Market Equilibrium under Separable, Piecewise-Linear, Concave Utilities

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Abstract

We consider Fisher and Arrow-Debreu markets under additively-separable, piecewise-linear, concave utility functions, and obtain the following results:

- For both market models, if an equilibrium exists, there is one that is rational and can be written using polynomially many bits.
- There is no simple necessary and sufficient condition for the existence of an equilibrium: The problem of checking for existence of an equilibrium is NP-complete for both market models; the same holds for existence of an ϵ-approximate equilibrium, for $\epsilon = O(n^{-5})$.
- Under standard (mild) sufficient conditions, the problem of finding an exact equilibrium is in PPAD for both market models.
- Finally, building on the techniques of [CDDT09] we prove that under these sufficient conditions, finding an equilibrium for Fisher markets is PPAD-hard.

1 Introduction

The following was the central question within mathematical economics for almost a century: Does a complex economy, with numerous goods and a large number of agents with diverse desires and buying powers, admit equilibrium prices? Its study culminated in the celebrated Arrow-Debreu Theorem [AD54] which provided an affirmative answer under some assumptions on the utility functions (they must satisfy non-satiation and be continuous and quasi-concave) and initial endowments of the agents (each agent must have a positive amount of each commodity); these are called *standard sufficient conditions*. Over the years, milder sufficient conditions were obtained for the existence of equilibrium, see e.g. [Max97] and the references therein. In some restricted cases, the sufficient conditions were also found to be necessary, i.e., they characterized the existence of equilibria in the corresponding markets.

Besides existence, another fundamental question is efficient computability of equilibria. We note that the proof of the Arrow-Debreu Theorem was based on the Kakutani's fixed point theorem and alternative proofs are based on Brouwer's theorem; they are all therefore highly non-constructive. In fact, theorems proving the existence of market equilibria and the existence of fixed points are closely

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related and in a sense equivalent: for excess demand functions that satisfy standard conditions, the existence of an equilibrium can be derived from Brouwer's theorem, and conversely Brouwer's theorem, for general continuous functions, can be derived from the equilibrium theorem [Uz62]; the sufficient conditions on an excess demand function are continuity, homogeneity and Walras' Law, i.e., that the inner product of the price vector and the excess demand function vector be zero. Furthermore, by the Sonnenschein-Mantel-Debreu theorem, all functions satisfying these standard conditions for excess demand functions can be realized by suitable utility functions.

Scarf [Sca73] initiated the development of algorithms for computing market equilibria, introducing a family of procedures that compute approximate price equilibria by pivoting in a simplicial subdivision of the price simplex. A number of other methods, including Newton-based, homotopy methods, etc., have been developed in the following decades. These algorithms perform well in practice for several markets, but their running time is not polynomially bounded. The study of efficient computability of equilibria, from the perspective of modern theory of computation, was initiated by Megiddo and Papadimitriou [MP91]; see also Megiddo [Meg88].

In recent years there has been a surge of interest in understanding computability of market equilibria, which is in part motivated by possible applications to markets on the Internet. This study has concentrated on the two fundamental market models of Fisher [BS00] and Arrow-Debreu [AD54] (the latter is also known as the Walrasian model or the exchange model, and is more general than the Fisher model) under increasingly general and realistic utility functions. For each class of utility functions, two main algorithmic questions arise: (1) Can we determine necessary and sufficient conditions for the existence of an equilibrium? A good characterization should be efficiently checkable, hence the question can be phrased algorithmically as: What is the complexity of checking for existence of an equilibrium? (2) If suitable sufficient conditions have been established for the existence of an equilibrium, what is the complexity of finding an equilibrium for an instance satisfying these conditions?

In a general setting, e.g., for markets satisfying standard sufficient conditions, and specified by demand functions given by polynomial-time Turing machines or by explicit algebraic formulae, the computation of equilibria is (apparently) hard [Pa94, EY10]. To have any hope of efficient algorithms, we need to restrict the class of demand/utility functions. Several important classes of functions have been studied over the years.

Not surprisingly, the first results were for linear utility functions [Gal60]. If the input parameters are rational (as is standard in computer science), then there is always a rational equilibrium for this case and there are simple, efficiently checkable necessary and sufficient conditions for the existence of an equilibrium; for the Fisher model, the conditions are straightforward, and for the Arrow-Debreu model, they were given by Gale [Ga76]. Moreover, for instances satisfying these conditions, polynomial time algorithms were obtained for finding equilibria [DPSV08, Jai04].

Complexity results were also obtained for some specific non-linear utility functions that are well-studied in economics, e.g., Cobb-Douglas, CES, and Leontief; the last case is particularly relevant to this discussion. For this case, the equilibria are in general irrational for both market models [Ea76, CV04]. For the Fisher model, assuming suitable sufficient conditions, the problem of approximately computing an equilibrium is polynomial time solvable [CV04, Ye07]. For the Arrow-Debreu model, checking existence of an equilibrium is NP-hard, and for instances satisfying the standard Arrow-Debreu sufficient conditions, the computation of approximate equilibria is PPAD-hard [CSVY06, HT07, DD08]. Note that these are hardness, rather than completeness, results because these problems for Leontief markets not lie necessarily in NP and PPAD. Also note the difference in the complexities of the two market models.

Within economics, concave utilities occupy a special place, since they capture the natural condition of decreasing marginal utilities. Hence, resolving their complexity has taken center stage over the last few years. Since we are dealing with a discrete computational model, it is natural to consider piecewise-linear, concave utilities. These can be further divided into two cases, non-separable and additively separable over goods; clearly, the latter is a subcase of the former. The non-separable case contains Leontief utilities and so the hardness results mentioned above for the Arrow-Debreu model carry over to this case. However, if the number of goods is a constant, then a polynomial time algorithm exists for both market models [DK08].

This leaves the case of additively separable piecewise-linear, concave utility functions. Recently, Chen, Dai, Du and Teng [CDDT09] made a breakthrough on this question by showing PPAD-hardness of computing equilibria, even approximate equilibria, for Arrow-Debreu markets with such utilities[1].

Our results for this class of utility functions are summarized below.

- For both market models, if an equilibrium exists, there is one that is rational and can be written using polynomially many bits.

- There is no efficiently checkable necessary and sufficient condition for the existence of an equilibrium: The problem of checking for existence of an equilibrium is NP-complete for both market models; the same holds for existence of an ϵ-approximate equilibrium, for $\epsilon = O(n^{-5})$.

- Under standard (mild) sufficient conditions, the problem of finding an exact equilibrium is in PPAD for both market models. We note that this is the first result showing membership in PPAD for a market model defined by an important, broad class of utility functions.

- Finally, building on the techniques of [CDDT09] we prove that under these sufficient conditions, finding an equilibrium for Fisher markets is PPAD-hard.

Observe that, unlike the Leontief case, the two market models turn out to have the same complexity in this case.

Thus the results of the present paper, together with [CDDT09, CT09], establish that the equilibrium computation problem for a broad, natural class of markets is characterized exactly by the class PPAD; an analogous role in game theory is played by the class of 2-player Nash equilibrium games.

A significant contribution of our work to complexity theory is a new way of proving membership of a problem in PPAD. Previous proofs, for other problems, involved either explicitly giving a path following algorithm, in the style of algorithm of Lemke-Hawson [LH], or reducing the given problem to a known problem in PPAD such a 2-player Nash. Our proof first uses a new characterization of PPAD given in [EY10] – this yields partial information about the sought-after market equilibrium. This information enables us to construct a certain linear program, which can be solved in polynomial time to obtain the equilibrium itself; interestingly enough, this LP also yields the result about rationality of equilibria mentioned above.

We remark that two of these results were obtained independently and concurrently by other authors: rationality was also proven by Devanur and Kannan for both market models [DK08] and

[1] Their initial claim, that the problem of finding an approximate equilibrium lies in PPAD, has been recently rescinded.

PPAD-hardness for Fisher markets was proven by Chen and Teng [CT09] (as noted in both these papers).

We also remark that in a recent paper, Ye showed a distinction between Fisher and Arrow-Debreu markets for a related, though different, model of piecewise-linear concave utility functions; in particular, he showed that the Fisher case can be solved in polynomial time, whereas the Arrow-Debreu case is equivalent to solving a linear complementarity problem [Ye07]. In Section 2 we will explain how his model is different from ours.

How does the "invisible hand of the market," in Adam Smith's famous words, find equilibria? The intractability results of [CDDT09, CT09] and the current paper make this question even more mysterious.

1.1 Techniques used

Our results involve several novel techniques; below we give an overview primarily for the positive results (the first and third results in the list in the Abstract).

The combinatorial algorithm for Fisher's linear case [DPSV08] gave new insights into the combinatorial structure underlying equilibrium prices and allocations. Given prices $\boldsymbol{p}$, [DPSV08] showed how to construct a suitable network such that a max-flow in it helped determine if $\boldsymbol{p}$ are equilibrium prices.

We first extend this structure to the case of separable, piecewise-linear, concave utilities; the main difference being that in this case, in general, at given prices $\boldsymbol{p}$, a buyer's optimal bundle *must* include certain quantities of certain goods – these are called *forced allocations*. The money that is left over after buying forced allocations is to be spent on buying *flexible allocations* from a suitable subset of goods with specified upper bounds on quantity, and *any* allocation exhausting the left-over money leads to an optimal bundle.

Our network is also a function of prices $\boldsymbol{p}$ and incorporates information about forced allocations and the choices available for flexible allocations. Again, a max-flow in this network helps determine if $\boldsymbol{p}$ are equilibrium prices (see Lemma 1). The problem of finding a max-flow in this network can be written as an LP in a straightforward manner.

The next transformation is the most interesting. We assume that prices $\boldsymbol{p}$ are now variables and the network is constructed for a *guess* on forced allocations and choices available for flexible allocations. It turns out that all edge capacities in this network are linear functions of the price variables. Moreover, max-flow in this network, which is a function of prices, can still be written as an LP. We then show if the guess is good, i.e., corresponds to an equilibrium, then the optimal solution to this new LP gives the corresponding equilibrium prices and allocations. Since the solution to an LP is rational, the theorem follows.

