Matching Markets with Transfers and Salaries (DRAFT: Not for distribution)

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1 Introduction

In the last two chapters, we studied how matches are made in matching markets based only on the preferences of agents over other agents or objects; in particular, a transfer of money was not involved in these markets. These are called *non-transferable utility (NTU) markets*. In contrast, there are fundamental matching markets, such as the labor market, in which monetary transfers, in the form of salaries, prices, or other terms (e.g., benefits) form an integral part of each match. These are called *transferable utility (TU) markets* and are the subject of this chapter. In most of this chapter, we will assume that utilities of the agents are stated in monetary terms, and that side payments are allowed in transactions.

The *core* is a quintessential solution concept in this theory. It captures all possible ways of distributing the total worth of a game among individual agents in such a way that the grand coalition remains intact, i.e., a sub-coalition will not be able to generate more profits by itself and therefore has no incentive to secede from the grand coalition. The core provides profound insights into the negotiating power of individuals and sub-coalitions. In particular, under a core imputation, the profit allocated to an agent is consistent with their negotiating power, i.e., their worth, see Section 2.3, and therefore the core is viewed as a "fair" profit-sharing mechanism.

In this chapter, we will study the core in the context of the *assignment game*, in particular, using the setting of a housing market. The pristine structural properties of this game make it a paradigmatic setting for studying the intricacies of this solution concept, with a view to tackling profit-sharing in real-life situations. We will analyze the core of the assignment game using ideas from matching theory and LP-duality theory and their non-trivial interplay.

The results of Section 2 naturally raise the question of viewing core imputations through the lens of complementarity; this is done in Section 2.1. It yields a relationship between the competitiveness of individuals and teams of agents and the amount of profit they accrue, where by

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competitiveness we mean whether an individual or a team is matched in every/some/no maximum matching. This viewpoint also sheds light on the phenomenon of *degeneracy* in assignment games, i.e., when the maximum weight matching is not unique.

The generalization of the assignment game to general graphs, which include bipartite as well as non-bipartite graphs, is called the *general graph matching game*. Whereas the core of this game is always non-empty, that of the general graph matching game can be empty. In Section 3, we show how to deal with this situation by using the notion of an approximate core.

In Section 4, we consider a many-to-one matching market, which may be interpreted as a labor market where monetary transfers represent salaries paid to workers. We first consider the case in which transfers are constrained to a discrete grid. Then, by taking the limit of an ever-finer discrete grid, we obtain a result for the model with a continuum of transfers. In Section 5, we introduce a model of matching with contracts in which the terms of employment between a firm and its workers may involve more than just salaries; in particular, it could involve benefits, medical leave, etc.

2 The Core Studied in a Paradigmatic Setting

We will study the following setting of the *housing market*. Let B be a set of n buyers and R a set of m sellers. Each seller $j \in R$ is attempting to sell her house, which she values at c_j dollars. Each buyer i has a valuation of h_{ij} dollars for each house j. If $h_{ij} \geq c_j$, then there is a price at which this trade can happen so that both agents are satisfied. Specifically, any p_j such that $c_j \leq p_j \leq h_{ij}$ is such a price, and it results in a *gain or profit* to agents i and j of:

$$v_i = h_{ij} - p_j \quad \text{and} \quad u_j = p_j - c_j,$$

respectively. If $h_{ij} < c_j$ then i and j will not be involved in this trade. If an agent does not trade at all, his/her gain will be zero. This motivates the following definition.

Definition 1. For $i \in B$ and $j \in R$, define the *worth* of the coalition $\{i, j\}$ to be the total gain from this possible trade, i.e.,

$$w(\{i, j\}) = a_{ij} = \max\{0, h_{ij} - c_j\}.$$

We will extend this definition to the of worth an arbitrary coalition $S \subseteq (B \cup R)$; intuitively, it is the maximum gain possible via trades made within the set S. Clearly, if $|S| \leq 1$, no trades are possible and therefore $v(S) = 0$. For the same reason, if $S \subseteq B$ or $S \subseteq R$, $v(S) = 0$.

Definition 2. Let $S \subseteq (B \cup R)$ be a coalition and let $k = \min(|S \cap B|, |S \cap R|)$. Find k disjoint pairs $(i_1, j_1), \ldots (i_k, j_k)$ of buyers and sellers from S such that the total gain from these k trades, i.e., $(a_{i_1 j_1} + \ldots + a_{i_k j_k})$, is maximized. Then the *worth of* S, $w(S)$, is defined to be this total gain, and $w : 2^{B \cup R} \to \mathcal{R}_+$ is the *characteristic function* of the housing game.

Among the possible coalitions, the most important one is of course $(B \cup R)$; this is called *the grand coalition*. A important problem in economics is: What are "good" ways of dividing the worth of the grand coalition among its agents? A quintessential solution concept in this respect is that of the *core*, which consists of ways of dividing the worth in such a way that no smaller coalition will have the incentive to secede and trade on its own.

However, before studying this concept, let us state explicitly some assumptions made in the housing game; these are implicit in the setting defined above. Despite these assumptions, the game still has a fair amount of flexibility, e.g., in the number of buyers and sellers participating and whether or not there is product differentiation.

1. The utilities of the agents are stated in monetary terms.
2. Side payments are allowed, in the form of prices.
3. The objects to be traded are indivisible.
4. The supply and demand functions are inflexible: each buyer wants at most one house and each seller has one house to sell.

The first two assumptions make this a *transferable utility (TU) market*. Recall that in the last two chapters we studied the notion of core in the context of NTU markets. We start by defining the core for the given TU setting.

Definition 3. An *imputation* for dividing the worth of the game, $w(B \cup R)$, among the agents consists of two non-negative vectors v and u specifying the gains conferred on agents, namely v_i for $i \in B$ and u_j for $j \in R$. An imputation (v, u) is said to be in the *core of the housing game* if for any coalition $S \subseteq (B \cup R)$, there is no way of dividing $w(S)$ among the agents in S in such a way that all agents are at least as well off and at least one agent is strictly better off than in the imputation (v, u).

An *assignment game* is defined as follows. Let $G = (B, R, E)$ be the complete bipartite graph over vertex sets B and R. The edge set E consists of edges (i, j) for all pairs $i \in B$ and $j \in R$. Let the weight function on E be given by a, i.e., the weight of (i, j) is a_{ij}. Finally, $w(B \cup R)$ is the weight of a maximum weight matching in G. The assignment game asks for an imputation in the core.

