{R}^f} q^{f,r}_{jk} \le L^f_k\tau^f_k \Rightarrow \sum_{j\in\mathcal{R}^f} \frac{C^f_{jk}}{p_j} p_j x^{f,r}_{jk} \le L^f_k \quad \text{(Using (29'))} \Rightarrow \sum_{j\in\mathcal{R}^f} C^f_{jk} x^{f,r}_{jk} \le L^f_k$$

If positive profit then the above inequality is tight due to (31') as required. Finally (32) ensures that total revenue equals total cost plus total profit. In other words, the amount produced is in proportion to the raw material used, and thus $\boldsymbol{P_4}$ follows and hence remaining part $\boldsymbol{P_0}$ as well. □

Next theorem is a consequence of Lemmas 8.4, 8.3, 8.10 and 8.11.

Theorem 8.12 *The solutions of* **NH-ADLCP** *capture exactly the equilibria of Leontief-free Arrow-Debreu market $\mathcal{M}$ up to scaling, assuming that for every good there is a non-satiated agent.*

As is the case with exchange, rationality of equilibrium prices in Arrow-Debreu markets follows using Theorem 8.12.

Theorem 8.13 *Leontief-free Arrow-Debreu market $\mathcal{M}$ with all parameters rational that has an equilibrium admits equilibrium prices which are polynomial sized rational numbers, assuming that for every good there is a non-satiated agent.*

9 Algorithm

From Theorem 8.12, computing an equilibrium of market $\mathcal{M}$ is equivalent to solving **NH-ADLCP**. The **NH-ADLCP** is in standard form except that variables τ'^f_ks are not part of any complementarity conditions, and it has equalities (32). We call τ'^f_ks as *abnormal* variables and rest as *normal* variables. In this section we design a Lemke-type algorithm to solve this LCP. For detailed description of Lemke's scheme refer to Appendix A. Like in Lemke, we augment the **NH-ADLCP** by adding a slack variable z in the inequalities with negative rhs, namely (34).

$$\begin{array}{rl} \forall i \in \mathcal{A}: & \sum_j W^i_j p'_j + \sum_f \Theta^i_f \phi^f - \sum_{j,k} q^i_{jk} - z \le -\sum_j W^i_j E_j \\ & \quad\text{and}\quad \lambda_i \left(\sum_j W^i_j (p'_j + E_j) + \sum_f \Theta^i_f \phi^f - \sum_{j,k} q^i_{jk} - z\right) = 0 \end{array} \tag{35}$$

We impose non-negativity on z and denote the augmented LCP by **NH-ADLCP'**. The algorithm will follow a path of solutions of **NH-ADLCP'**, hence all the equilibrium conditions except the market clearing will be satisfied through out the algorithm. We say that *segment (f,k) is undesired at a solution point* if all its $q^{f,r}_{jk}$s are zero; no production.

9.1 Inherent Degeneracy and Non-Degeneracy Assumption

There are two types of degeneracies possible at a solution of **NH-ADLCP'**. The first is for each undesired segment (f,k), the tight non-negativity constraints of $q^{f,r}_{jk}$s, $q^{f,p}_{jk}$s, δ^f_k, and equality (32) are linearly dependent. The second degeneracy comes about at a solution with $z=0$, because of the following fact established in Lemma 8.3: adding the constraints in (33) over all goods and those in (34) over all agents yields two identical equations. Henceforth, we will say that the polyhedron corresponding to **NH-ADLCP'** is non-degenerate if it has no other degeneracies.

A vertex with second type of degeneracy is also a solution of **NH-ADLCP** (market equilibrium), and hence a terminating point of the algorithm. At this vertex there will be a good j with $p'_j = 0$ to compensate, relaxing which leads to a ray of market equilibria; essentially scaled version of the equilibrium at the vertex. The first type of degeneracy arise at intermediate vertices of the algorithm as well. And to compensate we maintain the following invariant

(I): For each production segment (f, k) at least one of (29) hold with equality.

As a consequence the value corresponding to abnormal variable $\tau'^f_k + E^f_k$ always captures the minimum cost per unit raw material for segment (f, k). For forced and flexible segments (I) holds automatically. However, for an undesired segment (f, k) (I) holds in a *double label*[12], as for some $j \in \mathcal{R}^f$ (29) is tight and $q^{f,r}_{jk}$ is also zero. We call such a label an *extra double label*. The algorithm keeps a list $\mathcal{L}$ of *extra double labels*, exactly one for each undesired segment, and makes sure not to leave them explicitly in order to maintain (I). We note that, during the algorithm, relaxing inequalities corresponding to labels in $\mathcal{L}$ at any intermediate vertex may lead to a dead end. Further, maintaining (I) makes sure that even when $\tau^f_k \geq 0$ becomes tight at some vertex, there is a valid double label to leave corresponding to a normal variable.

Before presenting the algorithm, let us add slack variables to the constraints of **NH-ADLCP'** – assume that the slack variable that is added to the i^{th} constraint is v_i. This gives us an LCP in the form of the formulation given in (38). The algorithm appears in Table 12. Here T_0 is the solution vertex of **NH-ADLCP'** at which primary ray is incident, and $\mathcal{L}_0$ is the list of extra double labels at T_0. *flag* is set to one when there are two extra double labels for a segment, otherwise it is set to zero. Since, all the main variables except z are zero on primary ray, all production segments are undesired at T_0, and $\mathcal{L}_0$ contains exactly one double label for each. Hence flag is initialized to zero.

