Computational Complexity of the Hylland-Zeckhauser Scheme for One-Sided Matching Markets

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Abstract

In 1979, Hylland and Zeckhauser [HZ79] gave a simple and general scheme for implementing a one-sided matching market using the power of a pricing mechanism. Their method has nice properties – it is incentive compatible in the large and produces an allocation that is Pareto optimal – and hence it provides an attractive, off-the-shelf method for running an application involving such a market. With matching markets becoming ever more prevalent and impactful, it is imperative to finally settle the computational complexity of this scheme.

We present the following partial resolution:

1. A combinatorial, strongly polynomial time algorithm for the dichotomous case, i.e., 0/1 utilities, and more generally, when each agent's utilities come from a bi-valued set.
2. An example that has only irrational equilibria, hence proving that this problem is not in PPAD.
3. A proof of membership of the problem in the class FIXP.
4. A proof of membership of the problem of computing an approximate HZ equilibrium in the class PPAD.

We leave open the (difficult) questions of determining if computing an exact HZ equilibrium is FIXP-hard and an approximate HZ equilibrium is PPAD-hard.

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1 Introduction

In a brilliant and by-now classic paper, Hylland and Zeckhauser [HZ79] gave a simple and general scheme for implementing a one-sided matching market using the power of a pricing mechanism[1]. Their method produces an allocation that is Pareto optimal, envy-free [HZ79] and is incentive compatible in the large [HMPY18]. The Hylland-Zeckhauser (HZ) scheme can be viewed as a marriage between fractional perfect matching and a linear Fisher market, both of which admit not only polynomial time algorithms but also combinatorial ones. These facts have enticed numerous researchers over the years to seek an efficient algorithm for the HZ scheme. The significance of this problem has only grown in recent years, with ever more diverse and impactful matching markets being launched into our economy, e.g., see [ftToC19].

Our work on resolving this problem started with an encouraging sign, when we obtained a combinatorial, strongly polynomial time algorithm for the *dichotomous case*, in which all utilities are 0/1, by melding a prefect matching algorithm with the combinatorial algorithm of [DPSV08] for the linear Fisher market, see Section 4. This algorithm can be extended to solve a more general problem which we call the *bi-valued utilities case*, in which each agent's utilities can take one of only two values, though the two values can be different for different agents. However, this approach did not extend any further, as described in the next section.

One-sided matching markets can be classified along two dimensions: whether the utility functions are cardinal or ordinal, and whether agents have initial endowments or not. Under this classification, the HZ scheme is (cardinal, no endowments). Section 1.2 gives mechanisms for the remaining three possibilities as well as their game-theoretic properties. Ordinal and cardinal utility functions have their individual pros and cons, and neither dominates the other. Whereas the former are easier to elicit, the latter are far more expressive, enabling an agent to not only report if she prefers one good to another but also by how much, thereby producing higher quality allocations as illustrated in Example 1, which is taken from [GTV20].

Example 1. ([GTV20]) The instance has three types of goods, T_1, T_2, T_3, and these goods are present in the proportion of (1%, 97%, 2%). Based on their utility functions, the agents are partitioned into two sets A_1 and A_2, where A_1 constitute 1% of the agents and A_2, 99%. The utility functions of agents in A_1 and A_2 for the three types of goods are $(1,\ \epsilon,\ 0)$ and $(1,\ 1-\epsilon,\ 0)$, respectively, for a small number $\epsilon > 0$. The main point is that whereas agents in A_2 marginally prefer T_1 to T_2, those in A_1 overwhelmingly prefer T_1 to T_2. Clearly, the ordinal utilities of all agents in $A_1 \cup A_2$ are the same. Therefore, a mechanism based on such utilities will not be able to make a distinction between the two types of agents. On the other hand, the HZ mechanism, which uses cardinal utilities, will fix the price of goods in T_3 to be zero and those in T_1 and T_2 appropriately so that by-and-large the bundles of A_1 and A_2 consist of goods from T_1 and T_2, respectively.

While studying the dichotomous case of two-sided markets, Bogomolnaia and Moulin [BM04] called it an "important special case of the bilateral matching problem." Using the Gallai-Edmonds decomposition of a bipartite graph, they gave a mechanism that is Pareto optimal and group strategyproof. They also gave a number of applications of their setting, some of which are natural

[1]See Remark 5 for a discussion of the advantages of this mechanism.

applications of one-sided markets as well, e.g., housemates distributing rooms, having different features, in a house. Furthermore, they say, "Time sharing is the simplest way to deal fairly with indivisibilities of matching markets: think of a set of workers sharing their time among a set of employers." It turns out that the HZ (fractional) equilibrium allocation is a superior starting point for the problem of designing a randomized time-sharing mechanism; this is discussed in Remark 5 after introducing the HZ model. Roth, Sonmez and Unver [RSÜ05] extended these results to general graph matching under dichotomous utilities; this setting is applicable to the kidney exchange marketplace.

1.1 The gamut of possibilities

The most useful solution for practical applications would of course have been a combinatorial, polynomial time algorithm for the entire scheme. At the outset, this didn't seem unlikely, especially in view of the existence of such an algorithm for the dichotomous case. Next we considered the generalization of the bi-valued utilities case to tri-valued utilities, in particular, to the case of $\{0, \frac{1}{2}, 1\}$ utilities. However, even this case appears to be intractable and its status is discussed in Section 9.

Underlying the polynomial time solvability of a linear Fisher market is the property of weak gross substitutability[2]. We note that this property is destroyed as soon as one goes to a slightly more general utility function, namely piecewise-linear, concave and separable over goods (SPLC utilities), and this case is PPAD-complete[3] [VY11]; the class PPAD was introduced in [Pap94]. Since equilibrium allocations for the HZ scheme do not satisfy weak gross substitutability, e.g., see Example 10, we were led us to seek a proof of PPAD-completeness.

A crucial requirement for membership in PPAD is to show that there is always a rational equilibrium if all parameters of the instance are rational numbers. However, even this is not true; we found an example which admits only irrational equilibria, see Section 6. This example consists of four agents and goods, and hence can be viewed as belonging to the four-valued utilities case; see Remark 21 for other intriguing aspects of this example.

The irrationality of solutions suggests that the appropriate class for this problem is the class FIXP, introduced in [EY10]. The proof in [HZ79], showing the existence of an equilibrium, uses Kakutani's theorem and does not seem to lend itself in any easy way to showing membership in FIXP. For this purpose, we give a new proof of the existence of equilibrium. Our proof defines a suitable Brouwer function which adjusts prices and allocations in case they are not an equilibrium. It uses elementary arithmetic operations that improve their optimality or feasibility of the current prices and allocations. The adjustment scheme is such that the only stable prices and allocations are forced to be equilibria. The proof of the FIXP membership is presented in Section 7.

Next, we define the notion of an approximate equilibrium. This is still required to be a fractional perfect matching on agents and goods; agents' allocations are allowed to be slightly suboptimal and/or their cost is allowed to slightly exceed the budget of 1 dollar. We show in Section 8 that

[2]Namely, if you increase the price of one good, the demand of another good cannot decrease.
[3]Independently, PPAD-hardness was also established in [CT09].

the problem of computing such an approximate equilibrium is in PPAD. This involves relating approximate equilibria to the approximate fixed points of the Brouwer function we defined for our proof of membership in FIXP.

We leave open the questions of determining if the computation (to desired accuracy) of an exact HZ equilibrium is FIXP-hard, and if the computation of an approximate HZ equilibrium is PPAD-hard. We discuss briefly in Section 9 some of the obstacles in this regard and differences with other models of markets with hard equilibrium problems.

1.2 Related work

We first present mechanisms for the remaining three possibilities for the classification of one-sided matching market mechanisms given in the Introduction. The famous Top Trading Cycles mechanism is (ordinal, endowments) [SS74]; it is efficient, strategyproof and core-stable. Under (ordinal, no endowments) are Random Priority [Mou18], which is strategyproof though not efficient or envy-free, and Probabilistic Serial [BM01], which is efficient and envy-free but not strategyproof. Under (cardinal, endowments) is ϵ-Approximate ADHZ (for Arrow-Debreu Hylland-Zeckhauser) scheme [GTV20], which satisfies Pareto optimality, approximate envy-freeness and incentive compatibility in the large.

We are aware of only the following two computational results on the HZ scheme. Using the algebraic cell decomposition technique of [BPR95], [DK08] gave a polynomial time algorithm for computing an equilibrium for an Arrow-Debreu market under piecewise-linear, concave (PLC) utilities (not necessarily separable over goods) if the number of goods is fixed. One can see that their algorithm can be adapted to yield a polynomial time algorithm for computing an equilibrium for the HZ scheme if the number of goods is a fixed constant. Extending these methods, [AJKT17] gave a polynomial time algorithm for the case that the number of agents is a fixed constant.

There are several results establishing membership and hardness in PPAD and FIXP for equilibria computation problems in different settings. The quintessential complete problem for PPAD is 2-Nash [DGP09, CDDT09] and that for FIXP is multiplayer Nash equilibrium [EY10]. For the latter problem, computing an approximate equilibrium is PPAD-complete [DGP09].

For the case of market equilibria, in the economics literature, there are two parallel streams of results: one assumes that an excess demand function is given and the other assumes a specific class of utility functions. [EY10] proved FIXP-completeness of Arrow-Debreu markets whose excess demand functions are algebraic. This result is for the first stream and it does not establish FIXP-completeness of Arrow-Debreu markets under any specific class of utility functions. Results for the second stream include proofs of membership in FIXP for Arrow-Debreu markets under Leontief and piecewise-linear concave (PLC) utility functions in [Yan13] and [GMV16], respectively. This was followed by a proof of FIXP-hardness for Arrow-Debreu markets with Leontief and PLC utilities [GMVY17]. For the case of Arrow-Debreu markets with CES (constant elasticity of substitution) utility functions, [CPY17] show membership in FIXP but leave open FIXP-hardness.

For the CES market problem stated above, computing an approximate equilibrium is PPAD-complete, and the same holds more generally for a large class of 'non-monotonic' markets [CPY17]. Computing an (exact or approximate) equilibrium under separable, piecewise-linear, concave (SPLC) utilities for Arrow-Debreu and Fisher markets is also known to be PPAD-complete [CDT09, CT09, VY11].

In recent years, several researchers have proposed Hylland-Zeckhauser-type mechanisms for a number of applications, e.g., see [Bud11, HMPY18, Le17, McL18]. The basic scheme has also been generalized in several different directions, including two-sided matching markets, adding quantitative constraints, and to the setting in which agents have initial endowments of goods instead of money, see [EMZ19a, EMZ19b].

2 The Hylland-Zeckhauser Scheme

Hylland and Zeckhauser [HZ79] gave a general mechanism for a one-sided matching market using the power of a pricing mechanism. Their formulation is as follows: Let $A = \{1, 2, \ldots n\}$ be a set of n agents and $G = \{1, 2, \ldots, n\}$ be a set of n indivisible goods. The mechanism will allocate exactly one good to each agent and will have the following two properties:

- The allocation produced is Pareto optimal.
- The mechanism is incentive compatible in the large.

The Hylland-Zeckhauser scheme is a marriage between linear Fisher market and fractional perfect matching. The agents will reveal to the mechanism their desires for the goods by stating their von Neumann-Mogenstern utilities. Let u_{ij} represent the utility of agent i for good j. We will use language from the study of market equilibria to describe the rest of the formulation. For this purpose, we next define the linear Fisher market model.

A *linear Fisher market* consists of a set $A = \{1, 2, \ldots n\}$ of n agents and a set $G = \{1, 2, \ldots, m\}$ of m infinitely divisible goods. By fixing the units for each good, we may assume without loss of generality that there is a unit of each good in the market. Each agent i has money m_i and utility u_{ij} for a unit of good j. If x_{ij}, $1 \le j \le m$ is the *bundle of goods allocated to i*, then the utility accrued by i is $\sum_j u_{ij} x_{ij}$. Each good j is assigned a non-negative price, p_j. Allocations and prices, x and p, are said to form an *equilibrium* if each agent obtains a utility maximizing bundle of goods at prices p and the *market clears*, i.e., each good is fully sold and all money of agents is fully spent.

In order to mold the one-sided market into a linear Fisher market, the HZ scheme renders goods divisible by assuming that there is one unit of probability share of each good. An *allocation* to an agent is a collection of probability shares over the goods. Let x_{ij} be the probability share that agent i receives of good j. Then, $\sum_j u_{ij} x_{ij}$ is the *expected utility* accrued by agent i. Each good j has price $p_j \ge 0$ in this market and each agent has 1 dollar with which it buys probability shares. The entire allocation must form a *fractional perfect matching in the complete bipartite graph* over vertex sets A and G as follows: there is one unit of probability share of each good and the total probability share assigned to each agent also needs to be one unit. Subject to these constraints, each agent should buy a utility maximizing bundle of goods *having the smallest possible cost*. Note that the last condition is not required in the definition of a linear Fisher market equilibrium. It is needed here to guarantee that the allocation obtained is Pareto optimal, see Section 2.1 for an illustrative example. A second departure from the linear Fisher market equilibrium is that in the latter, each agent i must spend her money m_i fully; in the HZ scheme, i need not spend her entire dollar. Since the allocation is required to form a fractional perfect matching, all goods are fully sold. We will define these to be *equilibrium allocation and prices*; we state this formally below after giving some preliminary definitions.

Definition 2. Let x and p denote arbitrary (non-negative) allocations and prices of goods. By *size, cost and value* of agent i's bundle we mean

$$\sum_{j \in G} x_{ij}, \quad \sum_{j \in G} p_j x_{ij} \quad \text{and} \quad \sum_{j \in G} u_{ij} x_{ij},$$

respectively. We will denote these by $\text{size}(i)$, $\text{cost}(i)$ and $\text{value}(i)$, respectively.

Definition 3. (Hylland and Zeckhauser [HZ79]) Allocations and prices (x, p) form an *equilibrium* for the one-sided matching market stated above if:

1. The total probability share of each good j is 1 unit, i.e., $\sum_i x_{ij} = 1$.
2. The size of each agent i's allocation is 1, i.e., $\text{size}(i) = 1$.
3. The cost of the bundle of each agent is at most 1.
4. Subject to constraints 2 and 3, each agent i maximizes her expected utility at minimum possible cost, i.e., maximize $\text{value}(i)$, subject to $\text{size}(i) = 1$, $\text{cost}(i) \leq 1$, and lastly, $\text{cost}(i)$ is smallest among all utility-maximizing bundles of i.