Because of rationality, equilibria for these markets can be computed exactly and this leads to the possibility that these problems may lie in PPAD, under suitable sufficient conditions. We show that this is indeed the case for both market models; this is the technically most involved result of our paper.

There are very few ways for showing membership in PPAD. A promising approach for our case is to use the characterization of PPAD of [EY10] as the class of exact fixed point computation problems for piecewise-linear, polynomial time computable Brouwer functions. The Brouwer functions that have been proposed for market equilibria, such as those of Geneakoplos and McKenzie, are the obvious candidates. Unfortunately, we do not see how to do this: Although it is possible to show that these functions are polynomial time computable (this is nontrivial, e.g., for the Geneakoplos function), it is not clear how to transfer the piecewise-linearity of the utility functions to the Brouwer function.

Another approach is to reduce the problem to the computation of an approximate fixed point for a suitable general (not necessarily piecewise-linear) Brouwer function F that satisfies three conditions: (i) it is polynomially continuous (for example, Lipschitz continuous for a Lipschitz constant that is $O(2^{poly(n)})$), (ii) it is polynomial-time computable, and (iii) any weakly approximate fixed point of the function can be used to efficiently obtain a desired solution, e.g., a price equilibrium in our case (see [EY10] for a proof). By a *weakly approximate fixed point* we mean a point x such that $|F(x) - x|$ is small. However, such a point may be far from all the fixed points, and this makes task (iii) challenging[2].

The task is further complicated by the fact that, for given prices, the demand, i.e., optimal bundle, of an agent is in general not unique, i.e., it is a correspondence and a not function. Furthermore, this correspondence is very sensitive to the prices – an extremely small change in prices may lead to drastic changes in the demand.

Instead, we employ a combination of the two approaches. Let $\mathcal{M}$ be an instance of a market in the class defined above. We start with the correspondence F of a Kakutani Theorem-based proof of existence of equilibrium for $\mathcal{M}$; this is a correspondence on pairs of price and allocation vectors, (p, x), such that the price components of its fixed points correspond to the set of price equilibria for $\mathcal{M}$. We next obtain a piecewise-linear Brouwer function G that approximates F. The function G is easily computable, and hence finding an exact fixed point, (p^*, x^*), for it is in PPAD, by the characterization of PPAD given in [EY10]. Clearly, (p^*, x^*) may not be a fixed point of F. In addition, it may not even be close to any fixed point of F.

The heart of the proof lies in showing how to efficiently compute a price equilibrium p' for $\mathcal{M}$ from the fixed point (p^*, x^*) of G. For this, we show several properties of the fixed point (p^*, x^*) that allow us to identify which allocations should be forced and which flexible in an equilibrium, i.e., to pin down the combinatorial essence of the problem. We set up an LP, similar to the one used for proving rationality for the specification of flexible and forced allocations derived from (p^*, x^*), but with the constraints relaxed by a variable error amount ϵ. The objective function of the LP is to minimize ϵ. We use the properties of the fixed point (p^*, x^*) to show that it induces a feasible solution to the LP with a very small value of $\epsilon = 2^{-2m}$, where m is a parameter of the market instance $\mathcal{M}$ that upper bounds the bit complexity of an optimal solution to the LP, i.e., the size of the LP and bounds on its coefficients imply that the optimal solution to it must be either 0 or at least 2^{-m}; hence it must be zero. Therefore, solving the LP gives us an exact price equilibrium for market $\mathcal{M}$, say p'. Note that the entire computation involves finding a fixed point of G, a piecewise-linear Brower function, followed by a polynomial time computation. Since this can all be accomplished in PPAD, we get the desired membership result.

Observe that the function G is a Brouwer function, so it has a fixed point (p^*, x^*) regardless of whether the given market has an equilibrium or not. Obviously we cannot derive from (p^*, x^*) a market equilibrium if there is none, so the proof of correctness for the constructed price vector p' has to crucially use the fact that the given market instance satisfies the standard sufficient conditions for the existence of an equilibrium. Moreover, the proof must simultaneously show (constructively, in polynomial time) their sufficiency. Can we expect this procedure and proof to

[2]For example, this is the reason that we cannot place in PPAD the approximation of Nash equilibria in 3-player games. If we could do this, then this would have other important consequences; e.g., it would resolve the longstanding open problem of determining whether the square root sum problem is in NP [EY10].

work for all piecewise-linear markets that have an equilibrium, i.e., even ones not satisfying the sufficient conditions? In view of the NP-completeness of the existence problem, the answer is "No"; indeed, if this were the case, then NP would be contained in PPAD, which would imply NP=coNP.

We comment briefly on the negative results (the second and fourth results). We exploit the fact that the high sensitivity of the demands (optimal bundles) to small changes in prices can be combined with well-chosen "pieces" of the piecewise-linear utility functions to give the problems a discrete feel: an agent either buys a segment of a good completely or not at all, depending on how the prices of goods compare with each other. With a careful encoding, this discreteness can be reflected in the choices of the prices in the potential equilibria.

2 Fisher's model with piecewise-linear, concave utilities

Fisher's market model [BS00] is the following. Let G be a set of divisible goods and B be a set of buyers, $|G| = g$, $|B| = n$. Assume that the goods are numbered from 1 to g and the buyers are numbered from 1 to n. Each buyer $i \in B$ comes to the market with a specified amount of money, say $e(i) \in \mathbf{Q}^+$ dollars. We will assume w.l.o.g. that the amount of each good available is unit. For each buyer i and good j we are specified a function $f_j^i : \mathbf{R}^+ \to \mathbf{R}^+$ which gives the utility that i derives as a function of the amount of good j that she receives. Her overall utility, $u_i(x)$ for a bundle $x = (x_1, \ldots, x_g)$ of goods is additively separable over the goods, i.e., $u_i(x) = \sum_{j \in G} f_j^i(x_j)$. Let $M = \sum_{i \in B} e(i)$ denote the total money of all buyers.

In this paper, we will deal with the case that the f_j^i's are (non-negative) non-decreasing piecewise-linear, concave functions. Given prices $\boldsymbol{p} = (p_1, \ldots, p_g)$ for all the goods, consider bundles (baskets) of goods that make each buyer i happiest (there could be many such bundles). We will say that $\boldsymbol{p}$ are *market clearing prices* if there are choices of optimal bundles for the buyers, such that after each buyer is given an optimal bundle, there is no deficiency or surplus of any good, i.e., the market clears.[3] The problem is to find such market clearing or *equilibrium prices*.

Remark. Ye uses a somewhat different model of piecewise-linear concave utility functions in [Ye07]. Specifically, the utility function $u_i(x)$ of buyer i for a bundle $x = (x_1, \ldots, x_g)$ of goods is a function of the form $\min_k u_i^k(x)$, where each $u_i^k(x)$ is a homogeneous, linear function of the form $u_i^k(x) = \sum_{j \in G} u_{ij}^k x_j$. A utility function $u_i(x)$ in our model can be expressed as the minimum of a set of linear functions, but (i) an exponential number of functions will be needed in general, and (ii) the functions are not homogeneous. $\square$

We will call each piece of f_j^i a *segment*. The set of segments defined in function f_j^i will be denoted $\mathrm{seg}(f_j^i)$. The slope of a segment specifies the rate at which the buyer derives utility per unit of good received. Suppose one of these segments, s, has range $[a, b] \subseteq \mathbf{R}^+$, and a slope of c. Then, we will define $\mathrm{amount}(s) = b - a$, $\mathrm{slope}(s) = c$, and $\mathrm{good}(s) = j$. We will assume that for each segment s specified in the problem instance, $\mathrm{slope}(s)$ and $\mathrm{amount}(s)$ are rational numbers.

[3] In general markets, this requirement is imposed only on goods with positive prices, while for goods with zero price, it is only required that the demand does not exceed the supply, i.e., goods with zero price need not be fully sold. Since the utility functions here are non-decreasing, all the goods with zero price can be always fully distributed to the agents without decreasing their utility.

Let segments(i) denote the set of all segments of buyer i, i.e.,

$$\text{segments}(i) = \bigcup_{j=1}^{g} \text{seg}(f_j^i).$$

We will assume that the given problem instance satisfies the following (mild) condition; as shown in Section 6, under this assumption, the instance is guaranteed to have an equilibrium.

- For each buyer, there is a good whose entire unit and more gives her positive utility, i.e.,

$$\forall i \in B \ \exists j \in G : \sum_{s \in \text{seg}(f_j^i),\ \text{slope}(s) > 0} \text{amount}(s) > 1.$$

Under this condition, there is always a price equilibrium; our proof that the computation of an equilibrium is in PPAD includes also a proof of this fact.

3 Testing if given prices $\boldsymbol{p}$ are equilibrium prices

Given an instance $\mathcal{M}$ of Fisher's market with piecewise-linear, concave utilities and prices $\boldsymbol{p}$ of goods, we first show how to determine if $\boldsymbol{p}$ constitute equilibrium prices. We will assume that $\boldsymbol{p}$ satisfies the condition that the sum of prices of all goods equals the total money of the buyers, i.e.,

$$\sum_j p_j = \sum_i e(i).$$

3.1 Bang per buck and allocations

Given nonzero prices $\boldsymbol{p} = (p_1, \ldots, p_g)$, we characterize optimal baskets for each buyer relative to $\boldsymbol{p}$. Define the *bang per buck* relative to prices $\boldsymbol{p}$ for segment $s \in \text{seg}(f_j^i), j \neq 0$, to be $\text{bpb}(s) = \text{slope}(s)/p_j$. Sort all segments $s \in \text{segments}(i)$ by decreasing bang per buck, and partition by equality into classes: $Q_1, Q_2, \ldots$.