Table 12: Complementary Pivot Algorithm for a Leontief-free Market

Initialization: Let $T \leftarrow T_0$ and $\mathcal{L} \leftarrow \mathcal{L}_0$. flag $\leftarrow 0$
While $z > 0$ in the current solution T to **NH-ADLCP'**, **do**
Let $i \notin \mathcal{L}$ be the double label at solution T, i.e., $v_i = y_i = 0$ at T.
If flag $= 1$, **then** pivot by relaxing $v_i = 0$ and set flag $\leftarrow 0$.
Else if v_i just became 0 at the current vertex, **then** pivot by relaxing $y_i = 0$.
Else, pivot by relaxing $v_i = 0$.
Let T' be the solution to **NH-ADLPC'** at the newly reached vertex.
$\mathcal{L}' \leftarrow \mathcal{L}$, $T \leftarrow T'$ and $\mathcal{L} \leftarrow$ extra double labels at T.
If for an (f, k), $\mathcal{L}$ contains two labels l and l', and $l \in \mathcal{L}'$, **then** $\mathcal{L} \leftarrow \mathcal{L} \setminus \{l\}$ and flag $\leftarrow 1$.
Endwhile
Output solution T.

In the algorithm of Table 12, the last line of the While loop makes sure that $\mathcal{L}$ contains exactly one *extra double* label for each undesired segment. At vertex T if there are two for an undesired segment (f, k), then it must be the case that the new tight inequality also correspond to an *extra double* label, hence all the double labels at T are extra (assuming non-degeneracy). We set the *flag* to one when such a case arises. Only way to move away from T, while maintaining (I), is to leave

[12] Refer to Appendix A for definition

one of the two *extra double* label of (f,k). Further, since the segment (f,k) is undesired, $q_{jk}^{f,r}$ can not be made positive, and the only way to leave the label is by relaxing (29), as done in line three of the While loop. To move forward we have to leave the one which does not correspond to the newly tight inequality at T, and hence we set $\mathcal{L}$ accordingly.

10 Correctness

For convergence of the algorithm, two things have to be ensured; a double label not in $\mathcal{L}$, at every intermediate vertex (a unique edge to move forward while maintaining (I)), and no secondary ray in the polyhedron corresponding to **NH-ADLCP'**. The former is guaranteed in case of standard LCPs, however the latter has to be taken care of in general for Lemke-type schemes (see Appendix A).

Lemma 10.1 *At every intermediate vertex, there exist a unique double label, not in $\mathcal{L}$, to leave. In other words there exists a unique edge to move forward while maintaining (I).*

Proof : For existence we use induction. Let $\hat{i} \in \arg\max_i \sum_j W_j^i E_j$. Since on primary ray $\boldsymbol{y}$ is zero and z varies from ∞ to $\sum_j W_j^{\hat{i}} E_j$, both (34) and $\lambda_{\hat{i}} \geq 0$ are tight for $\hat{i}$, at its vertex T_0, and give a double label not in $\mathcal{L}$.

Suppose, the algorithm moves from vertex $\hat{T}$ to T by leaving a double label. Let $\hat{\mathcal{L}}$ and $\mathcal{L}$ be the sets of extra double labels at T' and T respectively. There are two cases: either $\mathcal{L} \subseteq \hat{\mathcal{L}}$ or not. In the latter case suppose the newly formed extra double label correspond to segment (f,k) and good $j \in \mathcal{R}^f$. If there was production happening on (f,k) at T' but not at T, then $q_{j'k}^{f,p} \geq 0$ for a $j' \in \mathcal{P}^f$ has become tight at T, giving a double label not in $\mathcal{L}$. If there was no production on (f,k) at T' as well, then $\hat{\mathcal{L}}$ also had an extra double label for (f,k), say l, which is removed by the last line in the While loop of the algorithm to form $\mathcal{L}$. Hence l is the double label to leave.

If $\mathcal{L} \subseteq \mathcal{L}'$, and the new tight inequality at T is other than $\tau_k'^f \geq 0$ then it gives a valid double label. Suppose for some (f,k), $\tau_k'^f \geq 0$ becomes tight at T. Recall that for every segment (f,k) we maintain one of (29) with equality, let it correspond to good $j \in \mathcal{R}^f$ on edge from T' to T. Since, it holds with equality at T as well, its rhs must be zero and $p_j' \geq 0$ must have become tight as well at T, giving a valid double label.

Uniqueness of a double label not in $\mathcal{L}$ follows from our non-degeneracy assumption. □

Market equilibrium may not exist even in exchange markets with SPLC utilities [60]. Maxfield gave a set of sufficiency conditions under which an equilibrium is guaranteed [39]. We have stated all of them except one in Section 2, which we discuss next. Like agents, a firm can also be non-satiated for a good. We say that a firm f is non-satiated for good j if there exists a segment (f,k) with $C_{jk}^f > 0$ and L_k^f being infinity.

Definition 10.2 (Strong Connectivity) *Construct a directed graph $G(\mathcal{M})$, where agents and firms are nodes. Put an edge from agent i to agent i' if i has a good in her initial endowment for which i' is non-satiated, i.e., $\exists j, W_j^i > 0, \exists k, U_{jk}^{i'} > 0, L_k^i = \infty$. Similarly, Put an edge from i to firm f, if f is non-satiated for a good that agent i has. An edge from firm f to firm f' or agent i appears if f can produce an infinite amount of a good for which f' or i is non-satiated respectively. We say that $\mathcal{M}$ satisfies strong connectivity if $G(\mathcal{M})$ has a strongly connected component containing all the agents.*

Henceforth, we assume that $\mathcal{M}$ satisfies strong connectivity. To prove that our algorithm terminates with an equilibrium, we will show that there are no secondary rays in **NH-ADLCP'**, and hence the algorithm has to terminate at a vertex with $z = 0$.