Using Kakutani's fixed point theorem, the following is shown:

Theorem 4. *[Hylland and Zeckhauser [HZ79]] Every instance of the one-sided market defined above admits an equilibrium; moreover, the corresponding allocation is Pareto optimal.*

Finally, if this "market" is large enough, no individual agent will be able to improve her allocation by misreporting utilities nor will she be able to manipulate prices. For this reason, the HZ scheme is incentive compatible in the large.

As stated above, Hylland and Zeckhauser view each agent's allocation as a lottery over goods. In this viewpoint, agents accrue utility in an *expected sense* from their allocations. Once these lotteries are resolved in a manner faithful to the probabilities, an assignment of indivisible goods will result. The latter can be done using the well-known Theorem of Birkhoff [Bir46] and von Neumann [VN53] which states that any doubly stochastic matrix can be written as a convex combination of permutation matrices, i.e., perfect matchings; moreover, this decomposition can be obtained efficiently. Next, pick one of these perfect matchings from the discrete distribution given by coefficients in the convex combination. As is well known, since the lottery over goods is Pareto optimal *ex ante*, the integral allocation, viewed stochastically, will also be Pareto optimal *ex post*.

Another viewpoint, forwarded by Bogomolnaia and Moulin [BM04], considers the fractional perfect matching, or equivalently the doubly-stochastic matrix, as the output of the mechanism, i.e., without resorting to randomized rounding. This viewpoint assumes that the agents are going to ”time-share” the goods or resources and the doubly-stochastic matrix, which is derived from a market mechanism, provides a “fair” way of doing so.

Remark 5. In their paper studying the dichotomous case of two-sided matching markets, Bogomolnaia and Moulin [BM04] state that the preferred way of dealing with indivisibilities inherent in matching markets is to resort to time sharing using randomization. Their method builds on the Gallai-Edmonds decomposition of the underlying bipartite graph; this classifies vertices into three categories: disposable, over-demanded and perfectly matched. This is a much more coarse insight into the demand structure of vertices than that obtained via the HZ equilibrium. The latter is the output of a market mechanism in which equilibrium prices reflect the relative importance of goods in an accurate and precise manner, based on the utilities declared by buyers, and equilibrium allocations are as equitable as possible across buyers. Hence the latter yields a more fair and desirable randomized time-sharing mechanism.

2.1 The importance of minimizing cost of bundles

Example 6. Our example has 3 agents A_1, A_2, A_3 and 3 goods g_1, g_2, g_3. The agents' utilities for the goods are given in Table 1.

Table 1: Agents' utilities.

	g_1	g_2	g_3
A_1	0	1	1
A_2	1	$\frac{1}{2}$	0
A_3	1	0	$\frac{1}{2}$

If the condition of minimizing the cost of agents' bundles is removed, the instance defined in Example 6 admits the following three solutions – the proof is quite straightforward.

Solution 1: The prices are $p_1 = 2, p_2 = 0, p_3 = 1$ and allocations as in Table 2.

Solution 2: The prices are $p_1 = 2, p_2 = 1, p_3 = 0$ and allocations as in Table 3.

Solution 3: The prices are $p_1 = 2, p_2 = 0, p_3 = 0$ and allocations as in Table 4.

Observe that Solution 3 Pareto dominates the other two. Also observe that in each of the first two solutions, agent A_1 buys a utility maximizing bundle which is not the cheapest. Therefore, Solution 3 is the only HZ equilibrium for this example.

Table 2: Allocations under Equilibrium 1.

	g_1	g_2	g_3
A_1	0	0	1
A_2	$\frac{1}{2}$	$\frac{1}{2}$	0
A_3	$\frac{1}{2}$	$\frac{1}{2}$	0

Table 3: Allocations under Equilibrium 2.

	g_1	g_2	g_3
A_1	0	1	0
A_2	$\frac{1}{2}$	0	$\frac{1}{2}$
A_3	$\frac{1}{2}$	0	$\frac{1}{2}$

Table 4: Allocations under Equilibrium 3.

	g_1	g_2	g_3
A_1	0	$\frac{1}{2}$	$\frac{1}{2}$
A_2	$\frac{1}{2}$	$\frac{1}{2}$	0
A_3	$\frac{1}{2}$	0	$\frac{1}{2}$

3 Properties of Optimal Allocations and Prices

Let p be given prices which are not necessarily equilibrium prices. An optimal bundle for agent i, x_i, is a solution to the following LP, which has two constraints, one for size and one for cost.

$$\begin{aligned} \max \quad & \sum_j x_{ij} u_{ij} && (1) \\ \text{s.t.} \quad & && (2) \\ & \sum_j x_{ij} = 1 && (3) \\ & \sum_j x_{ij} p_j \leq 1 && (4) \\ \forall j \quad & x_{ij} \geq 0 && (5) \end{aligned}$$

Taking μ_i and α_i to be the dual variables corresponding to the two constraints, we get the dual LP:

$$\begin{aligned} \min \quad & \alpha_i + \mu_i && (6) \\ \text{s.t.} \quad & && (7) \\ \forall i, j \quad & \alpha_i p_j + \mu_i \geq u_{ij} && (8) \\ & \alpha_i \geq 0 && (9) \end{aligned}$$

Clearly μ_i is unconstrained. μ_i will be called the *offset* on i's utilities. By complementary slackness, if x_{ij} is positive then $\alpha_i p_j = u_{ij} - \mu_i$. All goods j satisfying this equality will be called *optimal goods for agent i*. The rest of the goods, called *suboptimal*, will satisfy $\alpha_i p_j > u_{ij} - \mu_i$. Obviously an optimal bundle for i must contain only optimal goods.

The parameter μ_i plays a crucial role in ensuring that i's optimal bundle satisfies both size and cost constraints. If a single good is an effective way of satisfying both size and cost constraints, then μ_i plays no role and can be set to zero. However, if different goods are better from the viewpoint of size and cost, then μ_i attains the right value so they both become optimal and i buys an appropriate combination. We provide an example below to illustrate this.

Example 7. Suppose i has positive utilities for only two goods, j and k, with $u_{ij} = 10$, $u_{ik} = 2$ and their prices are $p_j = 2$, $p_k = 0.1$. Clearly, neither good satisfies both size and cost constraints optimally: good j is better for the size constraint and k is better for the cost constraint. If i buys one unit of good j, she spends 2 dollars, thus exceeding her budget. On the other hand, she can afford to buy 10 units of k, giving her utility of 20; however, she has far exceeded the size constraint. It turns out that her optimal bundle consists of 9/19 units of j and 10/19 units of k; the costs of these two goods being 18/19 and 1/19 dollars, respectively. Clearly, her size and cost constraints are both met exactly. Her total utility from this bundle is 110/19. It is easy to see that $\alpha_i = 80/19$ and $\mu_i = 30/19$, and for these settings of the parameters, both goods are optimal.

We next show that equilibrium prices are invariant under the operation of *scaling* the difference of prices from 1.

Lemma 8. *Let p be an equilibrium price vector and fix any $r > 0$. Let p' be such that $\forall j \in G$, $p'_j - 1 = r(p_j - 1)$. Then p' is also an equilibrium price vector.*

Proof. Consider an agent i. Clearly, $\sum_{j \in G} p_j x_{ij} \leq 1$. Now,

$$\sum_{j \in G} p'_j x_{ij} \;=\; \sum_{j \in G} (rp_j - r + 1) x_{ij} \leq 1,$$

where the last inequality follows by using $\sum_{j \in G} x_{ij} = 1$. □

Using Lemma 8, it is easy to see that if the allocation x provides optimal bundles to all agents under prices p then it also does so under p'. In the rest of this paper we will enforce that the minimum price of a good is zero, thereby fixing the scale. Observe that the main goal of the Hylland-Zeckhauser scheme is to yield the "correct" allocations to agents; the prices are simply a vehicle in the market mechanism to achieve this. Hence arbitrarily fixing the scale does not change the essential nature of the problem. Moreover, setting the minimum price to zero is standard [HZ79] and can lead to simplifying the equilibrium computation problem as shown in Remark 9.

Remark 9. We remark that on the one hand, the offset μ_i is a key enabler in construing optimal bundles, on the other, it is also a main source of difficulty in computing equilibria for the HZ scheme. We identify here an interesting case in which $\mu_i = 0$ and this difficulty is mitigated. In particular, this holds for all agents in the dichotomous case presented in Section 4. Suppose good j is optimal for agent i, $u_{ij} = 0$ and $p_j = 0$, then it is easy to check that $\mu_i = 0$. If so, the optimal goods for i are simply the maximum bang-per-buck goods; the latter notion is replete in market equilibrium papers, e.g., see [DPSV08].

Finally, we extend Example 7 to illustrate that optimal allocations for the Hylland-Zeckhauser model do not satisfy the weak gross substitutes condition in general.

Example 10. In Example 7, let us raise the price of k to 0.2 dollars. Then the optimal allocation for i changes to $4/9$ units of j and $5/9$ units of k. Notice that the demand for j went down from $9/19$ to $4/9$. One way to understand this change is as follows: Let us start with the old allocation of $10/19$ units of k. Clearly, the cost of this allocation of k went up from $1/19$ to $2/19$, leaving only $17/19$ dollars for j. Therefore size of j needs to be reduced to $17/38$. However, now the sum of the sizes becomes $37/38$, i.e., less than a unit. We wish to increase this to a unit while still keeping cost at a unit. The only way of doing this is to sell some of the more expensive good and use the money to buy the cheaper good. This is the reason for the decrease in demand of j.

4 Strongly Polynomial Algorithm for Bi-Valued Utilities

In this section, we will study the restriction of the HZ scheme to the bi-valued utilities case, which is defined as follows: for each agent i, we are given a set $\{a_i, b_i\}$, where $0 \leq a_i < b_i$, and the utilities u_{ij}, $\forall j \in G$, are picked from this set. However first, using a perfect matching algorithm and the combinatorial algorithm [DPSV08] for linear Fisher markets, we will give a strongly polynomial time algorithm for the dichotomous case, i.e., when all utilities u_{ij} are 0/1. Next we define the notion of equivalence of utility functions and show that the bi-valued utilities case is equivalent to the dichotomous case, thereby extending the dichotomous case algorithm to this case.

We need to clarify that we will not use the main algorithm from [DPSV08], which uses the notion of balanced flows and l_2 norm to achieve polynomial running time. Instead, we will use the "simple algorithm" presented in Section 5 in [DPSV08]. Although this algorithm is not proven to be efficient, the simplified version we define below, called Simplified DPSV Algorithm, is efficient; in fact it runs in strongly polynomial time, unlike the balanced-flows-based algorithm of [DPSV08]. Remark 9 provides an insight into what makes the dichotomous case computationally easier.

We note that recently, [GTV20] gave a *rational convex program (RCP)* for the dichotomous case of HZ. An RCP, defined in [Vaz12], is a nonlinear convex program all of whose parameters are rational numbers and which always admits a rational solution in which the denominators are polynomially bounded. An RCP can be solved exactly in polynomial time using the ellipsoid algorithm and diophantine approximation [GLS12, Jai07], and therefore directly implies the existence of a polynomial time algorithm for the underlying problem.

Notation: We will denote by $H = (A, G, E)$ be the bipartite graph on vertex sets A and G, and edge set E, with $(i, j) \in E$ iff $u_{ij} = 1$. For $A' \subseteq A$ and $G' \subseteq G$, we will denote by $H[A', G']$ the restriction of H to vertex set $A' \cup G'$. If ν is a matching in H, $\nu \subseteq E$, and $(i, j) \in \nu$ then we will say that $\nu(i) = j$ and $\nu(j) = i$. For any subset $S \subseteq A$ ($S \subseteq G$), $N(S)$ will denote the set of neighbors, in G (A), of vertices in S.

If H has a perfect matching, the matter is straightforward as stated in Steps 1a and 1b; allocations and prices are clearly in equilibrium. For Step 2, we need the following lemma.

Algorithm 12. Algorithm for the Dichotomous Case

1. If H has a perfect matching, say ν, then do:
   (a) $\forall i \in A$: allocate good $\nu(i)$ to i.
   (b) $\forall j \in G$: $p_j \leftarrow 0$. Go to Step 3.
2. Else do:
   (a) Find a minimum vertex cover in H, say $G_1 \cup A_2$, where $G_1 \subset G$ and $A_2 \subset A$. Let $A_1 = A - A_2$ and $G_2 = G - G_1$.
   (b) Find a maximum matching in $H[A_2, G_2]$, say ν.
   (c) $\forall i \in A_2$: allocate good $\nu(i)$ to i.
   (d) $\forall j \in G_2$: $p_j \leftarrow 0$.
   (e) Run the Simplified DPSV Algorithm on agents A_1 and goods G_1.
   (f) $\forall i \in A_1$: Allocate unmatched goods of G_2 to satisfy the size constraint.
3. Output the allocations and prices computed and Halt.

Lemma 11. *The following hold:*

1. *For any set $S \subseteq A_2$, $|N(S)| \geq |S|$.*
2. *For any set $S \subseteq G_1$, $|N(S) \cap A_1| \geq |S|$.*

Proof. 1). If $|N(S)| < |S|$ then $(G_1 \cup N(S)) \cup (A_2 - S)$ is a smaller vertex cover for H, leading to a contradiction.

2). If $|N(S) \cap A_1| < |S|$ then $(G_1 - S) \cup (A_2 \cup N(S))$ is a smaller vertex cover for H, leading to a contradiction. □

The first part of Lemma 11 together with Hall's Theorem implies that a maximum matching in $H[A_2, G_2]$ must match all agents. Therefore in Step 2a, each agent $i \in A_2$ is allocated one unit of a unique good from which it derives utility 1 and having price zero; clearly, this is an optimal bundle of minimum cost for i. The number of goods that will remain unmatched in G_2 at the end of this step is $|G_2| - |A_2|$.

Allocations are computed for agents in A_1 as follows. First, Step 2e uses the Simplified DPSV Algorithm, which we describe below, to compute equilibrium allocations and prices for the sub-market consisting of agents in A_1 and goods in G_1. At the end of this step, the money of each agent in A_1 is exhausted; however, her allocation may not meet the size constraint. To achieve the latter, Step 2f allocates the unmatched zero-priced goods from G_2 to agents in A_1. Clearly, the total deficit in size is $|A_1| - |G_1|$. Since this equals $|G_2| - |A_2|$, the market clears at the end of Step 2f. As shown in Lemma 13, each agent in A_1 also gets an optimal bundle of goods of minimum cost.