If for segment s, $\text{good}(s) = j$, then the *value of segment* s, $\text{value}(s) = \text{amount}(s) \cdot p_j$. For a class Q_l, define value(Q_l) to be the sum of the values of segments in it. At prices $\boldsymbol{p}$, goods corresponding to segments in Q_l make i equally happy, and those in Q_l make i strictly happier than those in Q_{l+1}.

Find $k_i \geq 1$ such that

$$\sum_{l < k_i} \text{value}(Q_l) \leq e(i) < \sum_{1 \leq l \leq k_i} \text{value}(Q_l).$$

At prices $\boldsymbol{p}$, i's optimal allocation must contain goods corresponding to all segments in $Q_1, \ldots, Q_{k_i - 1}$, and a bundle of goods worth $e(i) - (\sum_{1 \leq l \leq k_i - 1} \text{value}(Q_l))$ corresponding to segments in Q_{k_i}. We will say that for buyer i, at prices $\boldsymbol{p}$, $Q_1, \ldots, Q_{k_i - 1}$ are her *forced partitions*, Q_{k_i} is her *flexible partition*, and $Q_{k_i + 1}, \ldots$ are her *undesirable partitions*. Similarly, segments in these three sets will be called *forced, flexible and undesirable segments*, respectively.

For buyer i, we will denote the amount of money spent on forced segments by

$$\text{spent}(i) = \sum_{l < k_i} \text{value}(Q_l).$$

Define unspent$(i) = e(i) - \text{spent}(i)$. For good j, let forced(j) denote the amount of good j sold to all buyers under their forced allocations and let unsold$(j) = 1 - \text{forced}(j)$.

3.2 The network

First ensure that for each buyer, i, unspent$(i) \geq 0$ and for each good j, unsold$(j) \geq 0$; otherwise, $\boldsymbol{p}$ do not constitute equilibrium prices.

The network $N(\boldsymbol{p})$ is defined over vertices $\{s\} \cup G \cup B \cup \{t\}$, where s and t are its source and sink. For each good j, there is edge (s, j) with capacity unsold$(j) \cdot p_j$ and for each buyer i, there is edge (i, t) with capacity unspent(i). For each buyer i, $N(\boldsymbol{p})$ will contain an edge (j, i) corresponding to each segment s in its flexible partition, Q_{k_i}, where good$(s) = j$; the capacity of this edge is amount$(s) \cdot p_j$.

The following is straightforward.

Lemma 1 *Prices $\boldsymbol{p}$ constitute equilibrium prices iff max-flow in $N(\boldsymbol{p})$ is*

$$\sum_{i \in B} \text{unspent}(i).$$

Proof : We observe that the capacity of cut $(\{s\}, G \cup B \cup \{t\})$ is $\sum_i \text{unspent}(i)$. Furthermore, by the assumption $\sum_j p_j = \sum_i e(i)$, the cut $(\{s\} \cup G \cup B, \{t\})$ also has the same capacity.

Hence, the market clears at prices $\boldsymbol{p}$ iff max-flow in $N(\boldsymbol{p})$ is

$$\sum_{i \in B} \text{unspent}(i).$$

$\square$

4 Rationality of equilibrium prices for Fisher's model

We next prove that the given market $\mathcal{M}$ must have rational equilibrium prices which can be written using polynomially many bits.

Let $\boldsymbol{p}'$ be any equilibrium prices for $\mathcal{M}$. Consider all forced allocations made at equilibrium. For each buyer i, let Q_{k_i} denote i's flexible partition in this equilibrium and let L_i denote the set of goods of the segments in Q_{k_i}. Let $R_i = G - L_i$ be the remaining goods. For $j \in L_i$, let s_{ij} denote the segment of good j that is in Q_{k_i}. For $j \in R_i$, let s_{ij} denote the last segment of good j that is fully allocated to i and let s'_{ij} denote the next (unallocated) segment; if no segment of good j is allocated to i, then $s_{ij} = \phi$.

Next, we will construct an LP which will have a variable, p_j, corresponding to each good j, and any optimal solution to this LP will be equilibrium prices. The equilibrium $\boldsymbol{p}'$ considered above must be one of its solutions and since the LP has only rational parameters, it must have a rational solution as well, thereby completing the proof.

First write spent(i) and unspent(i) for each buyer as linear polynomials using the variables p_j's. For each good j, unsold(j) is a constant determined by the forced allocations and hence the left-over value of this good, unsold$(j) \cdot p_j$ is a linear expression. Construct the network, say N, described in Section 3.2, except that the capacities of edges will be linear polynomials in the p_j's.

We will add edge (t, s) of unbounded capacity to the network. This constitutes the set E if edges of N. Next, we introduce a variable f_e corresponding to each edge e in N, which will represent the flow on this edge.

We can finally describe the LP itself. Its objective is to maximize

$$f_{(t,s)} + \sum_{i \in B} \text{spent}(i),$$

subject to capacity constraints on each edge $e \in E - \{(t, s)\}$ and a flow conservation equation for each vertex in $\{s, t\} \cup G \cup B$.

In addition, for each buyer i, it has the following constraints to ensure that the forced and flexible segments of i satisfy desired properties; eventually this ensures that i indeed gets a utility maximizing bundle of goods. For each $j, j' \in L_i$, we have the equation:

$$\text{slope}(s_{ij}) \cdot p_{j'} = \text{slope}(s_{ij'}) \cdot p_j.$$

For each $j \in L_i$ and $j' \in R_i$, if $s_{ij'} \neq \phi$, we have the two inequalities:

$$\text{slope}(s_{ij}) \cdot p_{j'} \geq \text{slope}(s_{ij'}) \cdot p_j$$

$$\text{slope}(s_{ij}) \cdot p_{j'} \leq \text{slope}(s'_{ij'}) \cdot p_j.$$

If $s_{ij'} = \phi$, we have one inequality:

$$\text{slope}(s_{ij}) \cdot p_{j'} \leq \text{slope}(s'_{ij'}) \cdot p_j.$$

We also add the following constraints using the linear expressions derived above:

$$\forall i \in B : \quad \text{unspent}(i) \geq 0$$

$$\forall j \in G : \quad \text{unsold}(j) \geq 0$$

$$\sum_{j \in G} p_j = M,$$

where M is the total money of all buyers.

Finally, we add non-negativity constraints:

$$\forall e \in E : \quad f_e \geq 0$$

$$\forall j \in G : \quad p_j \geq 0.$$

Theorem 2 *Every Fisher market with additively-separable piecewise-linear, concave utilities and all parameters rational that has an equilibrium admits equilibrium prices which are rational numbers that can be written using polynomially many bits.*

Proof : Clearly, the starting equilibrium prices $\boldsymbol{p}'$ form an optimal solution, of value M, for the LP constructed. The theorem follows from Lemma 1 and the fact that this LP must have an optimal rational solution. □

5 Rationality for Arrow-Debreu Markets

An Arrow-Debreu market [AD54] under piecewise-linear, concave utilities differs from a Fisher market only in that the agents do not come to the market with money but with initial allocations of goods; each of the goods still totals 1 unit, w.l.o.g. For any prices of the goods, the agents sell all their initial endowments at these prices and use the money to buy optimal baskets. The problem again is to find market clearing prices.

The main change needed in the proof of Theorem 2 to prove an analogous statement for these markets as well, is that at given prices of goods, $\boldsymbol{p}$, we will let e_i denote the total value of i's initial endowment. If $\boldsymbol{p}$ is a vector of variables, then e_i will be a linear sum in these variables. The sum M of the prices can be set arbitrarily (if p is an equilibrium in an Arrow-Debreu market, then αp is also an equilibrium for all $\alpha > 0$), thus we can set w.l.o.g. $M = 1$. The rest of the proof is same as before. Hence we get:

Theorem 3 *Let $\mathcal{M}$ be an Arrow-Debreu market with additively-separable piecewise-linear, concave utilities and all parameters rational. If $\mathcal{M}$ has an equilibrium, then it admits an equilibrium in which prices are rational numbers that can be written using polynomially many bits.*

6 Membership in PPAD for Arrow-Debreu and Fisher Markets

Consider the Arrow-Debreu market with a set B of n agents (buyers) $1, \ldots, n$ and a set G of g goods $1, \ldots, g$. Each agent i has a given initial endowment (supply) vector $w(i) = (w_{i1}, \ldots, w_{ig}) \geq 0$ of goods. We may assume without loss of generality that the total initial supply of each good j is equal to 1, i.e., $\sum_i w_{ij} = 1$. We are also given for each $i \in B$ and $j \in G$, a (nonnegative, nondecreasing) concave piecewise linear function f^i_j which is the utility function of agent i for good j. The utility of agent $i \in B$ for an allocation vector $x = (x_1, \ldots, x_g)$ of goods is $u_i(x) = \sum_{j \in G} f^i_j(x_j)$.

We assume for computational purposes that all the input numbers are rationals with at most b bits in numerator and denominator. This includes the initial endowments w_{ij}, and the slopes and lengths (and breakpoints) of all the segments of all the utility functions f^i_j.

We want to compute a price equilibrium p, i.e., a vector $p = (p_1, \ldots, p_g)$ of prices for the goods that belongs to the unit g-simplex $S = \{p | p \geq 0, \sum_j p_j = 1\}$, such that there is an allocation $x = (x_{ij})$ of goods to the agents such that (a) each agent i receives an optimal bundle for his budget, i.e., the subvector $x(i) = (x_{i1}, \ldots, x_{ig})$ maximizes $u_i(x(i)) = \sum_{j \in G} f^i_j(x_{ij})$ subject to $\sum_{j \in G} p_j x_{ij} \leq \sum_{j \in G} p_j w_{ij}$ and $x(i) \geq 0$, and (b) the market clears, i.e. $\sum_{i \in B} x_{ij} = \sum_{i \in B} w_{ij} = 1$ for all $j \in G$.