10.1 No Secondary Rays

We denote the vector $(\boldsymbol{p'}, \boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{\gamma}, \boldsymbol{q}^p, \boldsymbol{q}^r, \boldsymbol{\delta}, \boldsymbol{\tau'})$ by $\boldsymbol{y}$. Suppose, there is a secondary ray in the polyhedron of **NH-ADLCP'**. Given a vertex $(\boldsymbol{y}^*, z^*)$ and direction vector $(\boldsymbol{y}^\circ, z^\circ)$ of a secondary ray, it is defined as

$$R = \{(\boldsymbol{y}, z) \mid \boldsymbol{y} = \boldsymbol{y}^* + \epsilon \boldsymbol{y}^\circ, z = z^* + \epsilon z^\circ, \epsilon \geq 0\}$$

By definition of secondary ray, we have $z^* > 0$. Due to non-negativity constraint on all the variables, we immediately get that all the coordinates of $\boldsymbol{y}^\circ$ and z° are non-negative. Based on the number of non-zero co-ordinates of vector $\boldsymbol{p'}^\circ$, the proof will have three cases. For the case when $\boldsymbol{p}^\circ = 0$ we show that R is the primary ray. For the other two cases, when $\boldsymbol{p'}^\circ > 0$ or $\boldsymbol{p'}^\circ \neq 0, \boldsymbol{p'}^\circ \not> 0$, we derive contradictions. In the former case we show that market clears in turn z^* is zero, and in the latter case we derive contradiction to the strong connectivity property of market $\mathcal{M}$.

Lemma 10.3 *At a solution of* **NH-ADLCP'**, *if λ_i is zero then so are all q^i_{jk}s of agent i. Further agents never spend more than their earnings and their surplus is at most z.*

Proof : The first part follows due to (27) and (27'). For the second part, there are two cases. For an agent i if λ_i is zero, then she does not spend anything and hence her spending is less than her earnings. For the case when $\lambda_i > 0$, z captures the surplus of agent i, which is always non-negative. □

Lemma 10.4 *If $\boldsymbol{p'}^\circ > 0$ then z^* is zero.*

Proof : Consider a point $v = (\boldsymbol{y}, z)$ on ray R. Since $\boldsymbol{p'}^\circ > 0$, we have $\boldsymbol{p'} > 0$. Then due to (33') market clears from goods side at v. Since no agent over spends at v (Lemma 10.3), the market has to clear from agents side as well. Earnings of agent i is at least $\sum_j W^i_j E_j$ and hence is non-zero. Since agents can spend only when their λ_i is non-zero (Lemma 10.3), z captures surplus of all the agents (34'), and hence is zero. □

Lemma 10.5 *At a solution of* **NH-ADLCP'**, *there are no production cycles and total production of every good is bounded.*

Proof : Suppose there is a production cycle $1, \ldots, d, 1$ of firms, where firm a produces good j_a using good j_{a-1} on segment (a, k_a) for $f > 1$, and firm 1 produces j_1 using j_d on segment $(1, k_1)$. Since firms operate as per optimal production plan at solutions of **NH-ADLCP'** (Lemma 8.11), production on every segment fetches non-negative profit. This implies $p_{j_a} D^a_{j_a k_a} \geq p_{j_{a-1}} / C^a_{j_{a-1} k_a} \Rightarrow p_{j_{a-1}} / p_{j_a} \leq C^a_{j_{a-1} k_a} D^a_{j_a k_a}$, for all $a > 2$, and for $a = 1$ replace $a - 1$ with d. Putting these together for all the firms we get a contradiction to no positive cycle condition.

There may be production paths, where a good produced by one firm is used as a raw material by another firm to produce another good and so on. However, since the good, getting used at the starting of such a path, comes from only initial endowments of agents, its available quantity is bounded. Hence, the maximum amount that can be produced at the end of such a path is bounded. □

Ray R is an edge of a polyhedron, defined by a set of tight inequalities. If a non-negativity inequality of a variable is tight on R then that variable remains zero on entire R. Also, if a variable is positive at a point on R, then it can never become zero.

Lemma 10.6 *If $\boldsymbol{p'}^\circ \neq 0$ and $\boldsymbol{p'}^\circ \not> 0$ then market $\mathcal{M}$ does not satisfy strong connectivity.*

Proof : Let $\mathcal{G}_1$ be the set of goods with $\boldsymbol{p}_j^\circ = 0$, and $\mathcal{G}_2$ be the rest of the goods. From the hypothesis, we know that both $\mathcal{G}_1$ and $\mathcal{G}_2$ are non-empty. Since prices of goods in $\mathcal{G}_2$ are strictly increasing, they are fully-sold on R (33'). Goods of $\mathcal{G}_1$ may be undersold and they contribute to the surplus of agents (Lemma 10.3). Since maximum available amount of every good is bounded (Lemma 10.5), the total surplus of agents is also bounded. The total surplus of agents is at most $m \cdot z$ (Lemma 10.3), so we get $z^\circ = 0$.

Using goods of $\mathcal{G}_2$ no good of $\mathcal{G}_1$ is produced on R, because such a production will eventually incur losses. Then in the absence of production cycles, some goods of $\mathcal{G}_2$ have to be consumed by agents in order to clear them. Let $\mathcal{A}_2$ be the set of such agents, and $\mathcal{A}_1$ be rest of them. Clearly, $\mathcal{A}_2 \neq \emptyset$. Agent $i \in \mathcal{A}_2$ has to be satiated for every good in $\mathcal{G}_1$ or else her bang-per-buck for such a good will surpass the bang-per-buck of the goods she is buying from $\mathcal{G}_2$. By construction agents of $\mathcal{A}_1$ do not buy goods of $\mathcal{G}_2$, hence their spending remains constant on R. In order to keep the surplus constant, their earnings should also be constant. Therefore, they can not own any of the goods from $\mathcal{G}_2$. Thus, there is no edge from an agent of $\mathcal{A}_1$ to an agent of $\mathcal{A}_2$ in $G(\mathcal{M})$.