(1) gives the LP-relaxation of the problem of finding a maximum wight matching. In this program, variable x_{ij} indicates the extent to which edge (i, j) is picked in the solution.

$$\begin{aligned} \max \quad & \sum_{i \in B, j \in R} a_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_j x_{ij} \le 1 \quad \forall i \in B, \\ & \sum_i x_{ij} \le 1 \quad \forall j \in R, \\ & x_{ij} \ge 0 \quad \forall i \in B, \ \forall j \in R \end{aligned} \tag{1}$$

Taking v_i and u_j to be the dual variables for the first and second constraints of (1), we obtain the dual LP:

$$\begin{aligned} \min \quad & \sum_{i \in B} v_i + \sum_{j \in R} u_j \\ \text{s.t.} \quad & v_i + u_j \ge a_{ij} \quad \forall i \in B, \ \forall j \in R, \\ & v_i \ge 0 \quad \forall i \in B, \\ & u_j \ge 0 \quad \forall j \in R \end{aligned} \tag{2}$$

Theorem 4. *The imputation (v, u) is in the core of the assignment game if and only if it is an optimal solution to the dual LP, (2).*

Proof. It is well known that the LP-relaxation of the bipartite graph maximum weight matching problem, namely (1), always has an optimal solution that is integral, i.e., there is always an optimal solution to this LP that is a maximum weight matching in G. Let W be the weight of such a matching; clearly, $W = w(B, R)$.

Let (v, u) be an optimal solution to the dual LP, (2). Then, by the LP-Duality Theorem,

$$\sum_{i \in B} v_i + \sum_{j \in R} u_j = W.$$

Therefore (v, u) is an imputation for distributing the worth of this game among the agents. Let $S \subseteq (B \cup R)$ and let $k = \min(|S \cap B|, |S \cap R|)$. By Definition 2, there are k disjoint buyer-seller pairs from S, say $(i_1, j_1), \ldots (i_k, j_k)$ such that $w(S) = (a_{i_1 j_1} + \ldots + a_{i_k j_k})$. By the first constraint of the dual LP, $v_{i_l} + u_{j_l} \geq a_{i_l j_l}$, for $1 \leq l \leq k$. Therefore, under imputation (v, u), the total gain of agents in S,

$$\sum_{i \in S} v_i + \sum_{j \in S} u_j \geq (a_{i_1 j_1} + \ldots + a_{i_k j_k}) = w(S).$$

Therefore the agents in S cannot improve on their gain under (v, u) by trading among themselves. Hence (v, u) is in the core.

Next, let (v, u) be in the core of this game. By definition of core, for the coalition $\{i, j\}$, $i \in B$ and $j \in R$, $v_i + u_j \geq w(\{i, j\}) = a_{ij}$. Therefore (v, u) is a feasible solution to the dual LP. Again by definition of core, the total gain of the grand coalition,

$$\sum_{i \in B} v_i + \sum_{j \in R} u_j \geq w(B \cup R) = W.$$

Therefore (v, u) is an optimal solution to the dual LP. □

2.1 The Core via the Lens of Complementarity

The worth of an assignment game is determined by an optimal solution to the primal LP, (1) and each core imputation distributes it using an optimal solution to the dual LP, (2). This fact naturally raises the question of viewing core imputations through the lens of complementarity[1]. This yields a relationship between the competitiveness of individuals and teams of agents and the amount of profit they accrue in imputations that lie in the core, where by *competitiveness* we mean whether an individual or a team is matched in every/some/no optimal assignment. Additionally, it sheds light on the phenomenon of degeneracy in assignment games, i.e., when the maximum weight matching is not unique.

Observe that in the housing market, the price at which a house is sold is completely determined by the core imputation that is chosen. In this section, we will use the following simpler setting to

[1] Recall that the complementary slackness conditions for a primal-dual pair of LPs relate primal variables with dual constraints and dual variables with primal constraints.

study the core of the assignment game. Suppose a coed tennis club has sets U and V of women and men players, respectively, who can participate in an upcoming mixed doubles tournament. Assume $|U| = m$ and $|V| = n$, where m, n are arbitrary. Let $G = (U, V, E)$ be a bipartite graph whose vertices are the women and men players and an edge (u, v) represents the fact that agents $u \in U$ and $v \in V$ are eligible to participate as a mixed doubles team in the tournament. Let w be an edge-weight function for G, where $w_{uv} > 0$ represents the expected earnings if u and v do participate as a team in the tournament. The total worth of the game is the weight of a maximum weight matching in G.

Assume that the club picks such a matching for the tournament. The question is how to distribute the total profit among the agents — strong players, weak players and unmatched players — so that no subset of players feel they will be better off seceding and forming their own tennis club.

Definition 5. By a *team* we mean an edge in G; a generic one will be denoted as $e = (u, v)$. We will say that e is:

1. *essential* if e is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that $e \in M$, and another, M' such that $e \notin M'$.
3. *subpar* if for every maximum weight matching M in G, $e \notin M$.

Definition 6. Let y be an imputation in the core of the game. We will say that e is *fairly paid in y* if $y_u + y_v = w_e$ and it is *overpaid* if $y_u + y_v > w_e$[2]. Finally, we will say that e is *always paid fairly* if it is fairly paid in every imputation in the core.

Definition 7. A generic player in $U \cup V$ will be denoted by q. We will say that q is:

1. *essential* if q is matched in every maximum weight matching in G.
2. *viable* if there is a maximum weight matching M such that q is matched in M and another, M' such that q is not matched in M'.
3. *subpar* if for every maximum weight matching M in G, q is not matched in M.

Definition 8. Let y be an imputation in the core. We will say that q *gets paid in y* if $y_q > 0$ and *does not get paid* otherwise. Furthermore, q is *paid sometimes* if there is at least one imputation in the core under which q gets paid, and it is *never paid* if it is not paid under every imputation.

Theorem 9. *The following hold:*

1. *For every team $e \in E$:*

$$e \text{ is always paid fairly} \iff e \text{ is viable or essential}$$

2. *For every player $q \in (U \cup V)$:*

$$q \text{ is paid sometimes} \iff q \text{ is essential}$$

Proof. The proofs follow by applying complementary slackness conditions and strict complementarity to the primal LP (1) and dual LP (2). We will use Theorem 4, stating that the set of imputations in the core of the game is precisely the set of optimal solutions to the dual LP.

[2]Observe that by the first constraint of the dual LP (2), these are the only possibilities.

1). Let x and y be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each $e = (u, v) \in E : \quad x_e \cdot (y_u + y_v - w_e) = 0$.

Suppose e is viable or essential. Then there is an optimal solution to the primal, say x, under which it is matched, i.e., $x_e > 0$. Let y be an arbitrary optimal dual solution. Then, by the Complementary Slackness Theorem, $y_u + y_v = w_e$. Varying y over all optimal dual solutions, we get that e is always paid fairly. This proves the forward direction.

For the reverse direction, we will use strict complementarity. It implies that corresponding to each team e, there is a pair of optimal primal and dual solutions x, y such that either $x_e = 0$ or $y_u + y_v = w_e$ but not both.

For team e, assume that the right hand side of the first statement holds and that x, y is a pair of optimal solutions for which strict complementarity holds for e. Since $y_u + y_v = w_e$ it must be the case that $x_e > 0$. Now, since the polytope defined by the constraints of the primal LP (**??**) has integral optimal vertices, there is a maximum weight matching under which e is matched. Therefore e is viable or essential and the left hand side of the first statement holds.

2). The proof is along the same lines and will be stated more succinctly. Again, let x and y be optimal solutions to LP (1) and LP (2), respectively. By the Complementary Slackness Theorem, for each $q \in (U \cup V) : \ y_q \cdot (x(\delta(q)) - 1) = 0$.