Let p be the prices of goods in G_1 at any point in this algorithm. As a consequence of the second part of Lemma 11, the equilibrium price of each good in G_1 will be at least 1. The Simplified

DPSV algorithm will initialize prices of goods in G_1 to 1 and declare all goods active. The algorithm will always raise prices of active goods uniformly[4].

For $S \subseteq G_1$ let $p(S)$ denote the sum of the equilibrium prices of goods in S. A key notion from [DPSV08] is that of a tight set; set $S \subseteq G_1$ is said to be *tight* if $p(S) = |N(S)|$, the latter being the total money of agents in A_1 who are interested in goods in S. If set S is tight, then the local market consisting of goods in S and agents in $N(S)$ clears. To see this, one needs to use the flow-based procedure given in [DPSV08] to show that each agent $i \in N(S)$ can be allocated 1 dollar worth of those goods in S from which it accrues unit utility. Thus equilibrium has been reached for goods in S.

As the algorithm raises prices of all goods in G_1, at some point a set S will go tight. The algorithm then *freezes* the prices of its goods and removes them from the active set. It then proceeds to raise the prices of currently active goods until another set goes tight, and so on, until all goods in G_1 are frozen.

We can now explain in what sense we need a "simplified" version of the DPSV algorithm. Assume that at some point, $S \subset G_1$ is frozen and goods in $G_1 - S$ are active and their prices are raised. As this happens, agents in $A_1 - N(S)$ start preferring goods in S relative to those in $G_1 - S$. In the general case, at some point, an agent $i \in (A_1 - N(S))$ will prefer a good $j \in S$ as much as her other preferred goods. At this point, edge (i, j) is added to the active graph. As a result, some set $S' \subseteq S$, containing j, will not be tight anymore and will be unfrozen. However, in our setting, the utilities of agents in $(A_1 - N(S))$ for goods in S is zero, and therefore no new edges are introduced and tight sets never become unfrozen. Hence the only events of the Simplified DPSV Algorithm are raising of prices and freezing of sets. Clearly, there will be at most n freezings. One can check details in [DPSV08] to see that the steps executed with each freezing run in strongly polynomial time, hence making the Simplified DPSV Algorithm a strongly polynomial time algorithm[5].

Lemma 13. *Each agent in A_1 will get an optimal bundle of goods of minimum cost.*

Proof. First note that for agents in A_1, there are no utility 1 goods in G_2 – this follows from the fact that no vertices from $A_1 \cup G_2$ are in the vertex cover picked. Therefore, for $i \in A_1$, an optimum bundle consists of the cheapest way of obtaining one dollar worth of goods from $N\{i\}$, which are in G_1, together with the right amount of zero-priced goods from G_2 to satisfy the size constraint.

Assume that the algorithm freezes k sets, $S_1, \ldots S_k$, in that order; the union of these sets being G_1. Let $p_1, p_2, \ldots p_k$ be the prices of goods in these sets, respectively. Clearly, successive freezings will be at higher and higher prices and therefore, $1 \leq p_1 < p_2 < \ldots < p_k$, and for $1 \leq j \leq k$, $p_j = |N(S_j)|/|S_j|$. If $i \in N(S_j)$, the algorithm will allocate $1/p_j$ amount of goods to i from S_j, costing 1 dollar.

By definition of neighborhood of sets, if $i \in N(S_j)$, then i cannot have edges to $S_1, \ldots S_{j-1}$ and

[4]In [DPSV08], prices of active goods are raised multiplicatively, which amounts to raising prices of active goods uniformly for our simplified setting.

[5]In contrast, in the general case, the number of freezings is not known to be bounded by a polynomial in n, as stated in [DPSV08].

can have edges to $S_{j+1}, \ldots, S_k$. Therefore, the cheapest goods from which it accrues unit utility are in S_j, the set from which she gets 1 dollar worth of allocation. The rest of the allocation of i, in order to meet i's size constraint, will be from G_2, which are zero-priced and from which i gets zero utility. Clearly, i gets an optimal bundle of minimum cost. □

Since all steps of the algorithm, namely finding a maximum matching, a minimum vertex cover and running the Simplified DPSV Algorithm, can be executed in strongly polynomial time, we get:

Lemma 14. *Algorithm 12 finds equilibrium prices and allocations for the dichotomous case of the Hylland-Zeckhauser scheme. It runs in strongly polynomial time.*

Definition 15. Let I be an instance of the HZ scheme and let the utility function of agent i be $u_i = \{u_{i1}, u_{i12}, \ldots, u_{in}\}$. Then $u'_i = \{u'_{i1}, u'_{i12}, \ldots, u'_{in}\}$ is *equivalent* to u_i if there are two numbers $s > 0$ and $h \geq 0$ such that for $1 \leq j \leq n$, $u'_{ij} = s \cdot u_{ij} + h$. The numbers s and h will be called the *scaling factor* and *shift*, respectively.

Lemma 16. *Let I be an instance of the HZ scheme and let the utility function of agent i be u_i. Let u'_i be equivalent to u_i and let I' be the instance obtained by replacing u_i by u'_i in I. Then x and p are equilibrium allocation and prices for I if and only if they are also for I'.*

Proof. Let s and h be the scaling factor and shift that transform u_i to u'_i. By the statement of the lemma, $x_i = \{x_{i1}, \ldots, x_{in}\}$ is an optimal bundle for i at prices p and hence is a solution to the optimal bundle LP (1). The objective function of this LP is

$$\sum_{j=1}^{n} u_{ij} x_{ij}.$$

Next observe that the objective function of the corresponding LP for i under instance I' is

$$\sum_{j=1}^{n} u'_{ij} x_{ij} \;=\; \sum_{j=1}^{n} (s \cdot u_{ij} + h) x_{ij} \;=\; h + s \cdot \sum_{j=1}^{n} u_{ij} x_{ij},$$

where the last equality follows from the fact that $\sum_{j=1}^{n} x_{ij} = 1$. Therefore, the objective function of the second LP is obtained from the first by scaling and shifting. Furthermore, since the constraints of the two LPs are identical, the optimal solutions of the two LPs are the same. Finally, for each $i \in A$: the bundle under allocation x is a minimum cost optimal bundle for I if and only if it is also for I'. The lemma follows. □

Next, let u_i be bi-valued with the two values being $0 \leq a < b$. Obtain u'_i from u_i by replacing a by 0 and b by 1. Then, u'_i is equivalent to u_i, with the shift and scaling being a and $b - a$, respectively. Therefore the bi-valued instance can be transformed to a unit instance, with both having the same equilibria. Now using Lemma 14 we get:

Theorem 17. *There is a strongly polynomial time algorithm for the bi-valued utilities case of the Hylland-Zeckhauser scheme.*

5 Characterizing Optimal Bundles

In this section we give a characterization of optimal bundles for an agent at given prices p which are not necessarily equilibrium prices. This characterization will be used critically in Section 7, 8 and to some extent in Section 6.

Notation: For each agent i, let $G_i^* \subseteq G$ denote the set of goods from which i derives maximum utility, i.e., $G_i^* = \arg\max_{j \in G}\{u_{ij}\}$. With respect to an allocation x, let $B_i = \{j \in G \mid x_{ij} > 0\}$, i.e., the set of goods in i's bundle.

We identify the following four types of optimal bundles.

Type A bundles: $\alpha_i = 0$ and $\text{cost}(i) < 1$.

By complementary slackness, optimal goods will satisfy $u_{ij} = \mu_i$ and suboptimal goods will satisfy $u_{ij} < \mu_i$. Hence the set of optimal goods is G_i^* and $B_i \subseteq G_i^*$. Note that the prices of goods in B_i can be arbitrary, as long as $\text{cost}(i) < 1$.

Type B bundles: $\alpha_i = 0$ and $\text{cost}(i) = 1$.

The only difference from the previous type is that $\text{cost}(i)$ is exactly 1. The reason for distinguishing the two types will become clear in Section 7.

Type C bundles: $\alpha_i > 0$ and all optimal goods for i have the same utility.

Recall that good j is optimal for i if[6] $\alpha_i p_j = u_{ij} - \mu_i$. Suppose goods j and k are both optimal. Then $u_{ij} = u_{ik}$ and $\alpha_i p_j = u_{ij} - \mu_i = u_{ik} - \mu_i = \alpha_i p_k$, i.e., $p_j = p_k$. Since $\alpha_i > 0$, by complementary slackness, $\text{cost}(i) = 1$. Further, since $\text{size}(i) = 1$, we get that each optimal good has price 1.

Type D bundles: $\alpha_i > 0$ and not all optimal goods for i have the same utility.

Suppose goods j and k are both optimal and $u_{ij} \neq u_{ik}$. Then $\alpha_i p_j = u_{ij} - \mu_i \neq u_{ik} - \mu_i = \alpha_i p_k$, i.e., $p_j \neq p_k$. Therefore optimal goods have at least two different prices. Since $\alpha_i > 0$, by complementary slackness, $\text{cost}(i) = 1$. Further, since $\text{size}(i) = 1$, there must be an optimal good with price more than 1 and an optimal good with price less than 1. Finally, if good z is suboptimal for i, then $\alpha_i p_z < u_{iz} - \mu_i$.

6 An Example Having Only Irrational Equilibria

Our example has 4 agents $A_1, \ldots, A_4$ and 4 goods $g_1, \ldots, g_4$[7]. The agents' utilities for the goods are given in Table 5, with rows corresponding to agents and columns to goods.

Thus, agents A_1 and A_2 like, to varying degrees, three goods only, g_1, g_2, g_4, while agents A_3 and A_4 like two goods each, $\{g_1, g_3\}$ and $\{g_2, g_3\}$, respectively. The precise values of the utilities are

[6] Note that under this case, optimal goods are not necessarily maximum utility goods; the latter may be suboptimal because their prices are too high.

[7] It can be shown, by analyzing relations in the bipartite graph on agents and goods with edges corresponding to non-zero allocations, that any instance with 3 agents and 3 goods and rational utilities has a rational equilibrium.

Table 5: Agents' utilities.

	g_1	g_2	g_3	g_4
A_1	2	4	0	8
A_2	2	3	0	8
A_3	2	0	5	0
A_4	0	4	5	0

not that important; the important aspects are: which goods each agent likes, the order between them, and the ratios $\frac{u_{14}-u_{12}}{u_{12}-u_{11}}$ and $\frac{u_{24}-u_{22}}{u_{22}-u_{21}}$. Notice that the latter are unequal.

We will show that this example has a unique equilibrium solution with minimum price 0. In this solution, good g_1 has price 0, and all the other goods have positive irrational values. Agents A_1, A_3 and A_4 buy the goods that they like, and A_2 buys g_1 and g_4 only.

Consider any equilibrium with minimum price 0. We will analyze its properties, and show eventually that they force specific prices and allocations.

Lemma 18. *Equilibrium prices satisfy:*

$$0 = p_1 < p_2 < 1 \text{ and } p_3,\ p_4 > 1.$$

The equilibrium bundle of each agent is of Type D and contains goods having positive utilities only.

Proof. Suppose $p_3 \leq 1$. Then agents A_3 and A_4 will demand 1 unit each of good g_3, leading to a contradiction. Similarly, if $p_4 \leq 1$ then A_1 and A_2 will demand 1 unit each of g_4. Therefore, $p_3, p_4 > 1$. Since the maximum utility goods of every agent have price > 1, all agents spend exactly 1. Therefore, the sum of the prices of the goods is 4.

Suppose $p_2 = 0 \leq p_1$. Then A_1, A_2, A_4 do not buy g_1, since they prefer g_2 and it is weakly cheaper than g_1. Therefore A_3 must buy the entire unit of g_1. Clearly A_1, A_2 do not buy g_3, since they prefer g_2. Therefore, the only agent who buys g_3 is A_4; however, she cannot afford the entire unit of g_3 since $p_3 > 1$, contradicting market clearing. Therefore $p_2 > 0$ and hence the 0-priced good is g_1 and $p_1 = 0 < p_2$. Furthermore, $p_2 + p_3 + p_4 = 4$.

Next suppose $p_2 \geq 3/4$. Then $p_4 = 4 - (p_2 + p_3) < 9/4$. For both agents A_1 and A_2, a combination of g_1 and g_4 in proportion 2:1 has a price less than 3/4 for one unit and utility 4, and is therefore preferable to g_2. Hence, A_1, A_2 will not buy any g_2, and since A_3 does not buy any g_2 either, since she prefers g_1, it follows that A_4 must buy the entire unit of g_2. This is possible only if $p_2 = 1$ and A_4 buys nothing else; in particular, she does not buy any g_3. Clearly, A_1, A_2 do not buy any g_3 since they prefer g_1. Therefore the entire unit of g_3 must be bought by A_3, which is impossible because $p_3 > 1$. Hence $p_2 < 3/4$. These facts together with $p_1 = 0 < p_2 < 1 < p_3, p_4$ imply that the agents' bundles are not Type B or C. Therefore they are all of Type D.

Finally we prove that none of the agents will buy an undesirable good (a good with utility 0). For A_1, A_2, A_3, such a good is dominated by another lower-priced good. Since $p_4 > 1$, A_4 does not buy g_4. Suppose agent A_4 buys good g_1. Since she spends 1 dollar, she must also buy g_3. Therefore we have: $\alpha_4 p_1 + \mu_4 = u_{41} = 0$. Therefore $\mu_4 = 0$. Also $\alpha_4 p_3 + \mu_4 = u_{43} = 5$; therefore

$\alpha_4 p_3 = 5$, which implies $\alpha_4 < 5$ since $p_3 > 1$. Furthermore, $\alpha_4 p_2 + \mu_4 \geq u_{42} = 4$, hence $p_2 > 4/5$, which contradicts $p_2 < 3/4$. Therefore, no agent buys any undesirable good. □

Lemma 19. *One of the agents A_1, A_2 buys all three desirable goods. If A_1 buys g_1, g_2, g_4, then A_2 buys g_1, g_4 only. If A_2 buys g_1, g_2, g_4, then A_1 buys g_2, g_4 only.*

Proof. Since all the bundles are of Type D, every bundle has at least two goods; clearly, every good is bought by at least two agents.