Let $\mathcal{F}_2$ be the set of firms non-satiated for goods in $\mathcal{G}_2$, and $\mathcal{F}_1$ be the rest of them. Production segments where a good $j' \in \mathcal{G}_2$ can be produced using a good $j \in \mathcal{G}_1$ will be profitable and hence utilized fully. By construction, a firm $f \in \mathcal{F}_1$ uses goods of $\mathcal{F}_1$ on non-satiated segments. Therefore, on such a segment it can not produce a good from $\mathcal{G}_2$. Thus, no edge from a firm of $\mathcal{F}_1$ or an agent of $\mathcal{A}_1$ to an agent of $\mathcal{A}_2$ or a firm of $\mathcal{F}_2$ in graph $G(\mathcal{M})$. In all there is no path from an agent of $\mathcal{A}_1$ to an agent of $\mathcal{A}_2$ in $G(\mathcal{M})$. Since, both $\mathcal{A}_1$ and $\mathcal{A}_2$ are non-empty, this violates strong connectivity of market $\mathcal{M}$. □

Lemma 10.7 *If $\boldsymbol{p'}^\circ = 0$ then $\boldsymbol{y}^*$ and $\boldsymbol{y}^\circ$ both are zero vectors, i.e., R is the primary ray.*

Proof : Recall that all the variables can only increase on R. Since, conditions for optimal production plan remain as they are in **NH-ADLCP**, at every point of R firms operate at optimal production plan (Lemma 8.11). Therefore, if prices are constant on R then so are the money spent on raw material, revenue and profits; $\boldsymbol{q}^{r^\circ} = 0, \boldsymbol{q}^{p^\circ} = 0$ and $\boldsymbol{\delta}^\circ = 0$.

This puts an upper bound on total money that agents can spend (33), $\boldsymbol{q}^\circ = 0$. Now since agent i is non-satiated for some good (strong connectivity condition), the corresponding segment will always be either flexible or undesired always, and hence its γ is always set to zero. Then λ_i can not increase in order to maintain (27); $\boldsymbol{\lambda}^\circ = 0$. Finally, (28) forces $\boldsymbol{\gamma}^\circ$ to be zero. In all we get $\boldsymbol{y}^\circ = 0$. Since the direction vector of a ray has to be non-zero we must have $z^\circ > 0$.

Since, only z is increasing on the ray, at any of its point $(\boldsymbol{y}, z)$ except vertex v^*, all the (34) hold with strict inequality. So then $\boldsymbol{\lambda} = \boldsymbol{\lambda}^*$ is zero using (34'). In that case $\boldsymbol{q}^* = 0$ (Lemma 10.3), and in turn $\boldsymbol{\gamma}^* = 0$ (due to (28)). If good j is neither produced nor consumed then clearly ${p'_j}^* = 0$, as it is under sold (33'). Among the goods that are produced or consumed by firms, there will be production paths at v^* but no cycles (Lemma 10.5). Consider one such path. Suppose, good j is getting produced at the end of this path. The ${p'_j}^* = 0$, as it is not consumed by any firm or agent. Since, a zero priced good can be produced only using zero priced raw material (no negative profit), the prices of goods, produced or used on this entire path, are zero. Essentially $\boldsymbol{p'}^* = 0$. Finally, using conditions for optimal production and the fact that when prices are E_js no firm makes profit, we get $\boldsymbol{\tau'}^* = 0$ (29), then $\boldsymbol{q}^{p^*} = 0$ (30,30'), $\boldsymbol{q}^{r^*} = 0$ and $\boldsymbol{\delta}^* = 0$ (32). Putting everything together we have $\boldsymbol{y}^* = 0$. □

Let the condition that for each good there exists an agent non-satiated for this good be called *enough demand*. Lemmas 10.4, 10.6, and 10.7 give:

Theorem 10.8 *The polyhedron of* **NH-ADLCP'**, *corresponding to a Leontief-free market $\mathcal{M}$, satisfying enough demand and strong connectivity, has no secondary rays.*

Lemma 10.1, and Theorems 8.12 and 10.8 directly yields:

Theorem 10.9 *If a Leontief-free market $\mathcal{M}$ satisfies strong connectivity and enough demand, then $\mathcal{M}$ admits an equilibrium and the algorithm in Table 12 terminates with one.*

Theorem 10.9 gives the first practical (see Section 12 for experimental results) algorithm for such a general class of production markets, and settles the appropriate case of the open problem, posed by Eaves (1975) [25]. The algorithm also gives a constructive proof of the existence of equilibrium for such markets.

Theorem 10.10 *Assuming strong connectivity and enough demand the problem of computing an equilibrium of a Leontief-free market is in PPAD.*

Proof : By Theorem 10.9, the algorithm in Table 12 must converge to an equilibrium. Now, by Todd's result [56] on the orientability of the path followed by a complementary pivot algorithm, we get a proof of membership of the problem in PPAD. □

Theorem 10.11 *If a Leontief-free market $\mathcal{M}$ satisfies strong connectivity and enough demand, and its polyhedron $\mathcal{P}'$ corresponding to* **NH-ADLPC'** *is non-degenerate, then $\mathcal{M}$ has an odd number of equilibria, up to scaling.*

Proof : The solutions of **NH-ADLCP'** satisfying $z = 0$ are precisely the solutions to **NH-ADLCP** and hence capture exactly the equilibria of $\mathcal{M}$ (Theorem 8.12). Given any solution of **NH-ADLCP'** we can convert it to a solution that satisfies (I), by just changing τ_k^fs. Let S be the set of solutions of **NH-ADLCP'** satisfying (I). Clearly, all the solutions of **NH-ADLCP'** are captured by S up to change of τ_k^fs, a non important quantity for equilibrium prices and allocations. As observed in Appendix A and using Lemma 10.1, the set S consists of paths and cycles. Now, the points of S satisfying $z = 0$ occur at endpoints of such paths (under non-degeneracy). One of the paths starts with the primary ray and ends with an equilibrium. Since by Theorem 10.8 the polyhedron of **NH-ADLCP'** has no secondary rays, the rest of the equilibria must be paired up. Hence there are an odd number of equilibria. □

Theorem 10.12 *The problem of computing an equilibrium of a Leontief-free market is PPAD-complete assuming the weakest known sufficiency conditions by Maxfield [39]. In general checking existence of an equilibrium in these markets is NP-complete.*

Proof : Containment in PPAD assuming the sufficiency conditions by Maxfield follows from Theorem 10.10. Given prices of goods, checking if they are equilibrium prices can be done in polynomial time using the characterization of Section 5, hence containment in NP follows in general.