As is well known, a Fisher market F can be reduced to an Arrow-Debreu market D with the same set G of goods, the same set B of agents, and the same utility functions. Assume w.l.o.g. that the total supply of each good in F is 1 and that the sum of the budgets of the agents is also 1. If an agent i has budget e_i in F, then his initial endowment $w(i)$ in D contains the same amount $w_{ij} = e_i$ for each good $j \in G$. Then a price vector p is an equilibrium in F if and only if p is an equilibrium in D. Thus, Fisher markets correspond essentially to the special case of Arrow-Debreu markets, where every agent's endowment contains the same amount of each good.

From the Arrow-Debreu theorem, a sufficient condition for the existence of an equilibrium for an Arrow-Debreu market in our setting is that (C1) all agents i have positive initial endowments w_{ij} for all goods j, and (C2) non-satiation of the agents' utility functions: for every bundle, there is another bundle that gives strictly more utility to each agent. In our case of piecewise linear functions, (C2) can be equivalently stated as: for every agent $i \in B$, there is a good $j \in G$ such that $\lim_{x\to\infty} f^i_j(x) = \infty$, i.e., the last (infinite) segment of f^i_j has positive slope; we will say that agent i is *non-satiated* with respect to good j, and the function f^i_j is non-satiated. Since the initial total supply of each good is assumed to be 1, it suffices actually to assume that each agent derives increasing utility from some good up to an amount greater than 1 (i.e. the utility function f^i_j could go flat after some value > 1).

Some weaker sufficient conditions for the existence of an equilibrium have been shown subsequently by other authors. In particular, Maxfield [Max97] showed a sufficient condition in terms of the following *economy graph*: The graph has a node for each agent $i \in B$ and has an arc $i \to j$ if there is a good $k \in G$ such that $w_{ik} > 0$ and j is non-satiated with respect to k. The sufficient condition is: (C') The economy graph is strongly connected. Clearly, (C') implies (C2), and the conjunction of (C1) and (C2) implies (C'). Note that in a Fisher market, each agent has a positive initial budget e_i, and thus, when we express a Fisher market in the Arrow-Debreu framework with an initial endowment $w(i) = (e_i, \dots, e_i)$, condition (C1) is automatically satisfied; in this case (C2) and (C') are equivalent.

We will show that, under the above sufficient conditions, the problem of computing a (exact) price equilibrium is in PPAD. As part of the proof, we will show also the sufficiency of the conditions for the existence of an equilibrium.

Theorem 4 *The problem of computing a (exact) price equilibrium for an Arrow-Debreu market with additively-separable, piecewise-linear concave utility functions, satisfying the condition (C'), is in PPAD. The same is true for the Fisher market under the condition (C2).*

In the rest of this section we will show Theorem 4. We are given an instance of an Arrow-Debreu market as above, satisfying the sufficient condition (C'). Let's trim each utility function f^i_j so that it goes flat after 1.1 unit of good j. Recall that the total supply of each good is 1, thus the trimming does not change the price equilibria. The purpose of the trimming is to get bounded allocations. Let S be the unit g-simplex for the prices, $S = \{p | p \geq 0, \sum_j p_j = 1\}$, and let D be the box $[0, 1.1]^{ng}$ of possible demand vectors of the agents. Use x_{ij} to denote the amount of good j bought by buyer i and let $x_j = \sum_i x_{ij}$ (the total demand for good j in x), and $x(i)$ the subvector of x for buyer i. We'll use the shorthand px for the cost of allocation x under prices p, i.e. $px = \sum_{ij} p_j x_{ij}$. Consider the correspondence mapping F from $S \times D$ to itself which takes a pair (p, x) consisting of a price vector p and demand vector x and maps it to the set of all pairs (p', x') where: p' is a price vector that maximizes $p'x = \sum_{ij} p'_j x_{ij}$ subject to $p' \in S$, and x' is a demand vector that consists of optimal budget-feasible bundles (in D) for the buyers under prices p. Since every buyer derives positive utility for a good only up to 1.1 unit, the demand for each good from each buyer is restricted to be ≤ 1.1 so $x' \in D$. We'll denote by $F1, F2$ the sets of $p-$ and $x-$ components of the mapping F.

A point (x, p) is a *fixed point* of F if $(x, p) \in F(x, p)$. All fixed points of F yield price equilibria (cf. [Sca73], section 5.4) and vice-versa. We include a proof below for reference (although we do not actually need it for proving Theorem 4).

Proposition 5 *1. If $(p^*, x^*) \in F(p^*, x^*)$ then p^* is a price equilibrium.*
2. If p^* is a price equilibrium then there exists a $x^* \in D$ such that $(p^*, x^*) \in F(p^*, x^*)$.

Proof : 1. Suppose $(p^*, x^*) \in F(p^*, x^*)$. For every point (p, x) and every image point $(p', x') \in F(p, x)$, the bundles x' bought by each buyer are within budget (according to p), hence $\sum_{ij} p_j x'_{ij} \leq$

$\sum_{ij} p_j w_{ij} = \sum_j p_j = 1$. Thus, $\sum p^*_j x^*_{ij} \le 1$. The vector p^* maximizes $\sum_{ij} p_j x^*_{ij}$ over all price vectors $p \in S$, thus, $\sum_{ij} p_j x^*_{ij} \le 1$ for all $p \in S$. In particular considering the vector p that has $p_j = 1$ for a good j and $p_k = 0$ for $k \ne j$ we conclude that the total demand for good j is $x^*_j = \sum_i x^*_{ij} \le 1$, and this holds for all goods j. That is, demand $x^* \le$ supply for all goods.

Suppose $x^*_r < 1$ for some r. Then either $p^*_r = 0$ or $\sum_j p^*_j x^*_j < 1$ (since all $x^*_j \le 1$ and $\sum_j p^*_j = 1$). Note: in the latter case we must have that all $x^*_j < 1$ because p^* maximizes the $\sum p^*_j x^*_j$ and if one x^*_j was =1 then we could set $p^*_j = 1$. In fact all the x^*_j with non-zero p^*_j must have the same value. If $\sum_j p^*_j x^*_j < 1$, this means that some buyers did not spend their whole budget; this contradicts non-satiation (of the original functions): if such a buyer i gets positive utility from some good j until 1.1 unit at least, then he would buy more of the good and either spend more money if $p^*_j > 0$, or reach 1.1 unit of the good if $p^*_j = 0$. We conclude that under non-satiation, we must have $\sum p^*_j x^*_j = 1$, which implies that if $x^*_r < 1$ then we must have $p^*_r = 0$. (This can happen, for example if nobody cares about good r.) We can modify x^* by assigning the remaining amount of any such good r to any agent, at 0 cost (and no change in utility), so that the market clears.

2. Conversely, suppose that p^* is a price equilibrium. Then there are optimal bundles for the buyers, resulting in a demand vector x^* that clears exactly the market. Because of non-satiation of the original utility functions, all buyers spend their money under p^*, because otherwise they could keep getting more utility either by spending more money if the good that gives them more utility has price > 0 , or by exceeding the 1 unit for the good if it has price =0. Since all buyers spend their money, $\sum_j p^*_j x^*_j = 1$. Clearly, for every p in S, $\sum_j p_j x^*_j \le 1$, since $x^*_j \le 1$ for all j, so p^* maximizes the sum and $(p^*, x^*) \in F(p^*, x^*)$. □

Recall that all input numbers are rationals with at most b bits in numerator and denominator. Assume wlog that the slopes of all non flat segments of the utilities are integers > 0. Let t be the total number of segments in the utility functions. Let m be the number of bits that suffice in the optimal solution of LPs with at most $3ng$ variables, $t^2 + 5ng$ constraints, and rational coefficients of bit complexity $2b$. Note that m is polynomially bounded in n, g, t, b, and $m >> n, g, t, b$. Let $\delta = 1/2^{10m}$.

Consider a regular simplicization of the domain $S \times D$ with resolution δ. Every cell (little simplex) in the simplicization has rational vertices which are equal in each coordinate or differ by δ. Define a function G which picks at each vertex (p, x) of the simplicization an arbitrary element of $F(p, x)$, and is extended to the domain $S \times D$ by linear interpolation. By definition, G is continuous but its modulus of continuity could be very large: any two vertices in the same simplex are within δ of each other in each coordinate, but their images may be very far apart; for example, a very small change in a price may change the relative bang-per-buck order of two segments for two goods in the utility function of a buyer, and thus cause a drastic change in the optimal bundle. Note that the vertices of the simplicization have rational coordinates of bit complexity polynomial in the input, and that for any given vertex (p, x) we can compute in polynomial time an optimal p' for x and an optimal x' for p, i.e., we can compute an element of $F(p, x)$. The function G is a polynomial piecewise linear function, so computing a (exact) fixed point (p^*, x^*) of it is in PPAD [EY10]. Note that a fixed point of G is not a fixed point of F, so we still have work to do. Let C be the simplex that contains (p^*, x^*). (It could be that it lies on a smaller dimensional face of the simplicization.) (p^*, x^*) can be written as a convex combination of the vertices of C, and $G(p^*, x^*) = (p^*, x^*)$ is also the same convex combination of the G-values of the vertices.

We show first that the demands of all goods in x^* are approximately bounded by the supplies.

Lemma 6 *For every vertex (p, x) of C, for every good j, the total demand for good j in x is $x_j \leq 1 + 4ng\delta$. Hence the same is true for x^*.*

Proof : Suppose some $x_j > 1 + 4ng\delta$ and let $G(p, x) = (p', x')$. Since p' maximizes $p'x$, it will give positive values only to goods j that have $x_j > 1 + 4ng\delta$. Every other vertex of the simplex differs at most by δ from (p, x) in each coordinate, hence it has also some goods with total demand $> 1 + (4ng - n)\delta$, and will be mapped by G to a p-vector that is positive only in such coordinates. We conclude that all the vertices of the simplex C are mapped by G to price vectors that are positive in goods for which the total demand in x^* is $> 1 + (4ng - 2n)\delta > 1 + 2ng\delta$. Hence the same is true for p^* (which is a convex combination of the G-images of the vertices) and thus $p^*x^* > 1 + 2ng\delta$.