Suppose that every agent buys two goods and every good is bought by two agents. If so, one of A_1, A_2 must buy g_1, g_4 and the other must buy g_2, g_4. Consider the graph with the goods as nodes and an edge joining two nodes if they are bought by the same agent. This graph must be the 4-cycle g_1, g_4, g_2, g_3, g_1. Therefore for some a, $0 < a < 1$, each agent buys a units of one good and $b = 1 - a$ units of the second good and each good is sold to two agents in the amounts of a and b.

Let $r_i = |1 - p_i|$. Observe that for every edge (g_i, g_j) of the cycle, one price is < 1 and the other price is > 1, and we have $ap_i + bp_j = 1$. Therefore $ar_i - br_j = 0$, and $\frac{r_i}{r_j} = \frac{b}{a}$. Hence

$$\frac{r_1}{r_4} = \frac{r_4}{r_2} = \frac{r_2}{r_3} = \frac{r_3}{r_1},$$

which implies that all the r_i are equal. Therefore $p_1 = p_2$, contradicting the previous claim that $p_1 < p_2$. Hence at least one of A_1, A_2 will buy all three of her desirable goods.

Suppose that A_1 buys all three desirable goods g_1, g_2, g_4. Then we have $\alpha_1 p_j + \mu_1 = u_{1j}$ for $j = 1, 2, 4$. Therefore, $(p_4 - p_1)/(p_4 - p_2) = (u_{14} - u_{11})/(u_{14} - u_{12}) = 3/2$. Agent A_2 buys g_4 and at least one of g_1, g_2. Suppose she buys g_2. Then $\alpha_2 p_j + \mu_2 = u_{2j}$ for $j = 2, 4$, hence $\alpha_2(p_4 - p_2) = u_{24} - u_{22} = 5$. This implies that $\alpha_2(p_4 - p_1) > 6 = u_{24} - u_{21}$, hence $\alpha_2 p_1 + \mu_2 < u_{21}$, a contradiction. Therefore A_2 does not buy g_2 and she buys g_1 and g_4 only.

Next suppose A_2 buys all three desirable goods g_1, g_2, g_4. By a similar argument we will prove that A_1 buys only two goods. We have $\alpha_2 p_j + \mu_2 = u_{2j}$ for $j = 1, 2, 4$. Therefore, $(p_4 - p_1)/(p_4 - p_2) = (u_{24} - u_{21})/(u_{24} - u_{22}) = 6/5$. Agent A_1 buys g_4 and at least one of g_1, g_2. Suppose that she buys g_1. Then $\alpha_2 p_j + \mu_2 = u_{2j}$ for $j = 1, 4$, hence $\alpha_2(p_4 - p_1) = u_{24} - u_{21} = 6$. This implies that $\alpha_2(p_4 - p_2) > 4 = u_{14} - u_{12}$, hence $\alpha_2 p_2 + \mu_2 < u_{12}$, a contradiction. Therefore, A_1 does not buy g_1, hence she buys g_2 and g_4 only. □

Theorem 20. *The instance of Table 5 has a unique equilibrium; the allocations to agents and prices of goods, other than the zero-priced good, are all irrational. The prices are as follows:*
$p_1 = 0, \quad p_2 = (23 - \sqrt{17})/32, \quad p_3 = (9 + \sqrt{17})/8, \quad p_4 = (69 - 3\sqrt{17})/32.$

Proof. Let $r_i = |1 - p_i|$. By Lemma 18, $r_1 = 1$. We consider the two cases established in Lemma 19. We will show that in Case 1 there is a unique equilibrium, while in Case 2 there is no equilibrium.

Case 1. A_1 buys g_1, g_2, g_4, and A_2 buys g_1, g_4.

Agent A_3 spends her dollar on goods g_1, g_3 in the proportion $r_3 : r_1$, i.e., $r_3 : 1$. Therefore, $x_{31} = \frac{r_3}{1+r_3}$, $x_{33} = \frac{1}{1+r_3}$. Agent A_4 buys goods g_2, g_3 in the proportion $r_3 : r_2$. Therefore, $x_{42} = \frac{r_3}{r_2+r_3}$, $x_{43} = \frac{r_2}{r_2+r_3}$. Since only agents A_3 and A_4 buy good g_3, we have $x_{31} = 1 - x_{33} = x_{43}$, and $x_{42} = 1 - x_{43} = x_{33}$. This implies $r_3^2 = r_2$ ... (1).

Since agent A_1 buys g_1, g_2, g_4, we have, $\frac{u_{14}-u_{12}}{u_{12}-u_{11}} = \frac{p_4-p_2}{p_2-p_1}$. Therefore $r_2 + r_4 = 2(1 - r_2)$ ... (2).

The sum of the prices is equal to 4, therefore $1 + r_2 - r_3 - r_4 = 0$ ... (3)

Now we have three equations, (1), (2) and (3), in three unknowns r_2, r_3, r_4. Using (1) and (2) we can express r_2 and r_4 in terms of r_3. Letting $r_3 = y$, we have from (1), $r_2 = y^2$, and from (2), $r_4 = 2 - 3r_2 = 2 - 3y^2$. Substituting into (3), we get $4y^2 - y - 1 = 0$.

The only positive solution is $y = \frac{1+\sqrt{17}}{8}$. Therefore,

$$p_1 = 0, \quad p_2 = 1 - r_2 = 1 - y^2 = \frac{23 - \sqrt{17}}{32}, \quad p_3 = 1 + r_3 = 1 + y = \frac{9 + \sqrt{17}}{8},$$

$$p_4 = 1 + r_4 = 3 - 3y^2 = \frac{69 - 3\sqrt{17}}{32}.$$

Once we have the value of y, we get:

$$r_1 = 1, \quad r_2 = y^2 = \frac{9 + \sqrt{17}}{32}, \quad r_3 = y = \frac{1 + \sqrt{17}}{8} \quad \text{and} \quad r_4 = 2 - 3y^2 = \frac{37 - 3\sqrt{17}}{32}.$$

We can compute then the allocations from the r_i. We already expressed the allocations for agents A_3, A_4 in terms of the r_i. Agent A_2 buys goods g_1, g_4 in the proportion $r_4 : r_1$, i.e., $r_4 : 1$. Therefore, $x_{21} = \frac{r_4}{1+r_4}$, $x_{24} = \frac{1}{1+r_4}$. Agent A_1 buys the remaining amount of each good g_1, g_2, g_4. Thus, the allocations of the agents in terms of the r_i are:

$$A_1: \quad x_{11} = 1 - \frac{r_3}{1+r_3} - \frac{r_4}{1+r_4}, \quad x_{12} = \frac{r_2}{r_2+r_3}, \quad x_{14} = \frac{r_4}{1+r_4}$$

$$A_2: \quad x_{21} = \frac{r_4}{1+r_4}, \quad x_{24} = \frac{1}{1+r_4}$$

$$A_3: \quad x_{31} = \frac{r_3}{1+r_3}, \quad x_{33} = \frac{1}{1+r_3}$$

$$A_4: \quad x_{42} = \frac{r_3}{r_2+r_3}, \quad x_{43} = \frac{r_2}{r_2+r_3}$$

We conclude that, if there is an equilibrium in Case 1, then there can be only one and it must have the above prices and allocations.

Conversely, we can verify that the above pair (p, x) is an equilibrium. First we note that all allocations are nonnegative. This is obvious for all the allocations, except for x_{11}, which, after plugging in the values for the r_i's evaluates to approximately 0.084. Second, note that every good has exactly one unit allocated: for good g_3 this follows from equation (1), and for the other goods it holds because A_1 buys the remaining amounts. Third, every agent buys a total of one unit of goods: this is obvious for agents A_2, A_3, A_4 from the allocations, and for agent A_1 it follows because exactly one unit is sold of each good. Fourth, every agent spends exactly one dollar: this holds for agents A_2, A_3, A_4 because they pay an average price of 1 for their goods, and for agent A_1 it follows from the fact that the total expenditure of the agents, which is equal to the sum of the prices of the goods, is 4 (equation (3)).

Finally, it can be shown that the bundle of every agent is optimal for these prices, using the dual LP and complementary slackness. The dual variables α_i can be calculated as $\alpha_i = \frac{u_{ij}-u_{ik}}{p_j-p_k}$, where g_k, g_j are (any) two goods bought by agent A_i; the shift $\mu_i = u_{ij} - \alpha_i p_j$ (which is equal to $u_{ik} - \alpha_i p_k$). Thus, for example $\alpha_1 = \frac{u_{12}-u_{11}}{p_2-p_1} = \frac{2}{p_2}$, and $\mu_1 = u_{11} - \alpha_1 p_1 = 2$. Note that $\frac{u_{12}-u_{11}}{p_2-p_1} = \frac{u_{14}-u_{12}}{p_4-p_2} = \frac{u_{14}-u_{11}}{p_4-p_1}$ by equation (2), so it does not matter which goods g_j, g_k in Agent A_1's bundle are used to calculate α_1. Also, for each agent A_i, it does not matter which good g_j in her bundle is used to calculate μ_i.

Clearly $\alpha_i \geq 0$ for all i. For all agents A_i and goods g_j in the bundle of A_i, we have $\alpha_i p_j + \mu_i = u_{ij}$, by construction. Furthermore, if good g_j is not in the bundle of A_i then $\alpha_i p_j + \mu_i > u_{ij}$: For agent A_1 and good g_3, note that g_3 has higher price and lower utility than good g_1 which is in the bundle of A_1, hence $\alpha_1 p_3 + \mu_1 > \alpha_1 p_1 + \mu_1 = u_{11} > u_{13}$. The same argument applies to agent A_2 and good g_3, agent A_3 and goods g_2, g_4 (they are both dominated by good g_1 in A_3's bundle), and to agent A_4 and good g_4 (it is dominated by good g_2). For agent A_2 and good g_2, note that $\alpha_2 = \frac{u_{24}-u_{21}}{p_4-p_1} = \frac{u_{14}-u_{11}}{p_4-p_1} = \alpha_1$, and $\mu_2 = u_{24} - \alpha_2 p_4 = u_{14} - \alpha_1 p_4 = \mu_1$. Therefore $\alpha_2 p_2 + \mu_2 = \alpha_1 p_2 + \mu_1 = u_{12} = 4 > 3 = u_{22}$. The only remaining case that needs to be checked numerically is agent A_4 and good g_1. Since $p_1 = 0$ and $u_{41} = 0$, the inequality $\alpha_4 p_1 + \mu_4 > u_{41}$ is equivalent to $\mu_4 > 0$. By construction, $\mu_4 = u_{44} - \alpha_4 p_3 = u_{44} - \frac{u_{43}-u_{42}}{p_3-p_2} p_3 = 5 - \frac{p_3}{p_3-p_2} = \frac{4p_3-5p_2}{p_3-p_2}$. Thus, $\mu_4 > 0$ is equivalent to $4p_3 > 5p_2$, which holds for the above values of p_2, p_3. Therefore, the values α_i, μ_i satisfy the constraints of the dual LP, and since they and the x_{ij} satisfy clearly also the complementary slackness conditions, it follows that the allocations x_{ij} give optimal bundles to the agents for the prices p_j. Therefore, (x, p) is an equilibrium.

Case 2. A_2 buys g_1, g_2, g_4, and A_1 buys g_2, g_4.

We will show that there is no equilibrium in this case. Specifically, we will show that if there is an equilibrium, it must have specific prices and allocations, and we will derive a contradiction.

Consider any equilibrium for Case 2. The allocations for agents A_3, A_4 are the same as in Case 1, i.e., $x_{31} = \frac{r_3}{1+r_3}$, $x_{33} = \frac{1}{1+r_3}$, and $x_{42} = \frac{r_3}{r_2+r_3}$, $x_{43} = \frac{r_2}{r_2+r_3}$. Again we have $x_{31} = x_{43}$ and $x_{42} = x_{33}$, which implies $r_3^2 = r_2$ ... (1).

Since agent A_2 buys g_1, g_2, g_4, we have, $\frac{u_{24}-u_{22}}{u_{22}-u_{21}} = \frac{p_4-p_2}{p_2-p_1}$, therefore $r_2 + r_4 = 5(1 - r_2)$ ... (2′).

The sum of the prices is 4, thus again $1 + r_2 - r_3 - r_4 = 0$ ... (3)

We can solve now for r_2, r_3, r_4. Using (1) and (2′) we can express r_2 and r_4 in terms of r_3. Letting $r_3 = y$, we have from (1), $r_2 = y^2$, and from (2′), $r_4 = 5 - 6r_2 = 5 - 6y^2$. Substituting into (3), we get $7y^2 - y - 4 = 0$.

The only positive solution is $y = \frac{1+\sqrt{113}}{14}$. Therefore,

$$p_1 = 0, \quad p_2 = 1 - r_2 = 1 - y^2 = \frac{41 - \sqrt{113}}{98}, \quad p_3 = 1 + r_3 = 1 + y = \frac{15 + \sqrt{113}}{14},$$

$$p_4 = 1 + r_4 = 6 - 6y^2 = \frac{246 - 6\sqrt{113}}{98}.$$

As in the previous case, the value of y gives:

$$r_1 = 1, \quad r_2 = y^2 = \frac{57 + \sqrt{113}}{98}, \quad r_3 = y = \frac{1 + \sqrt{113}}{14} \quad \text{and} \quad r_4 = 5 - 6y^2 = \frac{148 - 6\sqrt{113}}{98}.$$

If there is any equilibrium in Case 2, then it must have the above prices. We can compute again the allocations from the r_i. The allocations of A_3 and A_4 are as before. Agent A_1 buys goods g_2, g_4 in the proportion $r_4 : r_2$. Therefore, $x_{12} = \frac{r_4}{r_2+r_4}$, $x_{14} = \frac{r_2}{r_2+r_4}$. Substituting the values of the r_i's in the expressions for x_{12} and x_{42}, we get $x_{12} = \frac{r_4}{r_2+r_4} \approx 0.554$ and $x_{42} = \frac{r_3}{r_2+r_3} \approx 0.546$. Thus, $x_{12} + x_{42} \approx 1.1 > 1$, i.e., good g_2 is oversold. Therefore, there is no equilibrium in Case 2.

□

Remark 21. Observe that in the equilibrium, the allocations of all four agents are irrational even though each one of them spends their dollar completely and the allocations form a fractional perfect matching, i.e., add up to 1 for each good and each agent.