As discussed in Section 2 SPLC utilities and SPLC production are subclass of Leontief-free utilities and production respectively. The proof follows from the hardness results for SPLC utilities and SPLC production [10, 13, 60, 15, 31]. □

11 Strongly Polynomial Bound

In this section we show that our algorithm is strongly polynomial when the number of goods is constant. Recall that our algorithm traverses a path in the solution set S of NHAD-LCP'. For this, we first create a cell-decomposition in the price space by introducing strongly polynomially many hyperplanes. We note that the number of non-empty cells formed by N hyperplanes in $\mathbb{R}^d$ is at most $O(N^d)$. Thus we get strongly polynomial bound on number of cells. After this we show that every vertex of S maps to a cell thus created and at most two vertices of S can map to a same cell.

The idea is to use the fact that each vertex of S has a particular setting of forced, flexible and undesirable segments for both agents and firms. We essentially decompose price space into cells by a set of hyperplanes so that every non-empty cell thus created corresponds to a setting of forced, flexible and undesirable segments for both agents and firms. This gives a mapping between the two. The challenge here is to carefully construct the hyperplanes so that we get such a setting for every non-empty cell. [30, 31] also uses this approach for proving strongly polynomial bound for complementary pivot algorithms.

We also note that most of these cells do not map onto by any vertex of S due to an additional constraint on them that the surplus, i.e., spending – earning (captured by variable z), has to be same for all agents. Hence, our algorithm is much faster than enumerating all of them in a brute-force way to check for a solution. The same has been confirmed by the experiments.

Hyperplanes. Consider the cell decomposition in $(p_1, \ldots, p_n, z)$-space by adding hyperplanes as follows: For each 5-tuple (i, j, j', k, k'), introduce hyperplane $U^i_{jk} p_{j'} - U^i_{j'k'} p_j = 0$. Further, for each 4 tuple (f, j, j', k), introduce hyperplanes $C^f_{j'k} p_j - C^f_{jk} p_{j'} = 0$, $D^f_{jk} p_j - D^f_{j'k} p_{j'} = 0$ and $D^f_{jk} C^f_{j'k} p_j - p_{j'} = 0$. These hyperplanes divide the space into cells and each cell has one of the signs $<, =, >$ for each hyperplane.

For each segment of every firm, these signs give information about (i) most profitable raw good, i.e., $\min_j p_j/C^f_{jk}$, (ii) most profitable produced good, i.e., $\max_j D^f_{jk} p_j$, and (iii) whether positive, zero or negative profit (using the signs of hyperplanes $D^f_{jk} C^f_{j'k} p_j - p_{j'} = 0$). Let $\phi(f, k)$ denotes the maximum profit on segment (f, k), i.e., $\phi(f, k) = L^f_k(\max_j D^f_{jk} p_j - \min_j p_j/C^f_{jk})$. Recall that each firm produces on positive profit segments to their full limit, and no production on negative profit segments. Let ϕ^f denotes the profit of firm f (a placeholder variable). Using the information above, $\phi_f = \sum_{(f,k):\phi(f,k)\geq 0} \phi(f, k)$ can be obtained for each cell.

For each agent, these signs give partial order on the bang-per-buck of her segments. Using this information for a given cell, we can sort all segments of agent i by decreasing bang-per-buck, and partition them by equality into classes: $Q^i_1, Q^i_2, \cdots$. Let $Q^i_{<l}$ denote $Q^i_1 \cup Q^i_2 \cup \ldots \cup Q^i_{l-1}$. Similarly, we define $Q^i_{\leq l}$ and $Q^i_{>l}$.

Recall that every segment (i, k) of an agent i has an upper bound L^i_k on the maximum utility that can be obtained on that segment, and forced segments are fully bought, flexible are partially, and undesirable are not bought at all. Further on every segment, money is spent only on those goods which give maximum bang-per-buck. Define bang-per-buck of a segment (i, k) as $bpb(i, k) \stackrel{\text{def}}{=} \max_j U^i_{jk}/p_j$. Hence the total money spent on a segment (i, k) is: $L^i_k/bpb(i,k)$ (if forced), a value $\in [0, L^i_k/bpb(i,k)]$ (if flexible), and 0 (if undesirable).

Next we want to capture the flexible partition. To do this, we further subdivide a cell by adding hyperplane $\sum_{k\in Q^i_{<l}} L^i_k/bpb(i,k) = \sum_{j\in\mathcal{G}} W^i_j p_j + \sum_f \Theta^i_f \phi^f - z$, for each agent i and each of her partitions Q^i_l. For any given subcell, let $Q^i_{l_i}$ be the right most partition such that $\sum_{k\in Q^i_{<l_i}} L^i_k/bpb(i,k) < \sum_j W^i_j p_j - z$, then $Q^i_{l_i}$ is the flexible partition for agent i. In addition, we add hyperplanes

$p_j = E_j,\ \forall j \in \mathcal{G}$ and $z = 0$, and consider only those cells where $p_j \geq E_j$ and $z \geq 0$.