Suppose q is paid sometimes. Then, there is an imputation in the core, say y, such that $y_q > 0$. Therefore, for every primal optimal solution x, $x(\delta(q)) = 1$ and in every maximum weight matching in G, q is matched. Hence q is essential, proving the reverse direction.

Strict complementarity implies that corresponding to each player q, there is a pair of optimal primal and dual solutions x, y such that either $y_q = 0$ or $x(\delta(q)) = 1$ but not both. Since we have already established that the second condition must be holding for x, we get that $y_q > 0$ and hence q is paid sometimes. □

2.2 Consequences of Theorem 9

In this section, we will derive several useful consequences of Theorem 9.

1). Negating both sides of the first statement proved in Theorem 9 we get the following double-implication. For every team $e \in E$:

$$e \text{ is subpar} \iff e \text{ is sometimes overpaid}$$

Clearly, this statement is equivalent to the first statement of Theorem 9 and hence contains no new information. However, it may provide a new viewpoint. These two equivalent statements yield the following assertion, which at first sight seems incongruous with what we desire from the notion of the core and the just manner in which it allocates profits:

Whereas viable and essential teams are always paid fairly, subpar teams are sometimes overpaid.

How can the core favor subpar teams over viable and essential teams? Here is an explanation: Even though u and v are strong players, the team (u, v) may be subpar because u and v don't play well together. On the one hand, u and v are allocated high profits, since they generate large earnings while playing with other players. On the other hand, w_{uv} is small. Thus, this subpar team does not get overpaid while playing together, but by teaming up with others.

2). The second statement of Theorem 9 is equivalent to the following. For every player $q \in (U \cup V)$:

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Thus core imputations pay only essential players. Since we have assumed that the weight of each edge is positive, so is the worth of the game, and all of it goes to essential players. This gives the next conclusion; in contrast, the set of essential teams may be empty, as is the case in Examples 13 and 14 in Section 2.3.

Corollary 10. *In the assignment game, the set of essential players is non-empty and in every imputation, the entire worth of the game is distributed among essential players.*

Corollary 10 is of much consequence: It tells us that the players who are allocated profits are precisely the ones who always play, i.e., independent of which sets of teams the tennis club picks. Furthermore, the identification of these players, and the exact manner in which the total profit is divided up among them, follows the negotiating process described on Section 2.3, in which each player ascertains his/her negotiating power based on all possible sub-coalitions he/she participates in.

Thus by Theorem 4, each possible outcome of this very real process is captured by an inanimate object, namely an optimal solution to the dual LP, (2). This is perhaps the most remarkable aspect of this theorem.

3). Clearly the worth of the game is generated by teams that do play. Assume that (u, v) is such a team in an optimal assignment. Since $x_{uv} > 0$, by complementary slackness we get that $y_u + y_v = w_{uv}$, where y is a core imputation. Thus core imputations distribute the worth generated by a team among its players only. In contrast, the $\frac{2}{3}$-approximate core imputation for the general graph matching game given in Section 3 distributes the worth generated by teams which play to non-playing agents as well, thereby making a more thorough use of the TU aspect.

4). Next we use Theorem 9 to get insights into degeneracy. Examples 12 and 15 give a degenerate and a non-degenerate assignment game, respectively. Clearly, if an assignment game is non-degenerate, then every team and every player is either always matched or always unmatched in the set of maximum weight matchings in G, i.e., there are no viable teams or players. The next corollary characterizes the manner in which imputations in the core deal with players and teams in the presence of degeneracy.

Corollary 11. *In the presence of degeneracy, imputations in the core of an assignment game treat:*

- *Viable and essential teams in the same way, namely they are always fairly paid.*
- *Viable and subpar players in the same way, namely they are never paid.*

Example 12. The instance has $n = 3$ and $m = 2$. Table 1 gives the gains accrued by the various pairs of agents.

This instance is degenerate; it has two maximum weight matchings: $\{(b_1, r_1), (b_3, r_2)\}$ and $\{(b_2, r_1), (b_3, r_2)\}$, both of weight 190.

Table 1: Gains from all possible trades.

	r_1	r_2
b_1	100	0
b_2	100	70
b_3	0	90

2.3 Insights Provided by the Core into the Negotiating Power of Agents

Example 13. Consider an assignment game whose bipartite graph has two edges, $(u, v_1), (u, v_2)$ on the three agents u, v_1, v_2. Clearly, one of v_1 and v_2 will be left out in any matching. First assume that the weight of both edges is 1. If so, the unique imputation in the core gives zero to v_1 and v_2, and 1 to u. Next assume that the weights of the two edges are 1 and $1 + \epsilon$ respectively, for a small $\epsilon > 0$. If so, the unique imputation in the core gives $0, \epsilon$ and 1 to v_1, v_2 and u, respectively.

How fair are the imputations given in Example 13? As stated in the Introduction, imputations in the core have a lot to do with the negotiating power of individuals and sub-coalitions. Let us argue that when the imputations given above are viewed from this angle, they are fair in that the profit allocated to an agent is consistent with their negotiating power, i.e., their worth. In the first case, whereas u has alternatives, v_1 and v_2 don't. As a result, u will squeeze out all profits from whoever she plays with, by threatening to partner with the other player. Therefore v_1 and v_2 have to be content with no rewards! In the second case, u can always threaten to match up with v_2. Therefore v_1 has to be content with a profit of ϵ only.

In an arbitrary assignment game $G = (U, V, E), w$, by Theorem 9,

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Thus core imputations reward only those agents who always play. This raises the following question: Can't a non-essential player, say q, team up with another player, say p, and secede, by promising p almost all of the resulting profit? The answer is "No", because the dual (2) has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0$, $y_p \geq w_{qp}$, i.e., p will not gain by seceding together with q.

Example 14. Next, consider an assignment game whose bipartite graph has four edges, (u_1, v_1), (u_1, v_2), $(u_2, v_2), (u_2, v_3)$ on the five agents u_1, u_2, v_1, v_2, v_3. Let the wights of these four edges be $1, 1.1, 1.1$ and 1, respectively. The worth of this game is clearly 2.1.

In Example 14, at first sight, v_2 looks like the dominant player, since he has two choices of partners, namely u_1 and u_2, and because teams involving him have the biggest earnings, namely 1.1 as opposed to 1. Yet, the unique core imputation in the core awards $1, 1, 0, 0.1, 0$ to agents u_1, u_2, v_1, v_2, v_3, respectively.

The question arises: "Why is v_2 allocated only 0.1? Can't v_2 negotiate a higher profit, given its favorable circumstance?" The answer is "No". The reason is that u_1 and u_2 are in an even stronger position than v_2, since both of them have a ready partner available, namely v_1 and v_3, respectively, with whom each can earn 1. Therefore, the core imputation awards 1 to each of them, giving the leftover profit of 0.1 to v_2. Hence the core imputation has indeed allocated profits according to the negotiating power of each agent.

2.4 Extreme Imputations in the Core

In this section, we will build on Theorem 4 to characterize the extreme imputations that belong to the core of the housing game. First, observe that each imputation completely pins down the prices of all houses, since for each $j \in R$, $p_j = c_j + u_j$. Therefore, the prices contain no extra information and will be typically dropped from the discussion.