7 Membership of Exact Equilibrium in FIXP

In this section, we will prove that the problem of computing an HZ equilibrium lies in the class FIXP, which was introduced in [EY10]. This is the class of problems that can be cast, in polynomial time, as the problem of computing a fixed point of an algebraic Brouwer function. Recall that basic complexity classes, such as P, NP, NC and #P, are defined via machine models. For the class FIXP, the role of "machine model" is played by one of the following: a straight line program, an algebraic formula, or a circuit; further it must use the standard arithmetic operations of +, - * /, min and max. We will establish membership in FIXP using straight line programs. Such a program should satisfy the following:

1. The program does not have any conditional statements, such as if ... then ... else.
2. It uses the standard arithmetic operations of +, - * /, min and max.
3. It never attempts to divide by zero.

A *total problem* is one which always has a solution, e.g., Nash equilibrium and Hylland-Zeckhauser equilibrium. A total problem is in FIXP if there is a polynomial time algorithm which given an instance I of length $|I| = n$, outputs a polynomial sized straight line program which computes a function F_I on a closed, convex, real-valued domain $D(n)$ satisfying: each fixed point of F_I is a solution to instance I.

Algorithm 22. Straight line program for function F_p

1. For all $j \in [n]$ do: $p_j \leftarrow \min\{n, \max\{0, p_j + \sum_{i \in A} x_{ij} - 1\}\}$
2. $r \leftarrow \min_{j \in [n]}\{p_j\}$
3. For all $j \in [n]$ do: $p_j \leftarrow p_j - r$

Let p and x denote the allocation and price variables. We will give a function F over these variables and a closed, compact, real-valued domain D for F. The function will be specified by a polynomial length straight line program using the algebraic operations of $+, -, *, /, \min$ and $\max$, hence guaranteeing that it is continuous. We will prove that all fixed points of F are equilibrium allocations and prices, hence proving that Hylland-Zeckhauser is in FIXP.

Notation: We will denote the set $\{1, \ldots .n\}$ by $[n]$. x_i will denote agent i's bundle. For each agent i, choose one good from G_i^* and denote it by i^*. If e is an expression, we will use $(e)_+$ as a shorthand for $\max\{0, e\}$.

Domain $D = D_p \times D_x$, where D_p and D_x are the domains for p and x, respectively, with $D_p = \{p \mid \forall j \in [n], p_j \in [0, n]\}$ and $D_x = \{x \mid \forall i \in [n],\ \sum_{j \in G} x_{ij} = 1, \text{ and } \forall i, j \in [n],\ x_{ij} \geq 0\}$.

Let $(p', x') = F(p, x)$. (p, x) can be viewed as being composed of $n+1$ vectors of variables, namely p and for each $i \in [n]$, x_i. Similarly, we will view F as being composed of $n+1$ functions, F_p and for each $i \in [n]$, F_i, where $p' = F_p(p, x)$ and for each $i \in [n]$, $x_i' = F_i(p, x)$. The straight line programs for F_p and F_i are given in Algorithm 22 and Algorithm 23, respectively. It is easy to see that if F_i alters a bundle, the new bundle still remains in the domain; in particular, $\forall i \in [n],\ \text{size}(i) = 1$. Similarly, it is easy to see that the output of F_p is in the domain D_p.

Requirements on F: Observe that (p, x) will be an equilibrium for the market if, in addition to the conditions imposed by the domain, it satisfies the following:

1. $\forall j \in [n],\ \sum_{i \in A} x_{ij} = 1$.
2. $\forall i \in [n],\ \text{cost}(i) \leq 1$.
3. $\forall i \in [n]$, x_i is an optimal bundle for i. Furthermore, $\text{cost}(i)$ is minimum over all optimal bundles.

Function F has been constructed in such a way that if any of these conditions is not satisfied by (p, x), then $F(p, x) \neq (p, x)$, i.e., (p, x) is not a fixed point of F. Equivalently, every fixed point of F must satisfy all these conditions and is therefore an equilibrium. Conversely, every equilibrium (p, x) is a fixed point of F.

We will prove that if (p, x) is a fixed point, then no step of F will change (p, x), i.e., it couldn't be that some step(s) of F change (p, x) and some other step(s) change it back, restoring it to (p, x). This is easy to check for F_p, and is left to the reader. The proof for F_i is more delicate and uses a potential function argument based on the changes in $\text{value}(i) = \sum_j u_{ij} x_{ij}$ and $\text{cost}(i) = \sum_j p_j x_{ij}$

Algorithm 23. Straight line program for function F_i

1. $r \leftarrow (\sum_j p_j x_{ij} - 1)_+$.
2. For all $j \in [n]$ do: $x_{ij} \leftarrow \frac{x_{ij} + r \cdot (1-p_j)_+}{1 + r \cdot \sum_k (1-p_k)_+}$
3. $t \leftarrow (1 - \sum_j p_j x_{ij})_+$
4. For all $k \notin G_i^*$ do:
   (a) $d \leftarrow \min\{x_{ik}, \frac{t}{n^2}\}$
   (b) $x_{ik} \leftarrow x_{ik} - d$
   (c) $x_{ii^*} := x_{ii^*} + d$
5. For all pairs j, k of goods s.t. $u_{ij} \leq u_{ik}$ do:
   (a) $d \leftarrow \min\{x_{ij}, (p_j - p_k)_+\}$
   (b) $x_{ij} \leftarrow x_{ij} - d/n$
   (c) $x_{ik} \leftarrow x_{ik} + d/n$
6. For all triples j, k, l of goods such that $u_{ij} < u_{ik} < u_{il}$ do:
   (a) $d \leftarrow \min\{x_{ik}, ((u_{il} - u_{ik})(p_k - p_j) - (u_{ik} - u_{ij})(p_l - p_k))_+\}$
   (b) $x_{ik} \leftarrow x_{ik} - d$
   (c) $x_{ij} \leftarrow x_{ij} + \frac{u_{il} - u_{ik}}{u_{il} - u_{ij}} d$
   (d) $x_{il} \leftarrow x_{il} + \frac{u_{ik} - u_{ij}}{u_{il} - u_{ij}} d$
7. For all triples j, k, l of goods such that $u_{ij} < u_{ik} < u_{il}$ do:
   (a) $d := \min(x_{ij}, x_{il}, ((u_{ik} - u_{ij})(p_l - p_k) - (u_{il} - u_{ik})(p_k - p_j))_+)$
   (b) $x_{ik} := x_{ik} + d$
   (c) $x_{ij} := x_{ij} - \frac{u_{il} - u_{ik}}{u_{il} - u_{ij}} d$
   (d) $x_{il} := x_{il} - \frac{u_{ik} - u_{ij}}{u_{il} - u_{ij}} d$

caused by any change in the allocation x_i in every step of the algorithm for F_i, as stated in the following lemma.

Lemma 24. *Let (p,x) be such that $p \in D_p$, $x \in D_x$. Then,* $\text{cost}(i)$ *and* $\text{value}(i)$ *are modified by the steps of F_i as follows.*

1. *If Steps 1, 2 modify x_i, then the initial cost is > 1, and steps 1,2 decrease strictly* $\text{cost}(i)$.
2. *If steps 3, 4 modify x_i, then they increase strictly* $\text{value}(i)$ *while maintaining $cost(i) \leq 1$.*
3. *If anyone of steps 5, 6, 7 modifies x_i, then it increases weakly* $\text{value}(i)$ *and decreases strictly* $\text{cost}(i)$.

Proof. For the first part, note that if steps 1,2 modify the allocation x_i, then we must have $r\sum_k(1-p_k)_+ > 0$, hence $r > 0$ and $\sum_k(1-p_k)_+ > 0$. Therefore, the initial cost $\text{cost}(i) = \sum_j p_j x_{ij} = r+1$ is > 1. The new cost is $\frac{\sum_j p_j x_{ij} + r\sum_j p_j(1-p_j)_+}{1+r\sum_j(1-p_j)_+}$ which is $< \text{cost}(i)$, because $r\sum_j p_j(1-p_j)_+ \leq r\sum_j(1-p_j)_+ < (r\sum_j(1-p_j)_+)\text{cost}(i)$; the last inequality is strict because $r\sum_j(1-p_j)_+ > 0$, and $\text{cost}(i) > 1$.

For the second part, note that if the cost of the allocation, $\text{cost}(i) = \sum_j p_j x_{ij}$ before step 3 is ≥ 1, then $t = 0$ in line 3, and steps 3, 4 make no change. Suppose that the cost is < 1, i.e. $t > 0$. For every good $k \notin G_i^*$, if $x_{ik} = 0$ then no change is made for this good. Thus, if steps 3,4 change x_i, then there must be some good(s) $k \notin G_i^*$ with $x_{ik} > 0$. For every such good k, we swap d units of k for i^*, and as a result the value is increased by $d(u_{ii^*} - u_{ik}) > 0$, since $d > 0$ and $u_{ii^*} > u_{ik}$. The cost is increased at most by $d(p_{i^*} - p_k) \leq \frac{t}{n^2}n = \frac{t}{n}$. Hence, over all the goods $k \notin G_i^*$, the cost is increased by less than t, hence it remains < 1.

For the third part, we consider the following three cases.

- If Step 5 modifies x_i for a pair j,k of goods then we must have $p_j > p_k$ and $x_{ij} > 0$. Since $u_{ij} \leq u_{ik}$, step 5 weakly increases $\text{value}(i)$ and strictly decreases $\text{cost}(i)$.
- If Step 6 kicks in for a triple of goods j,k,l, then the net change in $\text{value}(i)$ is

$$d\frac{u_{il}-u_{ik}}{u_{il}-u_{ij}}u_{ij} + d\frac{u_{ik}-u_{ij}}{u_{il}-u_{ij}}u_{il} - du_{ik} = 0.$$

  The net change in $\text{cost}(i)$ is

$$d\frac{u_{il}-u_{ik}}{u_{il}-u_{ij}}p_j + d\frac{u_{ik}-u_{ij}}{u_{il}-u_{ij}}p_l - dp_k = \frac{d\Delta}{u_{il}-u_{ij}},$$

  where $\Delta = (u_{ik}-u_{ij})(p_l-p_k) - (u_{il}-u_{ik})(p_l-p_k) < 0$.

- If Step 7 kicks in for a triple of goods $j,k,l,$, then the net change in $\text{value}(i)$ is again 0, and the net change in cost is $\frac{-d\Delta}{u_{il}-u_{ij}} < 0$.

□

Corollary 25. *If (p,x) is a fixed point of F, then no step of F_i will change x_i.*

Proof. Suppose that some step(s) of F_i change the allocation x_i of fixed point (x, p), and consider the earliest such step. If it is step 2, then the initial $\text{cost}(i) > 1$, and step 2 decreases strictly the cost. Step 3,4 either do not change the allocation or if they do change it, the new cost is ≤ 1, i.e. still smaller than the initial one. Steps 5, 6, 7 do not increase the cost, hence the final cost is strictly smaller than the initial. Thus, the final allocation x_i cannot be the same as the initial.

If the earliest step that changes x_i is step 4, then it increases strictly the value and the subsequent steps do not decrease it, hence the final value is strictly higher than the initial. If the earliest modifying step is one of 5, 6, 7, then it decreases strictly the cost, and all other subsequent changes do not increase it. We conclude that no step of F_i can change the allocation x_i of a fixed point. □

Lemma 26. *If (p, x) is a fixed point of F, as defined in Algorithms 22 and 23, then*

1. $\exists z \in G$ *such that* $p_z = 0$.
2. $\forall i \in [n]$, $\text{cost}(i) \leq 1$.
3. $\forall j \in [n]$, $\sum_{i \in A} x_{ij} = 1$, *i.e. the market clears.*

Proof.
1. Steps 2 and 3 of F_p ensure that there is a good with price 0.
2. If for some $i \in [n]$, $\text{cost}(i) > 1$, then Steps 1 and 2 of F_i will modify x_i since $r = \text{cost}(i) - 1 > 0$, and $\sum_k (1 - p_k)_+ > 0$ because some good z has $p_z = 0$.
3. Suppose that there is a good j such that $\sum_i x_{ij} \neq 1$. Since $\sum_j x_{ij} = 1$ for all agents $i \in [n]$, there must be a good k such that $\sum_i x_{ik} < 1$, and another good l such that $\sum_i x_{il} > 1$.

   We claim that then $p_k = 0$. Since $\sum_i x_{ik} < 1$, if $p_k > 0$, then line 1 of F_p will strictly decrease p_k, and line 3 certainly does not increase it, contradicting $F_p(p, x) = p$. Thus, $p_k = 0$, the price p_k will stay 0 after line 1, hence $r = 0$ in line 2, and line 3 will not change any prices.

   On the other hand, we claim that $p_l = n$. Since $\sum_i x_{il} > 1$, if $p_l < n$, then line 1 of F_p will increase strictly p_l, and since line 3 has no effect, this contradicts $F_p(p, x) = p$.

   But $\text{cost}(i) = \sum_j p_j x_{ij} \leq 1$ for all $i \in [n]$ implies that $\sum_i \sum_j p_j x_{ij} \leq n$, which contradicts the fact that $p_l = n$ and $\sum_i x_{il} > 1$, hence $\sum_i p_l x_{il} > n$.

□

Lemma 27. *If (p, x) is a fixed point of F, as defined in Algorithms 22 and 23, then x_i is an optimal bundle for i at prices p. Furthermore,* $\text{cost}(i)$ *is minimum among optimal bundles.*

Proof. We will consider the following exhaustive list of cases. Each contradiction is based on applying Corollary 25. We will assume that α_i and μ_i are optimal variables of the dual to i's optimal bundle LP and that $u = \max_j \{u_{ij}\}$.

Case 1: Assume that $\text{cost}(i) < 1$. If $B_i \not\subseteq G_i^*$, then Steps 3 and 4 will kick in, contradicting the fact that (p, x) is a fixed point. Therefore $B_i \subseteq G_i^*$. Clearly, u is the maximum utility that i can derive from a bundle satisfying $\text{size}(i) = 1$ and $\text{cost}(i) \leq 1$. Therefore, x_i is an optimal bundle for i. Since step 5 does not modify x_i, all goods in B_i must have minimum price among the goods of G_i^*. Therefore, $\text{cost}(i)$ is minimum among the optimal bundles.

Henceforth, we will assume that $\text{cost}(i) = 1$.