On the other hand, at each vertex $v = (p_v, x_v)$, the demand vector of its G-image, x'_v is a budget-feasible allocation for p_v, thus $p_v x'_v \leq 1$. Since the p_v are within δ of each other in each coordinate, for any two vertices v, w we have $p_w x'_v \leq p_v x'_v + \delta \sum_j (x'_v)_j \leq 1 + 1.1ng\delta$ (since each coordinate of x'_v is ≤ 1.1). Since (p^*, x^*) is a fixed point of G, p^* is a convex combination of the p_v and x^* is the same convex combination of the x'_v, i.e. we can write p^*x^* as $(\sum \lambda_w p_w)(\sum \lambda_v x'_v)$ where the summations range over all vertices of C. Therefore, $p^*x^* \leq 1 + 1.1ng\delta$, contradiction. The claim follows $\square$

We show next that x^* is approximately budget-feasible for all agents with respect to prices p^*.

Lemma 7 *For each agent i, $\sum_j p^*_j x^*_{ij} \leq \sum_j p^*_j w_{ij} + 2.2g\delta$ (i.e. (p^*, x^*) is "almost" budget-feasible for each agent). Also, $\sum_j p^*_j x^*_{ij} \geq \sum_j p^*_j w_{ij} - 2.2g\delta$ and $\sum_{i,j} p^*_j x^*_{ij} \geq 1 - 2.2g\delta$.*

Proof : For each vertex (p, x) of the simplex C, the demand vector x' of its image $G(p, x) = (p', x')$ is budget feasible with respect to p. For every good j, $|p^*_j - p_j| \leq \delta$, thus $\sum_j p^*_j x'_{ij} \leq \sum_j p_j x'_{ij} + 1.1g\delta \leq \sum_j p_j w_{ij} + 1.1g\delta \leq \sum_j p^*_j w_{ij} + 2.2g\delta$. Since x^* is a convex combination of the x' for the vertices of C, the first claim follows.

Suppose that a vertex (p, x) is mapped by G to (p', x') with $\sum_j p_j x'_{ij} < \sum_j p_j w_{ij}$. Then x' must include all segments with positive slope of the utility functions of i and some good j will be at level 1.1 (because of non-satiation of the original untrimmed functions). Furthermore, since prices differ at most by δ at the vertices of C, if $\sum_j p_j x'_{ij} < \sum_j p_j w_{ij} - 1.1g\delta$ at some vertex, then $\sum_j p_j x'_{ij} < \sum_j p_j w_{ij}$ at all vertices, good j is bought at level 1.1 in the optimal bundle of buyer i at all vertices, and therefore also at x^*, contradicting Lemma 6. Therefore $\sum_j p_j x'_{ij} \geq \sum_j p_j w_{ij} - 1.1g\delta$ at each vertex of C. As above it follows then that $\sum_j p^*_j x'_{ij} \geq \sum_j p_j x'_{ij} - 1.1g\delta \geq \sum_j p_j w_{ij} - 2.2g\delta$ for every vertex, and hence $\sum_j p^*_j x^*_{ij} \geq \sum_j p^*_j w_{ij} - 2.2g\delta$. Therefore, $\sum_{i,j} p^*_j x^*_{ij} \geq 1 - 2.2g\delta$. $\square$

Consider the utility function f^i_j of agent i for good j, and the l-th segment of the function; let s_{ijl} be its slope and suppose the segment runs from amount c_{ijl} to $c_{ij,l+1}$ for the good j. For a demand vector x, we say that the segment is *empty* (resp. *full*) if $x_{ij} \leq c_{ijl}$ (resp. $\geq c_{ij,l+1}$); we say it is *partial* if x_{ij} is between the two amounts. For each good j, there is a last segment which is full and a first segment which is empty; either there is a partial segment which is between the two - we call this the *active* segment - or the two segments are consecutive and the amount x_{ij} is the common breakpoint. Let us say that a segment is *almost full* if it is full to a fraction $> 1 - 2^{-2m}$ of its length and *almost empty* if it has $< 2^{-2m}$ fraction of its length. Recall that $m >> n, g, t, b$. Condition (C') (or C1) is needed for the following lemma.

Lemma 8 *1. All agents have budget (income) at least $1/2^m$ at p^* and at all vertices of the simplex C.*
2. Suppose that $p^*_j < 2^{-3m}$ for some good $j \in G$. Then all the segments of good j that have positive slope are full in the demand vector $x' = G2(p,x)$ for every vertex (p,x) of the simplex C, and are also full in the demand vector x^*.

Proof : 1. If we assume condition (C1), then this is obvious: for any $p \in S$, at least one price $p_k \geq 1/g$, and since $w_{ik} \geq 2^{-b}$, it follows that $p_k w_{ik} \geq 1/(2^b g) \geq 1/2^m$.

For the weaker condition (C'), consider the economy graph Γ, and let z be an agent whose initial endowment has a positive amount of a good with price $p^*_j \geq 1/g$ in p^*; then the budget of z in p^* is at least $1/(2^b g)$, and the budget of z at every vertex of C differs at most by $g\delta$. Since Γ is strongly connected, all nodes can reach node z. We will show for each node y, by induction on the distance d from node y to z, that the budget of y in p^* is at least $2^{-(3d+1)b} \cdot (2g)^{-(d+1)}$, and again the budget at all vertices of C differs at most by $g\delta$. This implies that the budget of all agents at p^* and at all vertices of C is $\geq 1/2^m$.

The basis, $d = 0$, is clear. For the induction step, consider a shortest path from y to z and let i be the successor of y. By induction hypothesis, the budget of i in p^* is at least $2^{-(3d-2)b} \cdot (2g)^{-d}$. Let $j \in G$ be a good such that $w_{yj} > 0$ and i is non satiable with respect to j. If $p^*_j \geq 2^{-3db} \cdot (2g)^{-(d+1)}$ then the budget of y at p^* is at least $p^*_j w_{yj} \geq p^*_j 2^{-b} \geq 2^{-(3d+1)b} \cdot (2g)^{-(d+1)}$.

We argue that the opposite case, $p^*_j < 2^{-3db} \cdot (2g)^{-(d+1)}$, leads to a contradiction. Consider any vertex (p, x) of C. The price of j at the vertex is $p_j < 2^{-3db} \cdot (2g)^{-(d+1)} + \delta$, and the budget of i is at least $2^{-(3d-2)b} \cdot (2g)^{-d} - g\delta$. If in the utility function of agent i, a segment l for j has higher ratio p_j/s_{ijl} than that of a segment l' of another good j', then $p_{j'} \leq s_{i,j',l'} p_j/s_{ijl} \leq p_j 2^{2b} < 2^{-(3d-2)b} \cdot (2g)^{-(d+1)} + \delta 2^{2b}$. Buying all these segments up to 1.1 unit costs less than $\frac{1.1}{2} \cdot 2^{-(3d-2)b} \cdot (2g)^{-d} + 1.1 g\delta 2^{2b}$, which is less than $2^{-(3d-2)b} \cdot (2g)^{-d} - g\delta$ and thus less than the budget of agent i. Hence i will buy 1.1 unit of good j in the optimal allocation $x' = G2(p, x)$. This holds for every vertex of C, hence it holds also for x^* which is the convex combination of the x', contradicting Lemma 6.

2. The argument is similar to the last part of the argument for the first claim. Let (p, x) be a vertex of C. Consider a buyer i and all the goods with price $< 2^{-2m}$. Buying all of them up to 1.1 unit costs $< 1.1g2^{-2m} < 1/2^m$. The price p_j of the good j at the vertex is less than $2^{-3m} + \delta$. If the price of a segment l for j has higher ratio p_j/s_{ijl} than that of a segment l' of another good j', then $p_{j'} \leq s_{i,j',l'} p_j/s_{ijl} \leq p_j 2^{2b} < 2^{-2m}$. Since there is enough budget to buy all segments of such goods j', it follows that x' buys all segments of good j. This holds for all vertices (p, x) of C, hence it holds also for x^*. □

We show now that the allocation x^* is approximately consistent with the bang-per-buck order of all the segments in the utility functions of every buyer with respect to the prices p^*. Recall that t is the total number of segments of the utility functions.

Lemma 9 *The following holds for the demand vector x^* for each buyer i and each pair of goods $j1, j2$. If $l1$ is a full or partial segment of $j1$ and $l2$ is an empty or partial segment of $j2$, both with positive slopes, then the slopes of the segments and the vector p^* satisfy $p^*_{j1}/s_{i,j1,l1} \leq p^*_{j2}/s_{i,j2,l2} + 2t\delta$, unless both $l1, l2$ are partial and $l1$ is almost empty and $l2$ is almost full.*

Proof : If at each vertex of the simplex C the ratio for $l1$ is less than or equal the ratio of $l2$, then clearly the same is true at p^*. So, suppose that at some vertices the ratio for $l2$ is strictly smaller. Consider the ordering of segments of buyer i by ratio of price over slope (buck-per-bang) at each vertex and at the fixed point (p^*, x^*) of G. If two segments have different ordering at two vertices then their ratios must be within 2δ of each other (because the price coordinates are within δ and slopes are integer). So, if the claim is false, then the ratio for $l2$ is strictly smaller than $l1$ at all vertices of C.

Construct a directed graph $H1$ with the segments (with positive slope) of buyer i as the nodes and an arc from segment y to segment y' if the ratio of y is smaller than y' at all the vertices of C. Let $H2$ be the undirected complement of $H1$, i.e. there is an edge (y, y') if at some vertex of C, $y \leq y'$ and at some other vertex $y' \leq y$. Partition $H2$ into connected components. There is a total order of the components that holds for all the vertices of C. If two segments are in the same component, they are connected by a path of length at most t; the ratios of two adjacent segments differ by at most 2δ at every vertex of C, so the ratios of two segments in the same component differ at most by $2t\delta$.