If i and j do trade, then $v_i + u_j = a_{ij}$. Consequently, v_i and u_j both belong to the interval $[0, a_{ij}]$. The first question that arises is whether v_i, and therefore u_j, can take every value in this interval. The answer turns out to be very interesting, namely, it depends on the other options that i and j have available, even if those options are not good trades and therefore i or j may never be able to exercise such options. Example 15 illustrates this phenomenon.

Example 15. The instance has $n = m = 2$, with $c_1 = 800$ and $c_2 = 700$. The buyers' valuations for houses are given in Table 2. For the purpose of understanding the core of this game, the key information is contained in Table 3 which gives the gains accrued by the various pairs of agents from all possible trades.

Table 2: Buyers' valuations for houses.

	r_1	r_2
b_1	900	770
b_2	845	710

Table 3: Gains from all possible trades.

	r_1	r_2
b_1	100	70
b_2	45	10

It is easy to see that the trades that yield the maximum total gain are (b_1, r_2) and (b_2, r_1), with the total gain being 115. Since (b_1, r_2) and (b_2, r_1) are good trades, by Theorem **??**,

$$b_1 + r_2 = 70 \quad \text{and} \quad b_2 + r_1 = 45.$$

Therefore $b_1 \le 70$ and $r_1 \le 45$. Additionally, we have

$$b_1 + r_1 \ge 100 \quad \text{and} \quad b_2 + r_2 \ge 10.$$

Combining with the previous facts we get $b_1 \ge 55$ and $r_1 \ge 30$. These further give $b_2 \le 15$ and $r_2 \le 15$.

The interesting conclusion is b_1 and r_1 are tightly constrained, with $b_1 \in [55, 70]$ and $r_1 \in [30, 45]$. The constraints on b_2 and r_2 are $b_2 \in [0, 15]$ and $r_2 \in [0, 15]$. Observe that the trade (b_1, r_1) will not happen. Yet, the threat of this possible trade gives b_1 and r_1 negotiating power which ensures that they cannot be forced into zero gains; in fact the lower bounds on their gains are 55 and 30, respectively. On the other hand, the alternative trade which b_2 and r_2 have available is not competitive, thereby giving them little negotiating power and constraining their profits to intervals with upper bounds of 15 only.

We will show that the core contains two extreme imputations, one is maximally advantageous to buyers and the other to sellers; furthermore, the first is maximally disadvantageous to sellers and the second to buyers. For $i \in B$, let v_i^h and v_i^l denote the highest and lowest profits that i accrues among all imputations in the core. Similarly, for $j \in R$, let u_j^h and u_j^l denote the highest and lowest profits that j accrues in the core. Let v^h and v^l denote the vectors whose components are v_i^h and v_i^l, respectively. Similarly, let u^h and u^l denote vectors whose components are u_j^h and u_j^l, respectively. The following is a formal statement regarding the extreme imputations.

Theorem 16. *The two extreme imputations in the core are* (v^h, u^l) *and* (v^l, u^h).

We will start by proving Lemma 17. Let (q, r) and (s, t) be two imputations in the core. For each $i \in B$, let

$$\underline{v_i} = \min(q_i, s_i) \quad \text{and} \quad \bar{v}_i = \max(q_i, s_i).$$

Further, for each $j \in R$, let

$$\underline{u_j} = \min(r_j, t_j) \quad \text{and} \quad \bar{u}_j = \max(r_j, t_j).$$

Lemma 17. $(\underline{v}, \bar{u})$ *and* $(\bar{v}, \underline{u})$ *are imputations in the core.*

Proof. Consider the first imputation. We will show that for each $i \in B$ and $j \in R$, $\underline{v_i} + \bar{u}_j \geq a_{ij}$.

$$\underline{v_i} = \min(q_i, s_i) \geq \min(a_{ij} - r_j, a_{ij} - t_j) = a_{ij} - \max(r_j, t_j) = a_{ij} - \bar{u}_j.$$

Next, we need to show that the sum of the profits of all agents under this imputation equals the total worth of $(B \cup R)$. By renaming agents, we may assume that each pair in the maximum weight matching is of the form (l, l), for $l \in B$ and $l \in R$. Then, $q_l + r_l = a_{ll}$ and $s_l + t_l = a_{ll}$. Therefore, we get

$$\underline{v_i} = \min(q_i, s_i) = \min(a_{ij} - r_j, a_{ij} - t_j) = a_{ij} - \max(r_j, t_j) = a_{ij} - \bar{u}_j,$$

giving $\underline{v_l} + \bar{u}_l = a_{ll}$. Therefore, imputation $(\underline{v}, \bar{u})$ is in the core. An analogous statement about the second imputation follows in a similar manner. □

Let us call the process of obtaining imputations $(\underline{v}, \bar{u})$ and $(\bar{v}, \underline{u})$ from imputations (q, r) and (s, t) as *mating*. To obtain the extreme imputations promised in Theorem 16, we will start with an arbitrary imputation and keep mating it with an imputation that has v_i^l or u_j^h, for each value of i and j. This will give us (v^l, u^h). An analogous process will give us (v^h, u^l).

The extreme imputations in the core for Example 12 are $(0, 0, 0), (100, 90)$ and $(0, 0, 20), (100, 70)$, and those for Example 15 are $(55, 0), (45, 15)$ and $(70, 15), (30, 0)$.

3 Approximate Core for the General Graph Matching Game

The general graph matching game has numerous applications as well. Its underlying issues are nicely captured in the following setting. Suppose a tennis club has a set V of players who can play in an upcoming doubles tournament. Let $G = (V, E)$ be a graph whose vertices are the players and an edge (i, j) represents the fact that players i and j are compatible doubles partners. Let w be an edge-weight function for G, where w_{ij} represents the expected earnings if i and j do partner in the tournament. Then the total worth of agents in V is the weight of a maximum weight matching in G. Assume that the club picks such a matching M for the tournament. The question is how to distribute the total profit among the agents — strong players, weak players and unmatched players — so that no subset of players feel they will be better off seceding and forming their own tennis club.

Definition 18. The *general graph matching game* consists of an undirected graph $G = (V, E)$ and an edge-weight function w. The vertices $i \in V$ are the agents and an edge (i, j) represents the fact that agents i and j are eligible for an activity, for concreteness, let us say that they are eligible to participate as a doubles team in a tournament. If $(i, j) \in E$, w_{ij} represents the profit generated if i and j play in the tournament. The *worth* of a coalition $S \subseteq V$ is defined to be the maximum profit that can be generated by teams within S and is denoted by $p(S)$. Formally, $p(S)$ is defined to be the weight of a maximum weight matching in the graph G restricted to vertices in S only. The *characteristic function* of the matching game is defined to be $p : 2^V \to \mathcal{R}_+$.

Among the possible coalitions, the most important one is of course V, the *grand coalition*.