Case 2: Assume that i derives the same utility from all goods $j \in B_i$ and $B_i \subseteq G_i^*$. As in the previous case, x_i is an optimal bundle for i and hence each good in B_i is optimal. Furthermore, again since step 5 does not modify the allocation, as in Case 1, $\text{cost}(i)$ is minimum among the optimal bundles.

Case 3: Assume that i derives the same utility from all goods $j \in B_i$ and $B_i \nsubseteq G_i^*$. Let k be a good in B_i and let z be a good having price 0. Each good in B_i must be a minimum price good having utility u_{ik}, since otherwise Step 5 of F_i will alter the bundle. Since $\text{cost}(i) = 1$, $\text{size}(i) = 1$ and all goods in B_i have the same price, each good in B_i has price 1.

Let l be a good such that $u_{il} > u_{ik}$; observe that any good in G_i^* is such a good. We will prove that $p_l > 1 = p_k$. Clearly $u_{iz} < u_{ik}$, since otherwise Step 5 will kick in and change the bundle. Hence we have $u_{iz} < u_{ik} < u_{il}$. However, since Step 6 did not kick in, $(u_{il} - u_{ik})(p_k - p_z) \leq (u_{ik} - u_{iz})(p_l - p_k)$. Since $(u_{il} - u_{ik})(p_k - p_z) > 0$, we get that $(p_l - p_k) > 0$. Therefore $p_l > p_k = 1$. Hence we can conclude that the optimal bundle for i at prices p is not a Type A or Type B bundle.

Next, assume for the sake of contradiction that x_i is not an optimal bundle for i at prices p; in particular, this entails that the optimal bundle for i is not Type C. Therefore, i's optimal bundle must be Type D and k is a suboptimal good. As argued in Section 5, an optimal Type D bundle must contain a good of price < 1 and a good of price > 1; let j and l be such goods, respectively. Clearly $u_{iz} < u_{ik} < u_{il}$. Then we have,

$$\alpha_i p_j = u_{ij} - \mu_i, \quad \alpha_i p_k > u_{ik} - \mu_i \quad \text{and} \quad \alpha_i p_l = u_{il} - \mu_i$$

Subtracting the first from the second and the second from the third we get

$$\alpha_i(p_k - p_j) > (u_{ik} - u_{ij}) \quad \text{and} \quad \alpha_i(p_l - p_k) < (u_{il} - u_{ik})$$

This gives

$$(u_{il} - u_{ik})(p_k - p_j) - (u_{ik} - u_{ij})(p_l - p_k) > 0.$$

Therefore, Step 6 should kick in, leading to a contradiction. Hence x_i is a Type C optimal bundle. Since all goods in G_i^* have price > 1, every bundle with cost < 1 is suboptimal, thus x_i has minimum cost among optimal bundles.

Henceforth, we will assume that $\text{cost}(i) = 1$ *and* $\exists\, s, t \in B_i$ *with* $u_{is} < u_{it}$.

Case 4: Assume that the set $\{u_{ij} \mid j \in G\}$ has exactly two elements. Clearly, these utilities must be u_{is} and u_{it}. Now, s must be the zero-priced good, since otherwise Step 5 will kick in. Since $\text{cost}(i) = 1$ and $\text{size}(i) = 1$, $p_t > 1$. Again since Step 5 didn't kick in, s and t must be the cheapest goods having utilities u_{is} and u_{it}. Therefore, x_i is a Type D optimal bundle. It has minimum cost (=1) among optimal bundles for the same reason as in case 3.

Case 5: Assume that the set $\{u_{ij} \mid j \in G\}$ has three or more elements. Since $\text{size}(i) = 1$ and $\text{cost}(i) = 1$, $\exists\, t \in B_i,\ s.t.\ p_t > 1$. Now, any good having utility u must have price > 1, since otherwise Step 5 will alter the bundle. Therefore, x_i cannot be a Type A or Type B bundle. Therefore, $\alpha_i > 0$.

Suppose that x_i is not an optimal bundle. Then there are two cases: that the optimal bundle is Type C or Type D. In the first case, let $k \in G$ be an optimal good; $p_k = 1$. Let $j, l \in B_i$ with $p_j < 1 < p_l$ and at least one of j or l is suboptimal. Clearly, $u_{ij} < u_{ik} < u_{il}$, otherwise Step 5 will kick in. Therefore we have

$$\alpha_i p_j \geq u_{ij} - \mu_i, \quad \alpha_i p_k = u_{ik} - \mu_i \quad \text{and} \quad \alpha_i p_l \geq u_{il} - \mu_i,$$

with at least one of the inequalities being strict. Therefore,

$$(u_{ik} - u_{ij})(p_l - p_k) > (u_{il} - u_{ik})(p_k - p_j),$$

and Step 7 should kick in, leading to a contradiction. Hence x_i is a Type C optimal bundle.

Next suppose the optimal bundle is Type D. There are two cases. First, suppose $\exists\ k \in B_i$ such that k is a suboptimal good for i and there are optimal goods j and l with $u_{ij} < u_{ik} < u_{il}$. Then we have

$$\alpha_i p_j = u_{ij} - \mu_i, \quad \alpha_i p_k > u_{ik} - \mu_i \quad \text{and} \quad \alpha_i p_l = u_{il} - \mu_i$$

As before we get

$$(u_{il} - u_{ik})(p_k - p_j) - (u_{ik} - u_{ij})(p_l - p_k) > 0.$$

Therefore, Step 6 should kick in, leading to a contradiction.

Second, suppose that there is no such good $j \in B_i$. Let v and w be optimal goods with the smallest and largest utilities for i. Then all suboptimal goods in B_i have either less utility than u_{iv} or more utility than u_{iw}. Suppose there are both types of goods, say j and l, respectively. Then Step 7 should kick in with the triple j, v, l. Else there is only one type, say j with $u_j < u_v$. Then $\exists\ l \in B_i$ with $p_l > 1$. Now, Step 7 should kick in with the triple j, v, l. In the remaining case, $\exists\ j, l \in B_i$ with $p_j < 1$ and $u_{il} > u_{iw}$. Now, Step 7 should kick in with the triple j, w, l.

The contradictions give us that x_i does not contain a suboptimal good and is hence a Type D optimal bundle. The minimality of the cost holds for the same reason as in Cases 3, 4. ☐

Lemmas 26 and 27 give:

Theorem 28. *The problem of computing an exact equilibrium for the Hylland-Zeckhauser scheme is in FIXP.*

8 Membership of Approximate Equilibrium in PPAD

In this section we define approximate equilibria, and show that the problem of computing an approximate equilibrium is in PPAD.

First let us scale the utilities of all the agents so that they lie in $[0, 1]$. This can be done clearly without loss of generality without changing the equilibria.

Definition 29. A pair (p, x) of (non-negative) prices and allocations is an *ϵ-approximate equilibrium* for a given one-sided market if:

1. The total probability share of each good j is 1 unit, i.e., $\sum_i x_{ij} = 1$.
2. The size of each agent i's allocation is 1, i.e., $\text{size}(i) = 1$.
3. The cost of the allocation of each agent is at most $1 + \epsilon$.
4. The value of the allocation of each agent i is at least $v^*(i) - \epsilon$ where $v^*(i)$ is the value of the optimal bundle for agent i under prices p, i.e. the optimal value of the program: maximize $\text{value}(i)$, subject to $\text{size}(i) = 1$ and $\text{cost}(i) \le 1$. Furthermore, we require that the cost of the allocation x_i is at most $c^*(i) - \epsilon$, where $c^*(i)$ is the minimum cost of a bundle for agent i that has the maximum value $v^*(i)$.

The corresponding computational problem is: Given a one-sided matching market M and a rational $\epsilon > 0$ (in binary as usual), compute an ϵ-approximate equilibrium for M. Polynomial time in this context means time that is polynomial in the encoding size of the market M and $\log(1/\epsilon)$. We define also a more relaxed version, called a *relaxed ϵ-approximate equilibrium* where condition 1 is relaxed to $|\sum_i x_{ij} - 1| \le \epsilon$ for all goods j. It is easy to see that the two versions are polynomially equivalent, i.e., if one can be solved in polynomial time then so can the other.

Proposition 30. *The problems of computing an ϵ-approximate equilibrium and a relaxed approximate equilibrium are polynomially equivalent.*

Proof. Clearly, the relaxed version is no harder than the nonrelaxed version. On the other hand, if we have an algorithm for the relaxed version, then we can compute an ϵ-approximate equilibrium as follows. Given a one-sided market M and a rational $\epsilon > 0$, assume without loss of generality that $\epsilon \le 1$. Compute a relaxed δ-approximate equilibrium (p, x) where $\delta = \epsilon/4n$. In this equilibrium some goods may be oversold or undersold by an amount at most δ. Set up a bipartite transportation network with the goods and agents as nodes, where the edge connecting agent i with good j has capacity x_{ij}. For each good j that is oversold, the corresponding node is a source with supply $\sum_i x_{ij} - 1$; for each good j that is undersold, the corresponding node is a sink with demand $1 - \sum_i x_{ij}$. Since $\sum_j x_{ij} = 1$ for all agents i, the sum of the supplies over all sources is equal to the sum of the demands over all sinks. It is straightforward to construct a feasible flow that ships all the supply from the sources to the sinks, where the flow on each edge is at most δ. Combining this flow with the allocation x results in a new allocation x' that satisfies $\sum_j x'_{ij} = 1$ for all agents i, and $\sum_i x'_{ij} = 1$ for all goods j.

The flow on each edge is at most δ, so every allocation x_{ij} is changed at most by δ. Since $\sum_j x_{ij} p_j \le 1 + \delta$ for all agents i, $\sum_i \sum_j x_{ij} p_j = \sum_j p_j \sum_i x_{ij} \le n(1 + \delta)$. Since $\sum_i x_{ij} \ge 1 - \delta$ for all goods j, it follows that the sum of the prices $\sum_j p_j \le n(1 + \delta)/(1 - \delta) \le 2n$. Since $|x'_{ij} - x_{ij}| \le \delta$ for all i, j, it follows that $|cost'(i) - cost(i)| \le 2n\delta \le \epsilon/2$. Therefore, the new cost $cost'(i)$ of each agent's allocation is at most $1 + \delta + \epsilon/2 \le 1 + \epsilon$. Since all the utilities u_{ij} are in $[0, 1]$, the value of each agent's allocation is changed at most by $n\delta < \epsilon/2$. Therefore, the new $\text{value}'(i)$ is at least $v^*(i) - \delta - \epsilon/2 > v^*(i) - \epsilon$. Thus, the prices p and the new allocations x' form an ϵ-approximate equilibrium. □

Note however, that in general an ϵ-approximate equilibrium may not be close to an actual equilibrium of the matching market. This phenomenon is similar to the case of market equilibria for the standard exchange markets and to the case of Nash equilibria for games.

We will show membership of the approximate equilibrium problem in PPAD by showing that a relaxed approximate equilibrium can be obtained from an approximate fixed point of a variant of the function F defined in Section 7.

Definition 31. A *weak ϵ-approximate fixed point* of a function F (or *weak ϵ-fixed point* for short) is a point x such that $||F(x) - x||_\infty \le \epsilon$.

Let $\mathcal{F}$ be a family of functions, where each function F_I in $\mathcal{F}$ corresponds to an instance I of a problem (in our case a one-sided matching market) that is encoded as usual by a string. The function F_I maps a domain D_I, to itself. We assume that D_I is a polytope defined by a set of linear inequalities with rational coefficients which can be computed from I in polynomial time; this clearly holds for our problem. We use $|I|$ to denote the length of the encoding of an instance I (i.e., the length of the string). If x is a rational vector, we use $size(x)$ to denote the number of bits in a binary representation of x.

Definition 32. A family $\mathcal{F}$ of functions is *polynomially computable* if there is a polynomial q and an algorithm that, given the string encoding I of a function $F_I \in \mathcal{F}$ and a rational point $x \in D_I$, computes $F_I(x)$ in time $q(|I| + size(x))$.
A family $\mathcal{F}$ of functions is *polynomially continuous* if there is a polynomial q such that for every $F_I \in \mathcal{F}$ and every rational $\epsilon > 0$ there is a rational δ such that $\log(1/\delta) \le q(|I| + \log(1/\epsilon))$ and such that $||x - y||_\infty \le \delta$ implies $||F_I(x) - F_I(y)||_\infty \le \epsilon$ for all $x, y \in D_I$.

It was shown in [EY10] that, if a family of functions is polynomially computable and polynomially continuous, then the corresponding weak approximate fixed point problem (given I and rational $\delta > 0$, compute a weak δ-approximate fixed point of F_I) is in PPAD. The family $\mathcal{F}$ of functions for the online matching market problem defined in Section 7 is obviously polynomially computable. It is easy to check also that it is polynomially continuous.

We will use a variant F' of the function F of Section 7, where the functions F_i for the allocations are modified as follows. Step 5 for all pairs j, k of goods, and steps 6, and 7 for all triples j, k, l are applied all independently in parallel to the allocation that results after step 4. In order for the allocation to remain feasible (i.e. have $x_{ij} \ge 0$ for all i, j), we change line 5a in F'_i to $d \leftarrow \min\{\frac{x_{ij}}{3}, (p_j - p_k)_+\}$, change line 6a to $d \leftarrow \min\{\frac{x_{ik}}{3n^2}, ((u_{il} - u_{ik})(p_k - p_j) - (u_{ik} - u_{ij})(p_l - p_k))_+\}$, and we change line 7a to $d \leftarrow \min\{\frac{x_{ij}}{3n^2}, \frac{x_{il}}{3n^2}, ((u_{ik} - u_{ij})(p_l - p_k) - (u_{il} - u_{ik})(p_k - pj))_+\}$. In this way, a coordinate x_{ij} can be decreased by the operations of step 5 for all pairs j, k at most by $x_{ij}/3$ in total, and the same is true for the total decrease from the operations of steps 6 and 7 for all triples involving good j; therefore, the coordinates x_{ij} remain nonnegative. The function for the prices remains the same as before. All the properties shown in Section 7 for F hold also for F'.

The family $\mathcal{F}'$ of these functions F'_I is clearly also polynomially computable and polynomially continuous. We shall show that, given an instance I of the matching market problem and a rational $\epsilon > 0$, we can pick a $\delta > 0$ such that $\log(1/\delta)$ is bounded by a polynomial in $|I|$ and $\log(1/\epsilon)$, and every weak δ-approximate fixed point of F'_I is a relaxed ϵ-approximate equilibrium of the market I.