So assume that $l1$ and $l2$ are in different components; the component of $l2$ precedes that of $l1$. At every vertex (p, x) of C where $l1$ is full or partial with respect to $x' = G2(p, x)$, all segments in the preceding components (including the component of segment $l2$) must be full. Suppose that there is vertex $v = (p_v, x_v)$ of C with image $(p'_v, x'_v) = G(v)$ such that $l1$ is full or partial but more than almost empty wrt x'_v. Since $l2$ is not full at x^*, there is a vertex $w = (p_w, x_w)$ of the simplex C such that $l2$ is not full at $x'_w = G2(w)$ (hence $l1$ is empty at x'_w) . Let R be the segments in components strictly preceding that of $l1$. Then the total cost under p_w of all segments in R is more than the budget at w, whereas the total cost under p_v of R plus the portion 2^{-2m} of segment $l1$ is at most the budget at v. The difference in the cost of R between p_v and p_w is at most $1.1g\delta$, and the difference in the budgets is at most $g\delta$, so the cost under p_v of the portion 2^{-2m} of segment $l1$ must be $< 2.1g\delta$. The length of every segment is at least 2^{-b}, which implies that the price of good $j1$ in p_v is $< 2.1g\delta 2^b 2^{2m} < 2^{-4m}$. It follows that at all vertices of C the price of good $j1$ is $< 2^{-3m}$, hence by Lemma 8, the optimal bundle will buy all the segments with positive slope, including $l1$, contradiction. It follows that $l1$ is empty or almost empty at x'_v for all vertices v of C, hence also at x^*. By a similar argument, $l2$ is full or almost full at all vertices of C, and hence also at x^*. □
□

Assume we have a fixed point (p^*, x^*) of the function G. Compute for each buyer the full, partial, and empty segments wrt x^*. We will set up a Linear Program, whose solution will give us a (exact) price equilibrium. The variables of the LP are the same as for proving rationality, i.e. prices p_j, flows f_{ij} for buyer i, good j , corresponding to the costs of the allocations on the active segments , and in addition variable ϵ for the error (tolerance). The LP is: minimize ϵ subject to a set of constraints. For every pair of segments $(i, j1, l1), (i, j2, l2)$ of the same buyer i, if their slopes and the vector p^* satisfy $p^*_{j1}/s_{i,j1,l1} \leq p^*_{j2}/s_{i,j2,l2} + 2t\delta$, then we include a constraint $p_{j1}/s_{i,j1,l1} \leq p_{j2}/s_{i,j2,l2} + \epsilon$. For every buyer i and good j, let a_{ij} be the sum of the lengths of all full segments of good j wrt x^*. We have constraints $\sum_j a_{ij} p_j \leq \sum_j w_{ij} p_j + \epsilon$, for all buyers i. We set up the network as in the rationality proof, except that we add ϵ to all the capacities. If a segment is partial but almost empty, then we include the corresponding edge in the network with capacity ϵ. We have flow conservation constraints and capacity constraints. In addition we have constraints that say that the total flow out of s (or into t) is at least $1 - \sum_{i,j} p_j a_{ij} - \epsilon$ (i.e. Walras law is almost satisfied), And finally $\sum p_j = 1$, and all variables are ≥ 0.

The vector with $p = p^*$, and flow f = cost of active segments according to x^* and p^*, and $\epsilon = 2^{-2m}$ satisfies all the constraints. The segment comparison constraints are satisfied by construction, the flow conservation constraints by definition, the capacity constraints for edges incident to s and t (for the goods and the buyers) by Lemmas 6 and 7, for the other edges by definition, and the approximate saturation constraint by Lemma 7. The LP has less than $3nk$ variables, $t^2 + 5ng$ constraints, and rational coefficients of bit complexity $2b$. Thus, there is an optimal solution with bit complexity m, hence the optimal value is either 0 or at least 2^{-m}. Therefore, it is 0.

Consider an optimal solution $(\pi, \phi, 0)$. We claim that π is a price equilibrium. Let χ be the allocation induced by ϕ and π. That is, $\chi_{ij} = a_{ij} + \phi_{ij}/\pi_j$, if $\pi_j > 0$, and $\chi_{ij} = a_{ij}$ if $\pi_j = 0$. (Note that if $\pi_j = 0$, then $\phi_{ij} = 0$ for all i by the construction of the network.) Since $\epsilon = 0$ in the optimal solution, all capacities are exact, the total flow from s must be equal to $1 - \sum_{i,j} \pi_j a_{ij}$, the edges incident to s have flow $\pi_j(1 - \sum_j a_{ij})$ and are saturated. Therefore, $\sum_{i,j} \pi_j \chi_{ij} = 1$ (Walras law). Clearly no good is oversold and no buyer overspends because of the capacities of the network. (Some goods may be undersold, but then they have price $\pi_j = 0$, so we can allocate the leftovers arbitrarily at no cost).

It remains to show that χ is optimal for every buyer. Each buyer buys in χ completely all the segments that are full in x^* and partially some of the partial segments that are not almost empty in x^*; the almost empty segments of x^* are completely empty in χ (because $\epsilon = 0$ in the optimal solution). Consider two segments, $l1$ for good $j1$ and $l2$ for good $j2$. Suppose that $l1$ is full or partial and $l2$ is partial or empty in χ. Then $l1$ is full or partial but not almost empty in x^*, and $l2$ is partial or empty in x^*. From Lemma 9, the LP contains an inequality $p_{j1}/s_{i,j1,l1} \leq p_{j2}/s_{i,j2,l2} + \epsilon$, which the optimal solution satisfies with $\epsilon = 0$; hence $\pi_{j1}/s_{i,j1,l1} \leq \pi_{j2}/s_{i,j2,l2}$. In particular, this means that if both segments $l1, l2$ are partial in χ then the inequality holds in both directions, i.e. their ratios are equal, and they are at least as small as all the empty segments, and at least as large as all the full segments. We conclude that χ is an optimal allocation for the price vector π. This concludes the proof of Theorem 4.

7 PPAD-hardness for Fisher Markets

We will show in this section the following.

Theorem 10 *Computing a price equilibrium of a Fisher market with additively-separable piecewise-linear concave utilities that satisfies condition (C2) is PPAD-hard, and hence PPAD-complete. The computation of a ϵ-approximate equilibrium for $\epsilon = O(n^{-13})$ is also PPAD-complete.*

Our reduction builds on the construction of [CDDT09] which proves the hardness for Arrow-Debreu markets. Given a bimatrix game Γ, consisting of two payoff matrices A, B for the two players, they construct an instance D of an Arrow-Debreu market such that an approximate price equilibrium of D can be used to derive an approximate Nash equilibrium for the game. The notion of approximate equilibria used there is as follows: A price vector p is a ϵ-approximate market equilibrium if there is an allocation x for the agents such that each agent gets an optimal bundle with respect to p, and the market clears approximately in the sense that $|\sum_i x_{ij} - \sum_i w_{ij}| \leq \epsilon \sum_i w_{ij}$ for every good j. For the game Γ, a mixed strategy profile (pair of probability vectors) x, y is a ϵ-well supported approximate Nash equilibrium if for every pair i, j of strategies of player 1 (i.e. rows A_i, A_j of the payoff matrix A), $A_i y^T + \epsilon < A_j y^T$ implies that $x_i = 0$; and similarly for every pair i, j of strategies of player 2 (columns B_i, B_j of B), $xB_i + \epsilon < xB_j$ implies $y_j = 0$.

We outline briefly the structure of the reduction of [CDDT09]. Assume wlog that both players of the given game Γ have n strategies, and the entries of the payoff matrices are between -1 and

1. The constructed Arrow-Debreu market instance D has a set G of $g = 2n + 2$ goods; the first n goods correspond to the strategies of player 1 and the second set of n goods correspond to the strategies of player 2 (the final two goods are auxiliary). The set of agents is the union of two sets: a set B_0 of $g(g-1)$ "price-regulating" agents $\{(i,j)|1 \le i \ne j \le g = 2n+2\}$ and a set B_1 of $2n^2$ more agents. The agents in B_0 have the vast majority of the endowment: each agent (i,j) has a supply of $1/n$ units of good i (only); his utility functions are linear, with slope 2 for good i, slope 1 for good j, and slope 0 for all other goods. The other agents have much smaller endowments: for every agent in B_1 the total endowment is $O(1/n^4)$. The endowments and utility functions of the agents in B_1 incorporate the payoff matrices of the game Γ. It is not necessary for our purposes to describe them in any detail; our reduction will use them as a black box.

The role of the set B_0 of 'price-regulating' agents in the instance D is that they essentially dominate the market and impose a key property for any approximate price equilibrium: In any approximate market equilibrium p of D, the prices of all the goods are positive and within a factor of at most 2 of each other; that is, if the prices are scaled so that the smallest price is 1, then all prices are in $[1, 2]$. It is shown furthermore in [CDDT09] that if p is a n^{-13}-approximate market equilibrium of the instance D with all prices in the interval $[1, 2]$, and we form mixed strategy profiles x, y for the two players of the game Γ by subtracting 1 from the prices of the first $2n$ goods and normalizing them so that the strategy probabilities of each player sum to 1, then (x, y) is a n^{-6}-well supported approximate Nash equilibrium of the game Γ; constructing such an equilibrium is a PPAD-complete problem [CDT09].

We describe now the reduction which shows the PPAD-hardness of the Fisher market equilibrium problem (exact or approximate). It consists of a simpler gadget for the price regulation, and essentially a reduction from Arrow-Debreu to Fisher once price regulation is ensured.

The Fisher instance F has the same set G of $g = 2n + 2$ goods. The set of agents consists of a single agent 0 for the price regulation and the same remaining set B_1 of $2n^2$ agents as in the Arrow-Debreu instance D. The budget e_0 of agent 0 is $\frac{2n+1}{n} = 2 + \frac{1}{n}$, and his utility function for every good j has slope 2 until e_0 units and slope 1 from then on. Every agent k in B_1 is given budget e_k in instance F equal to the maximum amount of any good in his endowment in instance D; thus all agents in B_1 have budget at most $O(1/n^4)$.