Definition 19. An *imputation* gives a way of dividing the worth of the game, $p(V)$, among the agents. Formally, it is a function $v : V \to \mathcal{R}_+$ such that $\sum_{i \in V} v(i) = p(V)$. An imputation t is said to be in the *core of the matching game* if for any coalition $S \subseteq V$, there is no way of dividing $p(S)$ among the agents in S in such a way that all agents are at least as well off and at least one agent is strictly better off than in the imputation t.

The core of a non-bipartite game may be empty, as shown in Example 20.

Example 20. Consider the graph K_3, i.e., a clique on three vertices, i, j, k, with a weight of 1 on each edge. Any maximum matching in K_3 has only one edge, and therefore the worth of this game is 1. Suppose there is an imputation v which lies in the core. Consider all three two-agent coalitions. Then, we must have:

$$v(i) + v(j) \geq 1, \quad v(j) + v(k) \geq 1 \quad \text{and} \quad v(i) + v(k) \geq 1.$$

This implies $v(i) + v(j) + v(k) \geq 3/2$ which exceeds the worth of the game, giving a contradiction.

Observe however, that if we distribute the worth of this game as follows, we get a 2/3-approximate core allocation: $v(i) = v(j) = v(k) = 1/3$. Now each edge is covered to the extent of 2/3 of its weight. In Section 3.1 we show that such an approximate core allocation can always be obtained for the general graph matching game.

Definition 21. Let $p : 2^V \to \mathcal{R}_+$ be the characteristic function of a game and let $1 \geq \alpha > 0$. An imputation $t : V \to \mathcal{R}_+$ is said to be in the *α-approximate core* of the game if:

1. The total profit allocated by t is at most the worth of the game, i.e.,

$$\sum_{i \in V} t_i \leq p(V).$$

2. The total profit accrued by agents in a sub-coalition $S \subseteq V$ is at least α fraction of the profit which S can generate by itself, i.e.,

$$\forall S \subseteq V: \sum_{i \in S} t_i \geq \alpha \cdot p(S).$$

If imputation t is in the α-approximate core of a game, then the ratio of the total profit of any sub-coalition on seceding from the grand coalition to its profit while in the grand coalition is bounded by a factor of at most $\frac{1}{\alpha}$.

3.1 A 2/3-Approximate Core for the Matching Game

We will work with the following LP-relaxation of the maximum weight matching problem, (3). This relaxation always has an integral optimal solution in case G is bipartite, but not in general graphs. In the latter, its optimal solution is a maximum weight fractional matching in G.

$$\begin{aligned} \max \quad & \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall i \in V, \\ & x_{ij} \geq 0 \quad \forall (i,j) \in E \end{aligned} \tag{3}$$

Taking v_i to be dual variables for the first constraint of (3), we obtain LP (4). Any feasible solution to this LP is called a *cover* of G since for each edge (i,j), v_i and v_j cover edge (i,j) in the sense that $v_i + v_j \geq w_{ij}$. An optimal solution to this LP is a *minimum cover*. We will say that v_i is the *profit* of vertex i.

$$\begin{aligned} \min \quad & \sum_{i \in V} v_i \\ \text{s.t.} \quad & v_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\ & v_i \geq 0 \quad \forall i \in V \end{aligned} \tag{4}$$

We will say that a solution x to LP (3) is *half-integral* if for each edge (i,j), x_{ij} is 0, 1/2 or 1. By the LP Duality Theorem, the weight of a maximum weight fractional matching equals the total profit of a minimum cover. If for graph G, LP (3) has an integral optimal solution, then it is easy to see that an optimal dual solution gives a way of allocating the total worth which lies in the core; see the related Exercise 6. Otherwise, it must have a half-integral optimal solution as shown next.

Transform $G = (V,E)$ with edge-weights w to graph $G' = (V', E')$ and edge weights w' as follows. Corresponding to each $i \in V$, V' has vertices i' and i'', and corresponding to each edge $(i,j) \in E$, E' has edges (i', j'') and (i'', j') each having a weight of $w_{ij}/2$.

Since each cycle of length k in G is transformed to a cycle of length $2k$ in G', the latter graph has only even length cycles and is bipartite. A maximum weight matching and a minimum cover for G' can be computed in polynomial time, say x' and v', respectively. Next, let

$$x_{ij} \ = \ \frac{1}{2} \cdot (x_{i',j''} + x_{i'',j'}) \quad \text{and} \quad v_i = (v_{i'} + v_{i''}).$$

It is easy to see that the weight of x equals the value of v, thereby implying that v is an optimal cover.

Lemma 22. *x is a maximum weight half-integral matching and v is an optimal cover in G.*

Proof. We will first use the fact that v' is a feasible cover for G' to show that v is a feasible cover for G. Corresponding to each edge (i, j) in G, we have two edges in G' satisfying:

$$v'_{i'} + v'_{j''} \geq \frac{1}{2} \cdot w_{ij} \quad \text{and} \quad v'_{i''} + v'_{j'} \geq \frac{1}{2} \cdot w_{ij}.$$

Therefore, in G, $v_i + v_j \geq w_{ij}$, implying feasibility of v.

By the LP-duality theorem, the weight of x' equals the value of v' in G'. Corresponding to each edge (i, j) in G, we have:

$$x_{ij} \cdot w_{ij} \ = \ \left(\frac{1}{2} \cdot (x_{i',j''} + x_{i'',j'})\right) \cdot w_{ij} \ = \ x_{i',j''} \cdot w'_{ij} + x_{i'',j'} \cdot w'_{ij}.$$

Adding over all edges, we get that the weight of x in G equals the weight of x' in G'. Furthermore, the profit of i equals the sum of profits of i' and i''. Therefore the value of v in G equals the value of v' in G'.

Putting it together, we get that the weight of w equals the value of v, implying optimality of both. Clearly, x is half-integral. The lemma follows. ☐

Edges that are set to half in x form connected components which are either paths or cycles. For any such path, consider the two matchings obtained by picking alternate edges. The half-integral solution for this path is a convex combination of these two integral matchings. Therefore both these matchings must be of equal weight, since otherwise we can obtain a heavier matching. Pick any of them. Similarly, if a cycle is of even length, pick alternate edges and match them. This transforms x to a maximum weight half-integral matching in which all edges that are set to half form disjoint odd cycles. Henceforth we will assume that x satisfies this property.

Let C be a half-integral odd cycle in x of length $2k + 1$, with consecutive vertices $i_1, \ldots i_{2k+1}$. Let $w_C = w_{i_1,i_2} + w_{i_2,i_3} + \ldots + w_{i_{2k+1},i_1}$ and $v_C = v_{i_1} + \ldots v_{i_{2k+1}}$. On removing any one vertex, say i_j, with its two edges from C, we are left with a path of length $2k - 1$. Let M_j be the matching consisting of the k alternate edges of this path and let $w(M_j)$ be the weight of this matching.