Every utility u_{ij} is a rational number, without loss of generality in $[0, 1]$, which is given as the ratio of two integers represented in binary. Let m be the maximum number of bits needed to represent a utility. Note that every nonzero u_{ij} is at least $1/2^m$ and the difference between any two unequal utilities is at least $1/2^{2m}$. Given a positive rational ϵ (wlog in $[0, 1]$), let $\delta = \epsilon/(n^{10}2^{6m})$. We shall show that every weak δ-fixed point of F'_I is a relaxed ϵ-approximate equilibrium of the matching market I. The proof follows and adapts the proof in Section 7 of the analogous statement for the exact fixed points.

Lemma 33. *If (p, x) is a weak δ-fixed point of F', then*

1. *$\exists z \in G$ such that $p_z \leq \delta$.*
2. *$\forall i \in [n]$, $\text{cost}(i) \leq 1 + 2n^2\delta$.*
3. *$\forall j \in [n], 1 - 3n^3\delta \leq \sum_{i \in A} x_{ij} \leq 1 + 3n^2\delta$.*
4. *$\sum_j p_j < 2n$.*

Proof.

1. From Steps 2 and 3 of F_p, there is a good z such that the price of z in the output is 0. Therefore, $p_z \leq \delta$.

2. Suppose for some $i \in [n]$, $\text{cost}(i) > 1$. Then Steps 1 and 2 of F'_i will modify x_i since $r = \text{cost}(i) - 1 > 0$, and $\sum_k (1 - p_k)_+ > 0$ because some good z has $p_z \leq \delta$. The new cost is $\frac{\sum_j p_j x_{ij} + r \sum_j p_j (1-p_j)_+}{1 + r \sum_j (1-p_j)_+}$. This is at most $\frac{1 + r + r p_z (1-p_z)}{1 + r(1-p_z)} = 1 + \frac{r p_z (2-p_z)}{1 + r(1-p_z)} < 1 + 2\delta$. Steps 3, 4 of F'_i will either not change the allocation or if they do change it, the new cost will be less than 1. Steps 5, 6, 7 will not increase the cost. Thus, the final cost will be less than $1 + 2\delta$. Since F'_i changes each coordinate x_{ij} at most by δ and every price p_j is at most n, the total change in the cost is at most $n^2\delta$. Therefore, the initial $\text{cost}(i)$ is at most $1 + 2\delta + n^2\delta < 1 + 2n^2\delta$.

3. Suppose that there is a good l such that $\sum_i x_{il} > 1 + 3n^2\delta$. Since $\sum_j x_{ij} = 1$ for all agents $i \in [n]$, there must be a good k such that $\sum_i x_{ik} < 1 - 3n\delta$.

   We claim that then $p_k \leq \delta$, and that line 3 of F_p does not change the prices. Since $\sum_i x_{ik} < 1 - 3n\delta$, if $p_k > \delta$, then line 1 of F_p will decrease p_k by more than δ, and line 3 certainly does not increase it, contradicting $||F_p(p, x) - p||_\infty \leq \delta$. Thus, $p_k \leq \delta$, the price p_k will become 0 after line 1, hence $r = 0$ in line 2, and line 3 will not change the prices.

   On the other hand, we claim that $p_l \geq n - \delta$. Since $\sum_i x_{il} > 1 + 3n^2\delta$, if $p_l < n - \delta$, then line 1 of F_p will increase p_l by more than δ, and since line 3 does not change the prices, the final value of p_l exceeds the initial value by more than δ, contradicting the assumption that (p, x) is a δ-fixed point.

   But $\text{cost}(i) = \sum_j p_j x_{ij} \leq 1 + 2n^2\delta$ for all $i \in [n]$ implies that $\sum_i \sum_j p_j x_{ij} \leq n(1 + 2n^2\delta)$, which contradicts the fact that $p_l \geq n - \delta$ and $\sum_i x_{il} > 1 + 3n^2\delta$, hence $\sum_i p_l x_{il} > (n - \delta)(1 + 3n^2\delta) > n(1 + 2n^2\delta)$.

   We conclude that $\sum_i x_{il} \leq 1 + 3n^2\delta$ for all goods l. Since $\sum_i \sum_j x_{ij} = n$, it follows that $\sum_i x_{ij} \geq 1 - 3n^3\delta$ for all goods j.

4. From part (2), $\sum_j p_j x_{ij} \leq 1 + 2n^2\delta$ for all agents i, hence $\sum_i \sum_j p_j x_{ij} = \sum_j p_j \sum_i x_{ij} \leq n(1 + 2n^2\delta)$. From part (3), $\sum_i x_{ij} \geq 1 - 3n^3\delta$ for all j. Therefore, $\sum_j p_j \leq \frac{n(1+2n^2\delta)}{1-3n^3\delta} < 2n$.

□

In the case of approximate fixed points, it is possible that multiple steps of F'_i modify the allocation. However, as we will see, because of Lemma 24, none of the steps can change the value or the cost by a large amount, because then the other steps cannot reverse the change. Note that if two bundles of an agent differ by at most δ in every coordinate, then their values differ by at most $n\delta$ (because all utilities are in $[0, 1]$), and their costs differ by at most $2n\delta$ (because the sum of the prices is less than $2n$). This holds in particular for the values and the costs of the input and the output allocation of each function F'_i when the input is a weak δ-fixed point.

All the steps of F'_i weakly increase the value of the allocation, except possibly for step 2. Since r in step 1 is $(\text{cost}(i) - 1)_+ \leq 2n^2\delta$, the changes in each coordinate x_{ij} in step 2 are "small": From the update formula in step 2, x_{ij} can increase at most by $r \leq 2n^2\delta$. Thus, the value can increase in step 1 at most by $2n^3\delta$. On the other hand, coordinate x_{ij} may decrease at most by $x_{ij}(1 - \frac{1}{1+r\sum_k(1-p_k)_+}) \leq x_{ij}rn \leq x_{ij}2n^3\delta$. Therefore the value can decrease in step 1 also at most by $2n^3\delta$. As we observed above, the value of the output allocation of F'_i cannot differ from that of the input allocation by more than $n\delta$. Thus, we conclude:

Corollary 34. *If (p, x) is a weak δ-fixed point, then no step of F'_i changes the value of the allocation by more than $2n^3\delta + n\delta$.*

All steps of F'_i weakly decrease the cost, except possibly for step 4. We show that step 4 does not change the allocation significantly, and thus does not increase the cost very much.

Lemma 35. *Suppose that (p, x) is a weak δ-fixed point of F'. If t in Step 3 of F'_i satisfies $t > 3n^5\delta 2^{2m}$ then the value of x_i is within ϵ of the value $v^*(i)$ of the optimal bundle for agent i under prices p, and the cost of x_i is within ϵ of the minimum cost of an optimal bundle.*

Proof. Steps 1, 2 can decrease the cost at most by $1 + r - \frac{1+r+r\sum_j p_j(1-p_j)_+}{1+r\sum_j(1-p_j)_+} \leq r(1+r)\sum_j(1-p_j)_+ \leq r(1+r)n \leq 3n^3\delta$. Since t in step 3 exceeds $3n^3\delta$, it follows that the cost of the input allocation x_i is not greater than 1. Therefore, steps 1, 2 do not modify x_i.

Let $B'_i = \{j | x_{ij} > 3n^3\delta 2^{2m}\}$. Suppose that there is a good $k \in B'_i - G^*_i$. Then d in step 4 for good k satisfies $d \geq 3n^3\delta 2^{2m}$. The change of the allocation in step 4 increases the value by at least $d(u_{ii^*} - u_{ik}) \geq d/2^{2m} \geq 3n^3\delta$, contradicting Corollary 34.

Therefore, $B'_i \subseteq G^*_i$. Let u be the utility for agent i of the goods in G^*_i (the maximum utility). The goods $j \notin G^*_i$ have $x_{ij} \leq 3n^3\delta 2^{2m}$. Therefore the value of x_i is at least $u - 3n^4\delta 2^{2m} > u - \epsilon \geq v^*(i) - \epsilon$.

We show now the claim about the cost. If the min-cost optimal bundle has cost 1, then the claim follows from Lemma 33. So assume it has cost < 1, i.e. it is of type A and consists of goods in G^*_i. Let k be a good in G^*_i with minimum price. The minimum cost of an optimal bundle is p_k.

Step 4 may move some probability mass from goods that are not in G_i^*, hence not in B_i', to i^*. Since $x_{ij} \leq 3n^3\delta 2^{2m}$ for all $j \notin B_i'$, the total mass moved is at most $3n^4\delta 2^{2m}$, and the cost is increased at most by $3n^5\delta 2^{2m}$. Since the cost of the output allocation of F_i' is within $2n\delta$ of the cost of the input allocation, and all other steps of F_i' weakly decrease the cost, it follows that no step of F_i' can decrease the cost by more than $3n^5\delta 2^{2m} + 2n\delta$.

Let $s = 4n^3\sqrt{\delta}2^m$, and let $\hat{B}_i = \{j | x_{ij} > s\}$. Clearly, $s > 3n^3\delta 2^{2m}$, and thus $\hat{B}_i \subseteq B_i' \subseteq G_i^*$. We claim that every good $j \in \hat{B}_i$ has price $p_j \leq p_k + s$. If not, then step 5 for the pair j, k will have $d \geq s/3$, and it will decrease the cost by $\frac{d}{n}(p_j - p_k) > \frac{s^2}{3n} > 3n^5\delta 2^{2m} + 2n\delta$, a contradiction. Therefore, $p_j \leq p_k + s$ for all $j \in \hat{B}_i$. The allocation x_i has probability mass at most ns in the goods that are not in $\hat{B}_i$, and thus their cost is at most $n^2 s$. Therefore the cost of x_i is at most $p_k + s + n^2 s < p_k + \epsilon$. □

We assume henceforth that t in step 3 is at most $t_0 = 3n^5\delta 2^{2m}$. Step 4 increases the cost at most by t and the other steps of F_i' weakly decrease the cost. Since the difference between the final and the initial cost is at most $2n\delta$, we have:

Corollary 36. *No step of F_i' decreases the cost of the allocation by more than $t_0 + 2n\delta < 4n^5\delta 2^{2m}$.*

We show now the approximate optimality of the agents' bundles in an approximate fixed point.

Lemma 37. *If (p, x) is a weak δ-fixed point of F', then the value of x_i is within ϵ of the optimal value of a bundle for agent i at prices p, and the cost of x_i is within ϵ of the minimum cost among optimal bundles.*

Proof. Lemma 35 showed the result in the case that t in step 3 satisfies $t > t_0 = 3n^5\delta 2^{2m}$. So assume henceforth that $t \leq t_0$. Thus, the cost of the allocation after step 2 is $\geq 1 - t_0$. The cost of the input allocation x_i is at least as great, and is at most $1 + 2n^2\delta$ by Lemma 33. It follows that the cost of the input allocation x_i, as well as the allocations after step 2 and after step 4 are all in the interval $[1 - t_0, 1 + 2n^2\delta]$ (i.e., they are close to 1).

Let x_i' be the allocation after step 4. Let $\text{value}'(i)$ be the value of x_i' and $\text{value}(i)$ the value of x_i. As we observed earlier, steps 1, 2 change the value of the allocation at most by $2n^3\delta$, and step 3, 4 change each x_{ij} at most by t_0/n^2, hence they increase the value at most by t_0/n. Therefore, $\text{value}(i) \geq \text{value}'(i) - 2n^3\delta - (t_0/n) \geq \text{value}'(i) - \epsilon/2$.

Let $B_i' = \{j | x_{ij}' > s\}$, where $s = 4n^3\sqrt{\delta}2^m$. We start with some useful properties of the goods in B_i'.

Claim 38. *For every good $j \in B_i'$ and every good k with $u_{ik} \geq u_{ij}$, it holds that $p_j \leq p_k + s$.*

Proof. Suppose the claim is not true and consider step 5 for the pair j, k. We have $d \geq s/3$, and step 5 decreases the cost by $\frac{d}{n}(p_j - p_k) > \frac{s^2}{3n} > 4n^5\delta 2^{2m}$, in contradiction to Corollary 36. □

Thus, every good in B_i' has price that is close to the minimum price among goods with the same or higher utility. On the other hand, goods with strictly higher utility must have distinctly higher price:

Claim 39. *If $j \in B'_i$ and $p_j \geq 1/2$, then every good l with higher utility $u_{il} > u_{ij}$ has price $p_l > p_j + 2^{-2m-2}$. The same holds also for all goods j in an optimal bundle.*

Proof. Let j be a good in B'_i and let l be another good such that $u_{il} > u_{ij}$. Let z be a good with minimum price. By Lemma 33, $p_z \leq \delta$, hence $u_{iz} < u_{ij}$. Consider step 6 for the triple z, j, l. We have $d = \min\{\frac{x_{ij}}{3n^2}, \Delta\}$, where $\Delta = (u_{il} - u_{ij})(p_j - p_z) - (u_{ij} - u_{iz})(p_l - p_j)$. We know that $u_{il} - u_{ij} \geq 2^{-2m}$, $p_j - p_z \geq (1/2) - \delta$. If $p_l - p_j \leq 2^{-2m-2}$ then $\Delta \geq 2^{-2m-3} > 3n^2 s$. Thus, $d \geq \frac{s}{3n^2}$, step 6 will modify the allocation and decrease the cost by $\frac{d\Delta}{u_l - u_z} > s^2 > 4n^6\delta 2^{2m}$, contradicting Corollary 36. Therefore, $p_l > p_j + 2^{-2m-2}$.

The argument for the case that j is a good in an optimal bundle is similar. If $p_l \leq p_j + 2^{-2m-2}$, then applying step 6 for the triple z, j, l to the optimal bundle will reduce its cost while keeping the same value, and then its value can be increased by further transferring some probability mass from good j to l, contradicting the optimality of the bundle. □

We will prove now the approximate optimality of the allocations x'_i and x_i. We distinguish cases depending on the type of an optimal bundle for the prices p.