Given a fully-labeled vertex $(\boldsymbol{y}, z)$ of $\mathcal{P}'$, there is a natural cell associated with it, namely due to projection of it on $(\boldsymbol{p}, z)$-space by mapping p'_j to $p'_j + E_j$ and z to z itself.

Lemma 11.1 *At most two vertices of S can map to a cell. Furthermore, if a cell is mapped onto from two vertices, then they must be adjacent.*

Proof : Given a cell we specify one equality for every complementarity condition, to be satisfied by the fully-labeled vertex mapping to it. A fully labeled vertex v must satisfy the following equalities. In the cell,

- If $\frac{C^f_{jk}}{p_j} = \min_{j'} \frac{C^f_{j'k}}{p_{j'}}$ then $\tau'^f_k - \frac{p'_j}{C^f_{jk}} = \frac{E_j}{C^f_{jk}} - E^f_k$ else $q^{f,r}_{jk} = 0$ at v.
- If $D^f_{jk} p_j = \max_{j'} D^f_{j'k} p_{j'}$ then $D^f_{jk} p'_j - \tau'^f_k - \delta^f_k = E^f_k - D^f_{jk} E_j$ else $q^{f,p}_{jk} = 0$ at v.
- If $\max_j D^f_{jk} p_j \geq \min_j \frac{C^f_{jk}}{p_j}$ then $\sum_{j \in \mathcal{R}^f} q^{f,r}_{jk} - L^f_k \tau'^f_k = L^f_k E^f_k$ else $\delta^f_k = 0$ at v.
- If $\frac{U^i_{jk}}{p_j} \geq \frac{U^i_{j'k}}{p_{j'}}$ for a $Q^i_{l_i}$ then $U^i_{jk}(\lambda_i - \gamma^i_k) - p'_j = E_j$ else $q^i_{jk} = 0$ at v.
- If $\frac{U^i_{jk}}{p_j} \leq \frac{U^i_{j'k}}{p_{j'}}$ for a $Q^i_{l_i}$ then $\gamma^i_k = 0$ else $\sum_j q^i_{jk} = L^i_k(\lambda_i - \gamma^i_k)$ at v.
- If $\sum_j W^i_j p_j + \sum_f \Theta^i_f \phi^f - z \geq 0$ (second set of hyperplanes for the tuple $(i, 1)$) then $\sum_j W^i_j p'_j + \sum_f \Theta^i_f \phi^f - \sum_{j,k} q^i_{jk} = -\sum_j W^i_j E_j$ else $\lambda_i = 0$ at v.
- If $p_j > E_j$ then $\sum_{i,k} q^i_{jk} + \sum_{(f,k), j \in \mathcal{R}^f} q^{f,r}_{jk} - p'_j - \sum_{(f,k), j \in \mathcal{P}^f} q^{f,p}_{jk} \leq E_j$ else $p'_j = 0$ at v.

Since the above conditions enforces one equality from each complementary condition of NHAD-LCP', their intersection forms a line. If this line does not intersect $\mathcal{P}'$, no fully labeled vertex gets mapped to the given cell. If it does then intersection can be either a fully labeled vertex, say v, or a fully labeled edge – we say that an edge of the polyhedron $\mathcal{P}'$ is *fully labeled* if the solution represented by each point of this edge is fully labeled. In the former case only vertex v gets mapped to the cell and in the latter case endpoints of the fully labeled edge map to the cell. □

12 Experimental Results

We implemented our algorithm in Matlab and ran it on randomly generated instances of Leontief-free Arrow-Debreu and Leontief-free exchange markets. Number of segments are kept the same in all the utility functions and production sets, let it be denoted by $\#seg$. An instance is created by picking values uniformly at random – W^i_js from [0, 10], Θ^i_fs from [0, 1], U^i_{jk}s from [0, 1], L^i_ks from [0, $10/\#seg$], C^f_{jk} and D^f_{jk} from [0, 1] (in order to avoid production out of nothing) and L^f_ks from [0, $10/\#seg$]. For every firm j, Θ^i_fs are scaled so that they sum up to one. Typically, all the coefficients are non-zero in each segment. The experimental results are given in Tables 13 and 14. Note that, even in the worst case the number of iterations is always linear in the total number of parameters to represent the market. Total number of parameters in a market with n goods, m agents and o firms is $O((mn + on)\#seg)$.

Table 13: Experimental Results for Leontief-free *Exchange* Market
(Min, Max and Avg columns give number of iterations of the algorithm)

#agents, #goods, #seg	#Instances	Min	Max	Avg
5, 5, 5	100	39	51	42.6
5, 5, 10	100	57	107	73.1
5, 5, 20	100	141	189	161.2
5, 5, 30	100	185	276	229.8
5, 10, 10	100	109	163	138.2
5, 10, 20	100	179	304	254.7
5, 10, 30	20	336	430	377.5
10, 5, 10	100	89	124	103.1
10, 5, 20	100	163	205	178.3
10, 5, 30	100	207	285	249.8
10, 10, 10	100	167	221	189.2
10, 10, 20	20	297	401	346.9

Table 14: Experimental Results for Arrow-Debreu Market with Leontief-free Production
(Min, Max and Avg columns give number of iterations of the algorithm)

#agents, #goods, #firms, #seg	#Instances	Min	Avg	Max
5, 5, 5, 2	100	15	27.8	40
5, 5, 5, 5	100	57	100.5	118
10, 5, 5, 2	100	28	57.9	60
10, 5, 5, 5	100	42	158.8	189
10, 10, 10, 2	100	47	89.1	110
10, 10, 10, 5	20	104	284.4	362
10, 10, 10, 10	10	131	592.3	884
15, 5, 5, 2	100	57	69.5	86
15, 5, 5, 5	100	200	232.4	273
15, 10, 10, 2	100	87	118.7	149
15, 10, 10, 5	10	474	772.3	1132
15, 10, 10, 10	10	1071	1141.2	1376

13 Discussion

Considering the merits of complementary pivot algorithms, an obvious question is whether there are other classes of non-separable utilities that admit such algorithms – of course, this will require a proof of rationality for such classes. Another important question is to design practical, numerically stable algorithms for computing equilibria in Arrow-Debreu markets under classes of non-separable utilities that do not admit rational equilibria, most notably Leontief utilities. We note that Fisher markets under Leontief utilities admit a convex program for computing equilibrium prices and allocations, hence a practical algorithm is already known for this case.