Lemma 23. *Odd cycle C satisfies:*

1. $w_C = 2 \cdot v_C$
2. *C has a unique cover: $v_{i_j} = v_C - w(M_j)$, for $1 \leq j \leq 2k + 1$.*

Proof. **1).** We will use the fact that x and v are optimal solutions to LPs (3) and (4), respectively. By the primal complementary slackness condition, for $1 \leq j \leq 2k+1$, $w_{i_j,i_{j+1}} = v_{i_j} + v_{i_{j+1}}$, where addition in the subindices is done modulo $2k+1$; this follows from the fact that $x_{i_j,i_{j+1}} > 0$. Adding over all vertices of C we get $w_C = 2 \cdot v_C$.

2). By the equalities established in the proof of the first part, we get that for $1 \leq j \leq 2k+1$, $v_C = v_{i_j} + w(M_j)$. Rearranging terms gives the lemma. □

Let M' be heaviest matching among M_j, for $1 \leq j \leq 2k+1$.

Lemma 24.

$$w(M') \geq \frac{2k}{2k+1} \cdot v_C$$

Proof. Adding the equality established in the second part of Lemma 23 for all $2k+1$ values of j we get:

$$\sum_{j=1}^{2k+1} w(M_j) \ = \ (2k) \cdot v_C$$

Since M' is the heaviest of the $2k+1$ matchings in the summation, the lemma follows. □

Modify the half-integral matching x to obtain an integral matching T in G as follows. First pick all edges (i,j) such that $x_{ij} = 1$ in T. Next, for each odd cycle C, find the heaviest matching M' as described above and pick all its edges.

Definition 25. Let $1 > \alpha > 0$. A function $c : V \to \mathcal{R}_+$ is said to be an *α-approximate cover* for G if

$$\forall (i,j) \in E : \quad c_i + c_j \geq \alpha \cdot w_{ij}$$

Define function $f : V \to [\frac{2}{3}, 1]$ as follows: $\forall i \in V$:

$$f(i) = \begin{cases} \frac{2k}{2k+1} & \text{if } i \text{ is in a half-integral cycle of length } 2k+1. \\ 1 & \text{if } i \text{ is not in a half-integral cycle.} \end{cases}$$

Next, modify cover v to obtain an approximate cover c as follows: $\forall i \in V : \ c_i = f(i) \cdot v_i$.

Lemma 26. *c is a $\frac{2}{3}$-approximate cover for G.*

Proof. Consider edge $(i,j) \in E$. Then

$$c_i + c_j \ = \ f(i) \cdot v_i + f(j) \ \cdot v_j \ \geq \frac{2}{3} \cdot (v_i + v_j) \ \geq \ \frac{2}{3} \cdot w_{ij},$$

where the first inequality follows from the fact that $\forall i \in V, \ f(i) \geq \frac{2}{3}$ and the second follows from the fact that v is a cover for G. □

The mechanism for obtaining imputation c is summarized as Mechanism 27.

Theorem 28. *The imputation c is in the $\frac{2}{3}$-approximate core of the general graph matching game.*

Algorithm 27. (2/3-Approximate Core Imputation)

1. Compute x and v, optimal solutions to LPs (3) and (2), where x is half-integral.
2. Modify x so all half-integral edges form odd cycles.
3. $\forall i \in V$, compute:

$$f(i) = \begin{cases} \frac{2k}{2k+1} & \text{if } i \text{ is in a half-integral cycle of length } 2k+1. \\ 1 & \text{otherwise.} \end{cases}$$

4. $\forall i \in V$: $\quad c_i \leftarrow f(i) \cdot v_i$.

Output c.

Proof. We need to show that c satisfies the two conditions given in Definition 21, for $\alpha = \frac{2}{3}$.

1). By Lemma 24, the weight of the matched edges picked in T from a half-integral odd cycle C of length $2k+1$ is $\geq f(k) \cdot v_C \;=\; \sum_{i \in C} c(i)$. Next remove all half-integral odd cycles from G to obtain G'. Let x' and v' be the projections of x and v to G'.

By the first part of Lemma 23, the total decrease in weight in going from x to x' equals the total decrease in value in going from v to v'. Therefore, the weight of x' equals the total value of v'. Finally, observe that in G', T picks an edge (i, j) if and only if $x'_{ij} = 1$ and $\forall i \in G'$, $c_i = v'_i$.

Adding the weight of the matching and the value of the imputation c over G' and all half-integral odd cycles we get $w(T) \geq \sum_{i \in V} c_i$.

2). Consider a coalition $S \subseteq V$. Then $p(S)$ is the weight of a maximum weight matching in G restricted to S. Assume this matching is $(i_1, j_1), \ldots (i_k, j_k)$, where $i_1, \ldots i_k$ and $j_1, \ldots j_k \in S$. Then $p(S) = (w_{i_1 j_1} + \ldots + w_{i_k j_k})$. By Lemma 26,

$$c_{i_l} + c_{j_l} \;\geq\; \frac{2}{3} \cdot w_{i_l, j_l}, \quad \text{for } 1 \leq l \leq k.$$

Adding all k terms we get:

$$\sum_{i \in S} c_i \;\geq\; \frac{2}{3} \cdot p(S).$$

□

Because of the example given in Example 20, the factor of 2/3 cannot be improved. Observe that for the purpose of Lemma 26, we could have defined f simply as $\forall i \in V,\ f(i) = \frac{2}{3}$. However in general, this would have left a good fraction of the worth of the game unallocated. The definition of f given above improves the allocation for agents who are in large odd cycles and those who are not in odd cycles with respect to matching x. As a result, the gain of a typical sub-coalition on seceding will be less than a factor of $\frac{3}{2}$, giving it less incentive to secede.

4 Many-to-one matching with salaries

We turn to the study of a many-to-one labor market with salaries. Instead of houses, the agents buy and sell heterogeneous labor services. The set B of n buyers of labor services are now called *firms*. The set R of m sellers are *workers*. Each firm i has a *valuation* h_i, a function that takes subsets of workers as arguments. If firm i hires a set of workers $A \subset R$ paying salaries s_j (for $j \in A$) then its payoff is

$$h_i(A) - \sum_{j \in A} s_j.$$

Suppose that $h_i(\emptyset) = 0$, and that h_i is strictly monotonically increasing in the set of workers (so a firm would always like to add a worker at a salary of zero). With these definitions in place, we can talk about the *demand function* for firm i:

$$d_i(s) = \text{argmax}\{h_i(A) - \sum_{j \in A} s_j : A \subset R\},$$

for any vector $s \in \Re_+^R$ of salaries. Note that $d_i(s)$ may contain more than one set of workers.

A worker j who is employed by firm i suffers a loss, say $c_{j,i}$, from surrendering her labor services. This loss may depend on the identity of her employer because different firms require workers to perform different tasks, under different conditions. For example, assume that workers are academics and firms are universities. One university may demand a higher teaching load than another, or it may be located in a city that is more desirable to worker j than another. When worker j is employed by firm i at salary s, she obtains a utility, or payoff, $s - c_{j,i}$.

We need to make two assumptions. The first is relatively innocuous: it says that any firm would be willing to hire a worker at the salary that would compensate the worker for the disutility of working at the firm. Formally, suppose that, for any set of workers A,

$$h_i(A \cup \{j\}) - c_{j,i} \geq h_i(A \setminus \{j\}).$$

Call this assumption *acceptance*.