The utility function of an agent $k \in B_1$ in the Fisher market F is defined as follows. Let $w(k)$ be the endowment vector of agent k in the Arrow-Debreu instance D (thus, the maximum entry in $w(k)$ is e_k as defined above), let u^k_j be the utility function in D for each good $j \in G$, and let s_k be the maximum slope of any segment in these functions over all $j \in G$. The utility function f^k_j for good j in the Fisher instance F has slope $3s_k$ until $e_k - w_{kj}$, and from that point on, the additional utility is a copy of the function u^k_j. That is, $f^k_j(x) = 3s_k x$ if $x \le e_k - w_{kj}$, and $f^k_j(x) = 3s_k \cdot (e_k - w_{kj}) + u^k_j(x - (e_k - w_{kj}))$ if $x > e_k - w_{kj}$.

Let M be the sum of all the budgets; note that the total budget of the set B_1 of agents is $\le 2n^2 \cdot O(n^{-4}) = O(n^{-2})$, while the budget of agent 0 is $2 + n^{-1}$; thus $M = 2 + n^{-1} + O(n^{-2})$. The total supply of each good is set equal to M. This concludes the definition of the Fisher instance F.

Since there are M units of each good and a total budget of M, the sum of the prices of an equilibrium must satisfy $\sum_j p_j = 1$. We say that p is an ϵ-approximate equilibrium for the Fisher market if it satisfies $\sum_j p_j = 1$, and (as in the Arrow-Debreu case) there is an allocation x that consists of optimal bundles for all the agents with respect to p (subject to their budgets) such that $|\sum_{i \in B} x_{ij} - M| \le \epsilon M$ for all goods $j \in G$.

Lemma 11 *In any* 0.9*-approximate price equilibrium for the above Fisher market instance* F*, the prices of all the goods are positive and are within a factor 2 of each other.*

Proof : Let p be a 0.9-approximate price equilibrium and suppose wlog that good 1 has the maximum price, good 2 has the minimum price and $p_1 > 2p_2$. Consider the utility functions of agent 0 for goods 1 and 2. The first segment of good 1 has worse bang-per-back than the second (infinite) segment of good 2. Therefore, agent 1 will not buy any good 1. The total budget of the other agents is $O(n^{-2})$, and $p_1 > 1/g = 1/(2n+2)$, thus the other agents can buy at most $O(n^{-1})$ amount of good 1. Thus, almost the whole supply of good 1 is left over. □

Let p be a n^{-13}-approximate equilibrium for the Fisher market F. By the lemma, all prices are within a factor 2 of each other, and $\sum_j p_j = 1$. Let x be an allocation that witnesses the fact that p is a n^{-13}-approximate equilibrium: every agent i selects an optimal bundle $x(i)$ of goods for his budget, and the excess demand (or left-over supply) of each good has absolute value at most $n^{-13}M = O(n^{-13})$. We will show that the allocation x of the Fisher instance F can be mapped to an allocation y for the Arrow-Debreu instance D that satisfies the conditions witnessing that p is also a n^{-13}-approximate equilibrium for the instance D.

Let p_m be the minimum price of any good. All prices are between p_m and $2p_m$, and $1/2g < p_m \leq 1/g$. Partition the set G of goods into three sets: G_m is the set of goods with price equal to p_m, G_x is the set of goods with price equal to $2p_m$, and G_i is the remaining set of goods that have an 'intermediate' price, i.e., in the open interval $(p_m, 2p_m)$. Consider the operation of each agent in selecting their optimal bundle in the allocation x.

Consider first the operation of an agent $k \in B_1$. The agent will certainly select first all the first segments that have slope $3s_k$, since the prices are within factor 2 of each other. Thus, agent k will first buy $e_k - w_{kj}$ of each good j. After buying these goods, the amount e'_k left over from his budget e_k is $e'_k = e_k - \sum_j (e_k - w_{kj})p_j = \sum_j w_{kj}p_j$. At this point, the agent k will select an optimal bundle from the second and higher segments of the goods subject to the budget e'_k. Define the allocation $y(k)$ for the agent k in the Arrow-Debreu market D by letting $y_{kj} = x_{kj} - (e_k - w_{kj})$. It follows from our discussion and the definition of the utility functions that $y(k)$ is an optimal bundle for agent k in D with respect to the prices p.

Consider the operation of agent 0 . He has enough money to buy exactly the first segments of all the goods. If there is no price equal to $2p_m$ (i.e. if $G_x = \emptyset$), then this is exactly what agent 0 will buy. In general, agent 0 will certainly first buy all the first segments of all the goods in $G_m \cup G_i$. After this point there is a choice because the first segments of goods in G_x have the same bang-per-buck as the second segments of goods in G_m; thus, the allocation x will contain in general some portion of the first segments of goods in G_x and some portion of the second segments of goods in G_m.

Recall that x satisfies $|x_j - M| \leq n^{-13}M$ for all goods j, where $x_j = \sum_{i \in B} x_{ij}$, and $M = 2 + n^{-1} + O(n^{-2})$. Consider a good j and an agent $k \in B_1$: the agent will buy in the first phase $e_k - w_{kj}$ of good j, and in the second phase he may buy some more. The money available for the second phase is $e'_k = O(n^{-5})$, since all prices are $\Theta(n^{-1})$ and his total endowment is $O(n^{-4})$. Therefore, the maximum amount of good j that he can buy in the second phase is $O(n^{-4})$. Since also $w_{kj} = O(n^{-4})$, it follows that $|e_k - x_{kj}| = O(n^{-4})$. Summing over all agents in B_1, we have $|\sum_{k \in B_1} e_k - \sum_{k \in B_1} x_{kj}| = O(n^{-2})$. On the other hand we know that $|M - \sum_{k \in B} x_{kj}| = O(n^{-13})$, and $M = e_0 + \sum_{k \in B_1} e_k$, where $e_0 = 2 + n^{-1}$. Therefore $|e_0 - x_{0j}| = O(n^{-2})$. Thus, for the goods

$j \in G_m \cup G_x$, agent 0 has flexibility to get more or less than the e_0 units of the first segment in the optimal bundle $x(0)$, but the difference is at most $O(n^{-2})$.

We map now the bundle $x(0)$ of agent 0 in the Fisher instance F to bundles $y(i,j)$ for the agents $(i,j) \in B_0$ in the Arrow-Debreu instance D.[4] For all agents $(i,j) \in B_0$, except for the pairs (i,j) with $i \in G_x, j \in G_m$, we let agent (i,j) in market D buy back his endowment, i.e. $y(i,j)$ consists of $1/n$ unit for commodity i. Clearly, this is an optimal bundle since all prices are within factor 2 of each other. For goods $i \in G_i$, we know that $x_{0i} = e_0 = \frac{2n+1}{n}$, and there are $2n+1$ agents (i,j), thus x_{0i} is equal to the sum of the allocations of good i in y to the agents in B_0. For goods $i \in G_m$ we have allocated in y so far e_0 units of good i, and for goods $i \in G_x$ we have allocated $e_0 - \frac{|G_m|}{n}$.

For goods $i \in G_m$, let $z_i = x_{0i} - e_0$, and for goods $i \in G_x$, let $z_i = e_0 - x_{0i}$. We know that $z_i = O(n^{-2})$ for all i. Since agent 0 spends his budget e_0 exactly and goods in G_x are twice as expensive as those in G_m, we have $2\sum_{i \in G_x} z_i = \sum_{i \in G_m} z_i$. Set up a transportation problem on a complete bipartite graph with sets of nodes G_x and G_m on the two sides of the bipartition, supply $2z_i$ at each node $i \in G_x$ and demand z_i at each node $i \in G_m$. The total supply matches the total demand and we have a complete bipartite graph, so there is a feasible solution $\{h_{ij} | i \in G_x, j \in G_m\}$ such that $2z_i = \sum_{j \in G_m} h_{ij}$ for all $i \in G_x$, and $z_j = \sum_{i \in G_x} h_{ij}$ for all $j \in G_m$. For each agent (i,j) with $i \in G_x, j \in G_m$, let the bundle $y(i,j)$ consist of $\frac{1}{n} - \frac{h_{ij}}{2}$ units of good i and h_{ij} units of good j. Note that $\frac{1}{n} - \frac{h_{ij}}{2} > 0$ because $h_{ij} \le z_j = O(n^{-2})$. Since $p_i = 2p_j$, the cost of this bundle is $p_i(\frac{1}{n} - \frac{h_{ij}}{2}) + p_j h_{ij} = p_i \frac{1}{n}$, which is the income of the agent in D after selling his endowment. Also, the bundle $y(i,j)$ is clearly an optimal bundle for the agent (i,j).

For each good $i \in G_x$, the total allocation in y from agents $(i,j) \in B_0$ with $j \in G - G_m$ is $e_0 - \frac{|G_m|}{n}$, the allocation from agents (i,j) with $j \in G_m$ is $\frac{|G_m|}{n} - \sum_{j \in G_m} \frac{h_{ij}}{2} = \frac{|G_m|}{n} - z_i$, and the allocation from all other agents in B_0 is 0. Thus, the total allocation to good $i \in G_x$ from all agents in B_0 is $e_0 - z_i = x_{0i}$.

Similarly, for each good $j \in G_m$, the total allocation in y from agents (j,k) is e_0, the total allocation from agents (i,j) with $i \in G_x$ is $\sum_{i \in G_x} h_{ij} = z_j$, and it is 0 for the other agents $(i,j) \in B_0$. Thus, the total allocation to good $j \in G_m$ from all agents in B_0 is $e_0 + z_j = x_{0j}$.

We conclude that for every good $j \in G$, the excess demand $\sum_i x_{ij} - M$ of the allocation x (over all the agents) in the Fisher market F is equal to the excess demand $\sum_i y_{ij} - \sum_i w_{ij}$ of the allocation y in the Arrow-Debreu market D. The total supply of each good in both markets is $2 + n^{-1} + O(n^{-2})$. Since the excess demand of x is at most n^{-13} of the total supply in F, the same it true for the allocation y in the market D. Therefore, p is a n^{-13}-approximate equilibrium in D, and thus we can obtain from it a n^{-6}-well supported approximate Nash equilibrium of the game Γ. Theorem 10 follows.