Case 1. *The optimal bundle is of type A or B, i.e., there is a good $k \in G^*_i$ with price $p_k \leq 1$.*

Let p_k be the smallest price of a good in G^*_i; this is also the minimum cost of an optimal bundle. The value $v^*(i)$ of the optimal bundle is u, the maximum utility of a good. We argue that most of the probability mass of x'_i is allocated to goods in G^*_i. The goods not in B'_i have total size at most ns and cost at most $n^2 s$. The goods in B'_i have price at most $p_k + s$ by Claim 38. Since the cost of x'_i is close to 1, B'_i must contain goods with price close to 1, therefore p_k must be close to 1. Specifically, the cost of x'_i is at least $1 - t_0$ and at most $p_k + s + n^2 s$, hence $p_k \geq 1 - t_0 - n^2 s - s$. The goods in $B'_i \setminus G^*_i$ have price at most $p_k - 2^{-2m-2}$ by Claim 39. If the total size of the goods in $B'_i \setminus G^*_i$ is y, then the cost of x'_i is at most $n^2 s + y(p_k - 2^{-2m-2}) + (1 - y - ns)(p_k + s)$. Since the cost is at least $1 - t_0$, it follows that $y \leq 2^{2m+2}(n^2 s + t_0) \leq 2^{2m+3} n^2 s$. Therefore, the value of x'_i is at least $(1 - ns - y)u \geq u - \epsilon/2$. Hence the value of x_i is at least $v^*(i) - \epsilon$. The cost of x'_i is at most $p_k + s + n^2 s < p_k + \epsilon/2$, hence the cost of x_i is less than $p_k + \epsilon$.

We assume henceforth that the minimum price of a good in G^*_i is > 1, thus the optimal bundle is of type C or D and has cost=1. The claim of the lemma about the cost thus holds by Lemma 33, and we only need to prove the claim about the value.

Case 2. *The optimal bundle is of type C.*

Thus the optimal bundle has cost 1, and contains goods with the same utility, $v^*(i)$, and price 1. Let k be an optimal good. All the goods of B'_i with utility strictly smaller than u_{ik} have price $\leq 1 - 2^{-2m-2}$ (by Claim 39). Let y be the total size of these goods. The goods of B'_i with utility u_{ik} have price at most $1 + s$ (by Claim 38). Suppose that B'_i does not have any goods with utility $> u_{ik}$. Then the cost of x'_i is at most $n^2 s + y(1 - 2^{-2m-2}) + (1 - y - ns)(1 + s)$. Since the cost is at least $1 - t_0$, it follows that $y \leq 2^{2m+3} n^2 s$. Therefore, the value of x'_i is at least $(1 - ns - y)u_{ik} \geq u_{ik} - \epsilon/2$, from which it follows that value$(i) \geq v^*(i) - \epsilon$.

We assume thus that $y > 2^{2m+3} n^2 s$, which means that there are goods in B'_i with utility $> u_{ik}$, and there are also goods in B'_i with utility $< u_{ik}$ (since $y > 0$). Let $L = \{j \in B'_i | u_{ij} < u_{ik}\}$,

$R = \{l \in B'_i | u_{il} > u_{ik}\}$. By Claim 39, every good $j \in L$ has price $p_j < p_k - 2^{-2m-2} = 1 - 2^{-2m-2}$, and every good $l \in R$ has price $p_l > 1 - 2^{-2m-2}$.

For any good $j \in L$ and any good $l \in R$, consider Step 7 of F'_i for the triple of goods j, k, l. Let α_i, μ_i be the optimal dual values. We have:

$$a_i p_j = g_j + u_{ij} - \mu_i, \quad a_i p_k = \alpha_i = u_{ik} - \mu_i, \quad a_i p_l = g_l + u_{il} - \mu_i$$

where $g_j, g_l \geq 0$. We have $\alpha_i(1 - p_j) = u_{ik} - u_{ij} - g_j \leq 1$ and $1 - p_j \geq 2^{-2m-2}$ (by Claim 39), hence $\alpha_i \leq 2^{2m+2}$. The quantity $\Delta = (u_{ik} - u_{ij})(p_l - p_k) - (u_{il} - u_{ik})(p_k - p_j)$ is equal to $g_l(p_k - p_j) + g_j(p_l - p_k)$. If there is a $l \in R$ such that $g_l \geq n^2 s 2^{2m+2}$, and we let j be any element of L, then the quantity Δ for the triple j, k, l is at least $n^2 s$, thus $d \geq \frac{s}{3n^2}$, and step 7 will decrease the cost by $\frac{d\Delta}{u_{il} - u_{ij}} \geq \frac{s^2}{3} > 4n^6 \delta 2^{2m}$, contradicting Corollary 36. Similarly, if there is a $j \in L$ such that $g_j \geq n^2 s 2^{2m+2}$, and we take l to be any element of R, the quantity Δ for the triple j, k, l will be at least $n^2 s$, leading to the same contradiction. We conclude that $g_j < n^2 s 2^{2m+2}$ for all $j \in L \cup R$. Note that for all $j \in S = \{j \in B'_i | u_{ij} = u_{ik}\}$, we have $p_j \leq p_k + s = 1 + s$, hence $\alpha_i p_j \leq u_{ij} - \mu_i + \alpha_i s \leq u_{ij} - \mu_i + s 2^{2m+2}$, i.e. $g_j \leq s 2^{2m+2}$.

Thus, for all $j \in B'_i$, we have $\alpha_i p_j \leq u_{ij} - \mu_i + n^2 s 2^{2m+2}$. Multiplying each equation by x_{ij} and summing over all $j \in B'_i$ we get that $\alpha_i \sum_{j \in B'_i} x_{ij} p_j \leq \sum_{j \in B'_i} x_{ij} u_{ij} - \mu_i \sum_{j \in B'_i} x_{ij} + n^2 s 2^{2m+2} \sum_{j \in B'_i} x_{ij}$. The left hand side is α_i times the cost of x'_i, except for the goods that are not in B'_i, hence it is at least $\alpha_i(1 - t_0 - n^2 s)$. We have also $\sum_{j \in B'_i} x_{ij} \geq 1 - ns$. The value of x'_i is at least $\sum_{j \in B'_i} x_{ij} u_{ij}$ and the optimal value $v^*(i)$ is equal to $\alpha_i + \mu_i$. Thus the difference between $v^*(i)$ and the value of x'_i is $v^*(i) - \text{value}'(i) \leq \alpha_i(t_0 + n^2 s) + n^2 s 2^{2m+2} + \mu_i ns \leq 2^{2m+2}(t_0 + n^2 s) + n^2 s 2^{2m+2} + ns < \epsilon/2$. It follows then as before that the initial $\text{value}(i) > v^*(i) - \epsilon$.

Case 3. *The optimal bundle is of type D.*

The optimal bundle has cost 1 and contains some good l with price > 1 and some good j with price < 1. Clearly $u_{ij} < u_{il}$. Let α_i, μ_i be again the optimal dual values. We have

$$\alpha_i p_j = u_{ij} - \mu_i, \quad \alpha_i p_l = u_{il} - \mu_i, \quad \alpha_i = v^*(i) - \mu_i$$

Note that $\alpha_i = \frac{u_{il} - u_{ij}}{p_l - p_j} < 2^{2m+2}$, since $p_l - p_j > 2^{-2m-2}$ by Claim 39. For every other good k, we have $\alpha_i p_k = u_{ik} - \mu_i + g_k$, where $g_k \geq 0$. We say that k is a *near-optimal* good if $g_k \leq n^2 s 2^{2m+2}$, and k is *very suboptimal* if $g_k > n^2 s 2^{2m+2}$. Note that if a good $k \in B'_i$ has equal utility to u_{ij} or u_{il} then its price is within s of p_j or p_l respectively (by Claim 38), hence $g_k \leq \alpha_i s \leq s 2^{2m+2}$, i.e. k is near-optimal.

Claim 40. *Let y be the total size of the very suboptimal goods in B'_i. If $y \leq 2^{2m+3} n^2 s$, then $\text{value}'(i) \geq v^*(i) - \epsilon/2$ and $\text{value}(i) \geq v^*(i) - \epsilon$.*

Proof. Let N_i be the set of near-optimal goods of B'_i. For every $k \in N_i$ we have $\alpha_i p_k \leq u_{ik} - \mu_i + g$, where $g = n^2 s 2^{2m+2}$. Multiplying each equation by x_{ik} and summing up over all $k \in N_i$, we get

$$\alpha_i \sum_{k \in N_i} p_k x_{ik} \leq \sum_{k \in N_i} u_{ik} x_{ik} - \mu_i \sum_{k \in N_i} x_{ik} + g \sum_{k \in N_i} x_{ik}$$

The total size of the goods in N_i is $\sum_{k\in N_i} x_{ik} \geq 1 - ns - y$. Their cost, $\sum_{k\in N_i} p_k x_{ik}$ is at least $1 - t_0 - n^2 s - ny$. Since $\text{value}'(i) \geq \sum_{k\in N_i} u_{ik} x_{ik}$ and $v^*(i) = \alpha_i + \mu_i$, we have:

$$v^*(i) - \text{value}'(i) \leq \alpha_i(t_0 + n^2 s + ny) + \mu_i(ns + y) + g$$

Since $\alpha_i \leq 2^{2m+2}$, $\mu_i \leq 1$, and from the assumed upper bounds on y and g, we conclude that $v^*(i) - \text{value}'(i) \leq \epsilon/2$. This implies as before that $\text{value}(i) \geq v^*(i) - \epsilon$. □

Thus, assume that $y > 2^{2m+3} n^2 s$. This means in particular that B'_i contains some very suboptimal goods. We distinguish cases depending on how their utility compares to the utilities u_{ij}, u_{il} of the goods j, l in the optimal bundle. We will derive in each case a contradiction.

Subcase 1. There is a very suboptimal good $k \in B'_i$ such that $u_{ij} < u_{ik} < u_{il}$. Consider step 6 for the triple j, k, l. The quantity $\Delta = (u_{il} - u_{ik})(p_k - p_j) - (u_{ik} - u_{ij})(p_l - p_k)$ is equal to $g_k(p_l - p_j)$. We have $g_k \geq n^2 s 2^{2m+2}$ and $p_l - p_j \geq 2^{-2m-2}$ (by Claim 39), thus $\Delta \geq n^2 s$. Therefore the parameter d in step 6 is $d \geq s/3n^2$, and step 6 decreases the cost by $\frac{d\Delta}{u_{il}-u_{ij}} \geq s^2/3 > 4n^6 \delta 2^{2m}$, contradicting Corollary 36.

Subcase 2. There is a very suboptimal good $h \in B'_i$ such that $u_{ih} > u_{il}$. If all the goods in B'_i have utility $\geq u_{il}$, then the value of x'_i is $\text{value}'(i) \geq (1 - ns)u_{il} \geq u_{il} - \epsilon/2 \geq v^*(i) - \epsilon/2$, and the result follows. Thus, assume that B'_i has a good k with $u_{ik} < u_{il}$. Consider step 7 for the triple k, l, h. The quantity $\Delta = (u_{il} - u_{ik})(p_h - p_l) - (u_{ih} - u_{il})(p_l - p_k)$ is equal to $g_h(p_l - p_k) + g_k(p_h - p_l)$. Since $g_h \geq n^2 s 2^{2m+2}$ and $p_l - p_k \geq 2^{-2m-2}$ (by Claim 39), it follows that $\Delta \geq n^2 s$. Thus, $d \geq s/3n^2$, and step 7 will decrease the cost again by $\frac{d\Delta}{u_{il}-u_{ij}} \geq s^2/3$, contradicting Corollary 36.

Subcase 3. All very suboptimal goods k of B'_i have $u_{ik} < u_{ij}$. Note that then all very suboptimal goods k have price $p_k \leq u_{ij} - 2^{-2m-2} < 1 - 2^{-2m-2}$ by Claim 39. We claim that B'_i must contain a good h with utility $> u_{ij}$. For, if all goods in B'_i have utility $\leq u_{ij}$, then they all have price $\leq p_j + s < 1 + s$, and then we can argue as in Case 2 that the total size of the goods of B'_i with price $\leq 1 - 2^{-2m-2}$ must be at most $2^{2m+3} n^2 s$, contradicting the fact that the size y of the very suboptimal goods of B'_i is more than $2^{2m+3} n^2 s$.

Thus, let h be a good of B_i with utility $u_{ih} > u_{ij}$, and consider Step 7 for the triple k, j, h. The quantity $\Delta = (u_{ij} - u_{ik})(p_h - p_j) - (u_{ih} - u_{ij})(p_j - p_k)$ is equal to $g_h(p_j - p_k) + g_k(p_h - p_j)$. Since $g_k \geq n^2 s 2^{2m+2}$ and $p_h - p_j \geq 2^{-2m-2}$, it follows that $\Delta \geq n^2 s$. Thus, $d \geq s/3n^2$, and step 7 will decrease the cost again by $\frac{d\Delta}{u_{il}-u_{ij}} \geq s^2/3$, contradicting Corollary 36. □

9 Discussion

As stated in the Introduction, a conclusive proof of intractability of the HZ scheme, via either a proof of FIXP-hardness for exact equilibrium or PPAD-hardness for approximate equilibrium, has eluded us. One of the difficulties is the following: Optimal bundles of agents in an HZ equilibrium may include zero-utility zero-priced goods as "fillers" to satisfy the size constraint, e.g., observe their use in Algorithm 12, for the case of dichotomous utilities. In this easy setting, we knew which were the "filler" goods. However, when faced with a complex instance of HZ, we don't a priori know which zero-utility goods will be used as "fillers". Therefore, even though an agent may have very few positive utility goods, other goods are also in play, thereby giving no "control" on the equilibrium outcome.

We propose exploring the following avenue, in addition to the usual ones, for arriving at evidence of intractability: Relax the notion of polynomial time reducibility suitably and obtain a weaker result than FIXP-hardness or PPAD-hardness.

Other open problems related to our work are: obtain efficient algorithms for computing approximate equilibria, suitably defined, and identify other special cases, besides the bi-valued case, for which equilibrium is easy to compute. Additionally, generalizations and variants of the HZ scheme deserve attention, most importantly to two-sided matching markets [EMZ19a].

Encouraged by success on the bi-valued utilities case, we considered its generalization to the tri-valued utilities case, in particular, $\{0, \frac{1}{2}, 1\}$ utilities. We believe even this case has instances with only irrational equilibria. Finding such an example or proving rationality is non-trivial and we leave it as an open problem. Furthermore, it will not be surprising if even this case is intractable; resolving this is a challenging open problem.

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