Several optimization problems involve concave or convex objective functions and their efficient solvability sometimes hinges on assuming that the function is separable. It will be interesting to see whether Leontief-free non-separable functions are easy to handle in these situations. Promising candidates include min-cost flow and multicommodity flow problems [29, 2, 5, 4].

Finally, we mention a third dichotomy for an equilibrium problem, namely for *Eisenberg-Gale markets*, a notion defined in [34]. Such markets admit convex programs that yield equilibrium prices and allocations analogous to the way the classical Eisenberg-Gale convex program [27] does for linear Fisher markets [6]. [8] show that the class EG2 of Eisenberg-Gale markets for two agents admit rational convex programs, as defined in [57], i.e., convex programs which always admit rational solutions if all parameters are set of rational numbers, as well as polynomial time algorithms for exact computation of equilibrium prices and allocations. Furthermore, if the set of feasible utilities of the two agents can be described by a combinatorial LP (an LP whose matrix entries have encoding size a polynomial in the dimension), then they give a strongly polynomial time algorithm for finding equilibrium prices and allocations. In contrast, [34] give several examples of Eisenberg-Gale markets for three agents which have only irrational equilibria. The ellipsoid algorithm will find the equilibria to any required degree of accuracy in time polynomial in the number of bits of accuracy.

13.1 Submodular $\cap$ PLC: Irrational Example

In this section we illustrate a market with submodular, PLC utility functions, with an irrational equilibrium prices and allocation. Consider a market with two goods and three agents. All the agents initially have one unit of each good, *i.e.,* $\boldsymbol{w}^1 = \boldsymbol{w}^2 = \boldsymbol{w}^3 = (1, 1)$. The utility functions of the first two agents are linear,

$$U_1(x_1, x_2) = x_1 \quad \text{and} \quad U_2(x_1, x_2) = x_2$$

. For the third agent it is a PLC function defined by the following two hyper planes,

$$U_3 \leq \frac{5}{2}x_1 + \frac{7}{2}x_2; \quad U_3 \leq 2x_1 + x_2 + 1$$

Intersection of the above two hyper-planes is defined by line $0.5x_1 + 2.5x_2 = 1$ in (x_1, x_2)-space. Since, slope of this line is negative, it follows that the function is submodular.

Wlog we can assume that $p_1 = 1$, and let p_2 be p, which we will set latter. Note that, at these prices each agent earns $1 + p$. Since, the first agent is going to buy only good one, her market clearing condition ensures $x_{11} = 1 + p$ and $x_{12} = 0$. Similarly, for agent two we get $p_{21} = 0$ and $p_{22} = 1 + 1/p$.

Further, it so happens that at equilibrium the optimal bundle (x_{31}, x_{32}) of agent 3 is on the intersection of two hyper-planes, and therefore it satisfies $0.5x_{31} + 2.5x_{32} = 1$. In addition, to ensure market clearing, it should satisfy $x_{31} + px_{32} = 1 + p$. Using these two we get that $x_{31} = \frac{3p+5}{5-p}$ and $x_{32} = 1 - p5 - p$.

To ensure that all the goods are consumed, we have $x_{11} + x_{21} + x_{31} = 3$. Replacing all three by corresponding expressions in p, we get

$$p^2 - 10p + 5 = 0$$

The roots of the above polynomial are $5 \pm 2\sqrt{5}$. Setting p to $5 - 2\sqrt{5}$ in p_2 and all the xs gives an equilibrium. This is because, by construction they satisfy market clearing conditions, and for first two agents also the optimal bundle conditions. For the optimal bundle of the third agent, replacing p in x_{31} and x_{32} gives bundle $(2\sqrt{5} - 3, 1 - 2/\sqrt{5})$. One can check that this is the best affordable bundle for agent 3 at prices $p_1 = 1$ and $p_2 = 5 - 2\sqrt{5}$.

Since, in this example agent have the same initial endowment, if we consider a Fisher market instead, where buyers have one dollar each, the equilibrium prices will still remain irrational.

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A Linear Complementarity Problem and Lemke's Algorithm

Given an $n \times n$ matrix $\boldsymbol{M}$, and a vector $\boldsymbol{q}$, the linear complementarity problem asks for a vector $\boldsymbol{y}$ satisfying the following conditions:

$$\boldsymbol{M}\boldsymbol{y} \le \boldsymbol{q}, \quad \boldsymbol{y} \ge 0, \quad \boldsymbol{q} - \boldsymbol{M}\boldsymbol{y} \ge 0 \quad \text{and} \quad \boldsymbol{y} \cdot (\boldsymbol{q} - \boldsymbol{M}\boldsymbol{y}) = 0. \tag{36}$$

The problem is interesting only when $\boldsymbol{q} \not\geq 0$, since otherwise $\boldsymbol{y} = 0$ is a trivial solution. Let us introduce slack variables $\boldsymbol{v}$ to obtain the equivalent formulation

$$\boldsymbol{M}\boldsymbol{y} + \boldsymbol{v} = \boldsymbol{q}, \quad \boldsymbol{y} \geq 0, \quad \boldsymbol{v} \geq 0 \quad \text{and} \quad \boldsymbol{y} \cdot \boldsymbol{v} = 0. \tag{37}$$

Let $\mathcal{P}$ be the polyhedron in $2n$ dimensional space defined by the first three conditions; we will assume that $\mathcal{P}$ is non-degenerate. Under this condition, any solution to (37) will be a vertex of $\mathcal{P}$, since it must satisfy $2n$ equalities. Note that the set of solutions may be disconnected.