The second assumption is more substantive, and will severely restrict the functions h_i that are allowed. The idea is that if a firm hires worker i at some given vector of salaries, it cannot be because i is part of a team of complementary workers. In particular, if other workers' salaries are increased (and presumably some of those workers are laid off), worker i must continue to be employed by the firm. Formally, say that h_i satisfies the *substitutability* condition if $A \in d_i(s)$ and $s \leq s'$ then there is $A' \in d_i(s')$ so that A' contains all workers in A that have the same salary in s as in s'.[3]

The model determines a market outcome, which describes who works for whom, and the salaries earned by each of the workers. A *matching* is a function $\mu : R \to B \cup R$ with the property that $\mu(j) \in B \cup \{j\}$, so that a worker is either employed by firm $\mu(j) \in B$, or unemployed, which we

[3]Substitutability implies that the valuation is submodular, but it is a strictly stronger property than submodularity. One characterization of substitutability uses the indirect utility function defined by h_i – the function mapping each salary vector to the highest profit attainable by the firm at those salaries. Substitutability is equivalent to the submodularity of indirect utility.

denote as $\mu(j) = j$. A *market outcome* is a pair (μ, s), where μ is a matching and $s \in \Re^R_+$ is a vector of salaries.

We shall restrict the set of possible salaries to lie on a discrete grid. Suppose, in particular, that the set of possible salaries is $\mathbb{Z}_+$. Suppose also for convenience that each $c_{j,i} \in \mathbb{Z}_+$.

Finally, the firms and the workers in the model have some individual agency to opt out of the market. They have an *outside option* available. A worker can choose not to be employed, and perhaps take up another occupation, while firms can always opt to shut down operations. We assume that the utility of such an outside option is always zero.

Definition 29. We are interested in outcomes that leave no agent worse off than with their outside option, and where no re-contracting is desirable.

- An outcome (μ, s) is *individually rational* if, for all workers i, $s_i \geq c_{\mu(i),i}$; and for all firms j,

$$h_j(\mu^{-1}(j)) - \sum_{i \in \mu^{-1}(j)} s_i \geq 0.$$

- A firm i and a set of workers A *block* an outcome (μ, s) if there are salaries $(\hat{s}_j)_{j \in A}$ such that i would strictly prefer to hire the workers in A at salaries $(\hat{s}_j)_{j \in A}$ over the workers $\mu(i)$ at salaries $(s_j)_{j \in A}$, and all the workers $j \in A$ would strictly prefer to work for i at salaries $\hat{s}_j$ instead of for $\mu(j)$ at salary s_j.

Individual rationality provides workers an option to stay out of the market if they are not minimally compensated for their employment, and firms with the option to shut down: employ no workers and pay no salaries. Blocking, in turn, refers to a mutually beneficial recontracting between a firm and a set of workers. We have seen in Section 2 an instance of this phenomenon, where it is enough to be concerned about recontracting by limited subsets of agents – the only sets of agents that can generate value in the model.

Definition 30. An outcome is in the *core* if it is individually rational, and there is no firm and set of workers that blocks it.

4.1 The salary adjustment process

We propose a variation of the Deferred Acceptance Algorithm that we term the *Salary Adjustment Process*. The algorithm operates iteratively, with firms proposing employment to a set of workers at certain fixed salaries.

Initialize the algorithm by setting the salary offered by firm i to worker j at $s_{i,j}(0) = c_{j,i}$. In each iteration k of the algorithm, the algorithm carries out these steps:

1. Each firm offers employment to a set of workers $A \in d_i((s_{i,j}(k))_{j \in R})$, as long as A contains all the workers that i offered employment to in the previous iteration of the algorithm and whose salaries did not change between iterations.

2. Each worker tentatively accepts the best offer that they have received, rejecting all others.

3. If a worker rejects an offer from a firm, then the salary proposed by that firm increases by one. So, if firm i made an offer to worker j, and j accepted another offer, then $s_{i,j}(k+1) = s_{i,j}(k) + 1$.

The algorithm ends when no worker rejects any more offers. The outcome of the algorithm is a matching, determined by the last accepted offers by each worker, and a salary for each worker defined as the last salary that was proposed and accepted.

Observe:

- The algorithm asks firms to keep their offers to workers who did not reject them in prior rounds, and for whom salaries have been kept the same. This is made possible by the substitutability assumption.
- We may wlog assume that, in the first iteration of the algorithm, all firms make an offer to all workers. Indeed, by our acceptance assumption, no firm makes any losses by including worker j at salary $c_{j,i}$.

Note also that at each step of the algorithm, whenever some offer is rejected, the salary of some worker increases. By the nature of firms' profits, salaries can get high enough that no worker is hired. So it is also clear that the algorithm stops after a finite number of steps (a number of steps bounded by $m \times \bar{w}$, where $\bar{w}$ is a *choke salary* at which $d_i(\bar{w}, \ldots, \bar{w}) = \emptyset$).

When the algorithm starts, we may define $\mu(j)$ to be the firm whose offer worker j last tentatively accepted, and set s_j to be the salary that was offered (so if the algorithm stopped in iteration k, $s_j = s_{\mu(j),j}(k)$). Let (μ, s) be the outcome of the salary adjustment process.

Theorem 31. *The outcome of the salary adjustment process is in the core.*

Proof. First, it is clear that the outcome is individually rational. Indeed, offers start at salaries that compensate workers for the disutility involved in accepting the offer; firms, in turn, only make offers to sets of workers that maximize their profits. So they would never make an offer that is worse than shutting down.

Second, lets establish that there are no blocks. Suppose that worker j would rather work for firm i at salary $\hat{s}_i$, over the terms of her employment at the outcome of the salary adjustment process. Since all firms make an offer to each worker in the initial round of the algorithm, j would have made i an offer at all salaries (in the grid $\mathbb{Z}_+$) from $s_{j,i}(0)$ and up to some $s'_i < \hat{s}_i$. This offer must have been rejected by j.

Now, if A is a set of workers in this situation, and $(\hat{s}_i)_{i \in A}$ the salaries at which they would prefer to work for firm j. Then we have that

$$h_i(\mu^{-1}(i)) - \sum_{l \in \mu^{-1}(i)} s_i \geq h_i(A) - \sum_{i \in A} s_i \geq h_i(A) - \sum_{i \in A} \hat{s}_i.$$

Thus the set of workers A together with firm j do not constitute a block. □

4.2 A model with a continuum of salaries

We have taken salaries to be in a discrete grid, but it is easy to see that the argument we have laid out implies the existence of core outcomes for a model with continuum salaries. In the rest of our discussion we shall be somewhat informal, but the ideas are very simple.

First, as an exercise, we invite the reader to modify the description of the model to allow salaries to take any value in the compact interval $[0, \bar{\omega}]$. The model remains essentially the same, but salaries are allowed to take any real values in this interval. Note that $\bar{\omega} > 0$ should be chosen large enough to accommodate all values of $c_{j,i}$, and so as to be larger than the choke salary for all firms (so, large enough to make employment not profitable at such high salaries).

Now, it should be easy to see that the argument proving Theorem 31 does not depend on the grid being $\mathbb{Z}_+$. Indeed, for any n we may consider the finite grid $W_n = \{0, \bar{\omega}/n, 2\bar{\omega}/n, \ldots, \bar{\omega}\} \cup \{c_{j,i} : j \in B, i \in R\}$. The salary adjustment process determines an outcome (μ^n, s^n) that is in the core of the appropriate discrete version of the model. By compactness of $[0, \bar{\omega}]$, there is (after going to a subsequence) $(\mu, s) = \lim_{n\to\infty}(\mu^n, s^n)$. We leave the next result as an exercise:
Theorem 32. *The outcome (μ, s) is in the core of the continuum salary many-to-one market.*

5 Matching with contracts

We next turn to a generalization of the model with salaries. Again we shall focus on many-to-one matchings between workers and firms, but now instead of a simple salary we shall allow for complex terms of employment. In addition to a salary, a firm may offer its workers health insurance, day-care, or a retirement plan. Such additional terms are captured through an abstract model of contracts.

Suppose that, as before R is the set of workers and B the set of firms. The finite set X contains all possible contracts. Each contract $x \in X$ specifies a unique worker $x_R \in R$ and firm $x_B \in B$, the idea being that these are the two parties to the contract. There will generally be many contracts in X available to each worker-firm pair, and we can interpret the model as one of multilateral bargaining over terms of employment. Workers and firms interact with contracts in somewhat different ways.

First, we restrict attention to many-to-one matching, which means that workers can sign at most one contract. Indeed, for worker i and a set of contracts X', we let $c_i(X')$ be the most preferred contract by worker i out of the set X'. The function c_i is a *choice function*. There are some implicit assumptions here that we should mention. The worker has a strict preference over contracts, and when we write "most preferred" we have this preference in mind. The worker always has the option of refusing all contracts in X', which is denoted as $c_i(X') = \emptyset$. Finally, obviously the chosen contract must involve worker i.

Firms can choose a set of contracts, each involving different workers. So firm j when facing a set X' of contracts will choose $c_j(X') \subseteq X'$ as its most preferred set of contracts. Similarly to workers, these contracts must all involve j, and when j refuses all contracts in the set we shall write $c_j(X') = \emptyset$. Firms are assumed to have a strict preference over sets of contracts, so that

$c_j(X')$ is the most preferred set of contracts involving firm j, out of all the contracts in X'. Finally, there can be no ambiguity in the terms of employment, so any two different contracts contained in $c_j(X')$ must involve two different workers.

In the model of matching with contracts, an *outcome* is a set of contracts such that each worker is involved in at most one contract. Clearly such an outcome specifies a matching between workers and firms because it describes which firm any worker who holds a contract is employed with. Workers who hold no contracts are unemployed. Moreover, for each employed worker, the terms of employment are determined by the unique contract that she holds.

For succinctness, lets denote by X'_a the set of contracts in X'_a that involve agent a, who may be a worker or a firm. In an allocation, for a worker i, X'_i is either a single contract or $\emptyset$.

Definition 33. We are interested in outcomes that leave no agent worse off than with their outside option, and where no re-contracting is desirable.

- An outcome X' is *individually rational* if $c_a(X') = X'_a$ for all $a \in R \cup B$.
- A firm j and a set of workers A *block* an outcome X' if there is a set of contracts $X'' \neq X'$ for which $X'' = c_j(X' \cup X'')$, and $X''_i = c_i(X' \cup X'')$ for all $i \in A$.

Individual rationality here gives workers and firms agency to unilaterally turn down a contract. So a set of contracts respects individual rationality when no worker or firm prefers to drop one or more of the contracts in the outcome.

Blocking captures, as before, the idea of recontracting. The existence of a block means that a firm, together with a set of workers, and using the vehicle of an alternative set of contracts, can all be made better off. Note that we allow for some workers to receive the same contract in X' and in X'', so that they are not made strictly better off.

Definition 34. An outcome is *stable* if it is individually rational and admits no blocks.

Similarly to the model of matching with salaries, we shall rule out that firms regard workers as complements. Specifically, we say that a choice function c_j for firm j satisfies *substitutability* if, for any set of contracts $X' \subseteq X''$, if $x \in c_j(X'')$ and $x \in X'$ then $x \in c_j(X')$. In words, x cannot be chosen in X'' due of the presence of some complementary contracts that are absent from X'.

Theorem 35. *There exists a stable outcome.*

6 Exercises

1. ([SS71]) Let $S \subseteq (B \cup R)$ and q, r be agents not in S. We will say that q and r are of the same type if they are both buyers or both sellers, and of different types otherwise. Let $S \cup \{q\}$ be denoted by S^q. Define the *marginal value of agent q to S* to be $w(S^q) - w(S)$, and denote it by $m(q, S)$. Prove:

1. If q and r are of the same type then $m(r, S^q) \leq m(r, S)$.
2. If q and r are of different types then $m(r, S^q) \geq m(r, S)$.

2. Give an efficient algorithm for checking the following: Given an instance of the housing market and an imputation (v, u), is it in the core?

3. Consider the instance of housing market with $n = m = 4$ and gains of pairs of agents given in Table 1; blanks represent zeros.

Table 4: Gains from all possible trades.

	r_1	r_2	r_3	r_4
b_1	100	110	51	
b_2	80	100		51
b_3	50			
b_4		50		

Show that the total profit distributed to agents b_3, b_4, r_3, r_4 by any imputation in the core is 2, even though they are matched in every maximum weight matching. What are the extreme imputations in the core?

4. In the instance of Example 12, assume that edge (b_3, r_1) has weight 120, keeping all other weights the same. How many maximum weight matchings are there now and how many imputations are there in the core?

5. ([vBB23]) Consider an instance of the housing market in which there is no product differentiation, i.e., all houses are identical for all buyers. However, different buyers have different values for a house. Also, different sellers have a different value for their own house. Find a way of computing the worth of this game via a process that is easier than computing a maximum weight matching. Also, prove that there is a uniform market price for all houses. Find the interval in which this price lies.

6. ([DIN97]) Prove that the core of the general graph matching game is non-empty if and only if LP (1) has an integral optimal solution.

7. One way of formally stating an improved factor beyond that given in Theorem 28 is the following: Assume that the underlying graph G has no odd cycles of length less than $2k + 1$. Then imputation c computed by Algorithm 27 is in the $\frac{2k}{2k+1}$-approximate core of the matching game for G. Prove this statement.

8. Formalize the model in Section 4.2 and the proof of Theorem 32.

7 Notes

Theorems 4 and 16 are from the classic work of Shapley and Shubik [SS71] on the core of the assignment game. Theorems 9 and 28 are due to Vazirani, and are taken from [Vaz22a] and [Vaz22b], respectively.

Section 4.1 is due to Kelso and Crawford [KC82], while the model of matching with contracts was developed by Hatfield and Milgrom [HM05], in part as a generalization of matching markets with salaries. Under their assumptions, however, the two models turn out to be equivalent: see Echenique [Ech12] for a discussion. An early model of matching with contracts was due to Roth [Rot84].

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