An ingenious idea of Lemke was to introduce a new variable and consider the system:

$$\boldsymbol{M}\boldsymbol{y} + \boldsymbol{v} - z\mathbf{1} = \boldsymbol{q}, \quad \boldsymbol{y} \geq 0, \quad \boldsymbol{v} \geq 0, \quad z \geq 0 \quad \text{and} \quad \boldsymbol{y} \cdot \boldsymbol{v} = 0. \tag{38}$$

Let $\mathcal{P}'$ be the polyhedron in $2n+1$ dimensional space defined by the first four conditions; again we will assume that $\mathcal{P}'$ is non-degenerate. Since any solution to (38) must still satisfy $2n$ equalities, the set of solutions, say S, will be a subset of the one-skeleton of $\mathcal{P}'$, i.e., it will consist of edges and vertices of $\mathcal{P}'$. Any solution to the original system must satisfy the additional condition $z = 0$ and hence will be a vertex of $\mathcal{P}'$.

Now S turns out to have some nice properties. Any point of S is *fully labeled* in the sense that for each i, $y_i = 0$ or $v_i = 0$. We will say that a point of S *has double label* i if $y_i = 0$ and $v_i = 0$ are both satisfied at this point. Clearly, such a point will be a vertex of $\mathcal{P}'$ and it will have only one double label. Since there are exactly two ways of relaxing this double label, this vertex must have exactly two edges of S incident at it. Clearly, a solution to the original system (i.e., satisfying $z = 0$) will be a vertex of $\mathcal{P}'$ that does not have a double label. On relaxing $z = 0$, we get the unique edge of S incident at this vertex.

As a result of these observations, we can conclude that S consists of paths and cycles. Of these paths, Lemke's algorithm explores a special one. An unbounded edge of S such that the vertex of $\mathcal{P}'$ it is incident on has $z > 0$ is called a *ray*. Among the rays, one is special – the one on which $\boldsymbol{y} = 0$. This is called the *primary ray* and the rest are called *secondary rays*. Now Lemke's algorithm explores, via pivoting, the path starting with the primary ray. This path must end either in a vertex satisfying $z = 0$, i.e., a solution to the original system, or a secondary ray. In the latter case, the algorithm is unsuccessful in finding a solution to the original system; in particular, the original system may not have a solution.

Remark: Observe that $z\mathbf{1}$ can be replaced by $z\boldsymbol{a}$, where vector $\boldsymbol{a}$ has a 1 in each row in which $\boldsymbol{q}$ is negative and has either a 0 or a 1 in the remaining rows, without changing its role; in our algorithm, we will set a row of $\boldsymbol{a}$ to 1 if and only if the corresponding row of $\boldsymbol{q}$ is negative. As mentioned above, if $\boldsymbol{q}$ has no negative components, (36) has the trivial solution $\boldsymbol{y} = 0$. Additionally, in this case Lemke's algorithm cannot be used for finding a non-trivial solution, since it is simply not applicable.

B Notations

Notation	Description
$\mathbb{R}_+^n$	Set of non-negative n-dimensional real vectors
$\mathcal{A}$	Set of agents
$\mathcal{G}$	Set of goods
$\mathcal{F}$	Set of firms
$\mathcal{R}^f$	Set of raw goods, firm f can use
$\mathcal{P}^f$	Set of goods, firm f can produce
m	Number of agents in the market
n	Number of goods in the market
o	Number of firms in the market
$\mathbf{W}$	Endowment matrix
W_j^i	Initial endowment of good j with agent i
U_{jk}^i	Utility per unit of good j on segment k of agent i
Θ_f^i	Agent i's profit share in firm f
C_{jk}^f	Constant essentially captures usefulness of raw good j on segment k of firm f production capability
D_{jk}^f	Constant essentially captures capability of firm f to produce good j on segment k
L_k^f	Upper bound on production on k^{th} segment of firm f
L_k^i	Upper bound on utility on k^{th} segment of agent i
T_k^i	Constant used in k^{th} hyperplane of agent i's PLC utility function
T_k^f	Constant used in k^{th} hyperplane of firm f's PLC utility function
E_j	Lower bound imposed on price of good j to make LCP non-homogeneous (obtained using Farkas' Lemma)
E_k^f	Lower bound imposed on τ_k^f to make algorithm converge
$\boldsymbol{p}$	Vector of price variables $(p_1, \ldots, p_n)$
p_j	price of good j
$\boldsymbol{x}^i$	Vector of amount allocated to agent i $(x_1^i, \ldots, x_n^i)$
x_j^i	Amount of good j allocated to agent i
x_{jk}^i	Amount of good j allocated to agent i on segment k
$x_j^{f,r}$	Amount of raw good j used by firm f
$x_j^{f,p}$	Amount of produced good j by firm f
q_{jk}^i	Money spent on good j by agent i on segment k
$q_{jk}^{f,r}$	Money spent on raw good j by firm f on segment k
$q_{jk}^{f,p}$	Money earned from produced good j by firm f on segment k
τ_k^f	Variable to capture the most profitable raw good to use on segment k by firm f
δ_k^f	Variable to capture the profit on segment k of firm f
ϕ^f	Placeholder variable to capture total profit earned by firm f
λ_i	Variable to essentially capture the bang-per-buck of agent i
γ_k^i	Dual variable for k^{th} hyperplane of agent i' PLC utility function