8 NP-completeness of Existence of Equilibrium

In this section we will show the following:

Theorem 12 *The problem of determining whether a given Fisher or Arrow-Debreu market with additively-separable piecewise linear concave utilities has an equilibrium is NP-complete. The same holds for the existence of a ϵ-approximate equilibrium with $\epsilon = O(n^{-5})$.*

[4]One could have used similarly a single agent for price regulation in the Arrow-Debreu reduction also, which would make this mapping immediate, but since we don't want to redo the AD proof and are using the construction of [CDDT09] as a black box, we have to construct a suitable mapping.

Membership in NP follows from the analysis of Sections 4 and 5 and Theorems 2 and 3. We show the NP-hardness in the following. We reduce from the *Exact Cover by 3-Sets* (X3C) problem [GJ79]. In this problem, we are given a family $\mathcal{C}$ of n sets $C_1, \ldots, C_n$, where each set C_i is a 3-element subset of a set $X = \{x_1, \ldots, x_n\}$. The question is whether there exists a subfamily $\mathcal{C}'$ of $\mathcal{C}$ which covers X exactly, i.e. every element $x_j \in X$ belongs to exactly one set in $\mathcal{C}'$; such a subfamily is called an *exact cover*.

Given an instance of the X3C problem, we will construct an instance D of an Arrow-Debreu market and a corresponding instance F of a Fisher market such that the X3C instance has a solution iff D and F have an equilibrium. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ be the given collection of 3-sets, which are subsets of the set $X = \{x_1, \ldots, x_n\}$. We may assume without loss of generality that n is a multiple of 3 (otherwise the X3C instance has no solution), that $n > 35$, and that the union of the sets C_i is X. We construct a Fisher market F as follows. We have $2n + 1$ goods: n goods $C_1, \ldots, C_n$ corresponding to the sets, another n goods $x_1, \ldots, x_n$ corresponding to the elements, and an additional good 0.

There are $2n + 2$ agents: an agent 0 that serves a price regulating role (as in Section 7) and an additional set B_1 of agents that encode the X3C instance, consisting of n agents C_i corresponding to the sets, n agents x_j corresponding to the elements, and a final "extra" agent. The price-regulating agent 0 has budget $e_0 = n^3$. In an Arrow-Debreu market, the corresponding agent has an endowment consisting of e_0 units of each good. The utility function (in either market) of agent 0 for every good has slope 2 until e_0 units, and slope 1 from then on until infinity (i.e., it is the same as in the proof of Theorem 10).

For the other $2n + 1$ agents in B_1, it may be helpful to think of them first as Arrow-Debreu agents, which are then transformed to Fisher agents as in the PPAD-hardness proof. We will describe each of them, first as an agent in an Arrow-Debreu market D, and then in the Fisher market F.

- Agent C_i. In the AD market, his endowment consists of 1 unit of good C_i. His utility function has a segment of slope 1 and length 1/2 for good 0, a segment of slope 1/3 and length 1/6 for each good x_j that belongs to set C_i, and a segment of slope 1/9 and length 1/4 for good C_i. Apart from these segments, the functions are flat (have slope 0).

  The corresponding agent in the Fisher market has budget 1. His utility function in the Fisher market starts with a segment of slope 3 and length 1 for all goods except for good C_i; after that, it is a copy of the utility function in the AD market. Thus, for example the utility function for good 0 consists of the initial segment that has slope 3 and length 1, followed by a segment of slope 1 and length 1/2; after that the function goes flat (has slope 0).

- Agent x_j. In the AD market, his endowment consists of 1/6 unit of good x_j. The utility function has just one segment of slope 1 and length 1/12 for good 0.

  In the Fisher market, the budget of agent x_j is 1/6. The utility functions start with a segment of slope 3 and length 1/6 for all goods except x_j; the function for good 0 has then an additional segment of slope 1 and length 1/12.

- Extra agent. His endowment in the AD market consists of $n/2$ units of good 0. The utility function has a segment of slope 1 and length $3n/4$ for each good C_i.

  In the Fisher market, the budget of the extra agent is $n/2$. The utility functions starts with a segment of slope 3 and length $n/2$ for all goods except good 0; the functions for the goods

C_i have then an additional segment of slope 1 and length $3n/4$.

The above description concludes the specification of the Arrow-Debreu market D. For the Fisher market F, let $M = n^3 + n + \frac{n}{6} + \frac{n}{2} = n^3 + \frac{5n}{3}$ be the sum of the budgets of all the agents. There is a supply of M units of each good. Thus, in an equilibrium, the sum of the prices must be equal to 1.

Lemma 13 *Suppose that the X3C instance has a solution. Then the corresponding Arrow-Debreu and Fisher markets have an equilibrium.*

Proof : Let $\mathcal{C}'$ be an exact cover. Let $p_m = \frac{3}{7n+6}$. Assign the following prices to the goods. Good 0 is assigned price $p(0) = 2p_m$, each good x_j is assigned price $p(x_j) = p_m$, each good $C_i \in \mathcal{C}'$ is assigned price $p(C_i) = 2p_m$, and each good $C_i \notin \mathcal{C}'$ is assigned price $p(C_i) = p_m$. The sum of the prices is $(2 + n + 2 \cdot \frac{n}{3} + \frac{2n}{3})p_m = 1$.

We verify that these prices form an equilibrium in the Arrow-Debreu market D. For a set $C_i \notin \mathcal{C}'$, the corresponding agent C_i receives income p_m after selling his endowment, which is enough to buy only the segment of slope 1 and length 1/2 of good 0. For a set $C_i \in \mathcal{C}'$, the corresponding agent C_i receives income $2p_m$ after selling his endowment, which is enough to buy all the segments in his utility function that have positive slope: they cost $p_m + 3 \cdot \frac{1}{6}p_m + 2 \cdot \frac{1}{4}p_m = 2p_m$. An agent x_j receives income $p_m/6$, with which he can buy the segment of length 1/12 of good 0. The extra agent receives income $n \cdot p_m$; this is just enough to buy the segments of length 3/4 of all the goods C_i which cost $\frac{3}{4}(\frac{n}{3} \cdot 2p_m + \frac{2n}{3} \cdot p_m) = n \cdot p_m$. Thus, from the agents in the set B_1, there is an excess demand of $n/12$ units of good 0 and a surplus of 1/4 unit of each good $C_i \notin \mathcal{C}'$; the rest of the goods, i.e., the goods $x_j \in X$ and $C_i \in \mathcal{C}'$ balance out.

Consider agent 0 now. His income is e_0. There is a tie in the bang-per-buck ratio between the first segment of good 0 (and the goods $C_i \in \mathcal{C}'$) and the second segments of the goods $C_i \notin \mathcal{C}'$ (and the goods x_j). One optimal allocation for agent 0 is to buy back e_0 units of all the goods $x_j \in X$ and $C_i \in \mathcal{C}'$, buy $e_0 + \frac{1}{4}$ of each good $C_i \notin \mathcal{C}'$, and $e_0 - \frac{n}{12}$ units of good 0. The cost of this allocation is $e_0 + \frac{1}{4} \cdot \frac{2n}{3}p_m - \frac{n}{12} \cdot 2p_m = e_0$. Thus, the market clears and every agent receives an optimal allocation. Hence the prices form an equilibrium.

The proof for the Fisher market F is similar. Since all goods have prices within a factor of 2 of each other, the starting segments with slope 3 in the utility functions of each agent in B_1 have the best bang-per-buck ratio, and thus each agent will first buy these segments (there is enough money). After an agent buys these segments, the remaining amount of money from his budget is equal to the income of the corresponding agent in the Arrow-Debreu market D. Also the remaining supplies of the goods are the same as the supplies in the AD market. The proof from this point on is the same as in the Arrow-Debreu case. $\square$

We show in the remainder the converse; in fact even an approximate equilibrium yields a solution to the X3C instance.

Lemma 14 *If the Fisher market F, or the Arrow-Debreu market D, has an equilibrium, or even a ϵ-equilibrium with $\epsilon = n^{-5}$, then the X3C instance has a solution.*

Proof : The arguments for the two markets are similar. Suppose that there is a ϵ-equilibrium with $\epsilon = n^{-5}$. Let $p(j)$ be the price of each good j, where $\sum p(j) = 1$, and let p_m be the minimum price. By Lemma 11, all the prices are within a factor of 2 each other, i.e. they are all between p_m and $2p_m$. Let G_m be the set of goods with price p_m, let G_x be the set of goods with price $2p_m$, and let G_i be the remaining set of goods. As in the proof of Theorem 10, the price-regulating agent will buy exactly the first segment (i.e. e_0 units) of each good in G_i, will buy at least the first segments and possibly parts of the second segments of the goods in G_m, and will buy at most the first segments of the goods in G_x. Since the prices are within a factor of 2 of each other, the agents of B_1 in the Fisher market will buy first the whole starting segments with slope 3 of all the goods. After this point, the Fisher and the Arrow-Debreu markets coincide, and the proof is basically the same. For concreteness, we give the arguments in the following for the Arrow-Debreu case. Fix an optimal allocation α to the agents according to prices p, such that the absolute value of the excess demand for each good is bounded by an ϵ factor of the supply. We assumed without loss of generality that $n > 35$. The total excess supply or demand of each good in the allocation α must be at most $\epsilon M < \frac{1}{30n}$.

Claim 15 *Good 0 has price* $2p_m$.

Proof : Every agent C_i has income (remaining budget) at least p_m, thus he will buy first the whole segment of length 1/2 of good 0. Also each agent x_j has income at least $p_m/6$, enough to buy the segment of length 1/12 of good 0. Therefore the agents in B_1 buy $\frac{n}{2} + \frac{n}{12}$ of good 0, whereas they supply only $\frac{n}{2}$ units (in the endowment of the extra agent). If $p(0) < 2p_m$, then agent 0 buys at least e_0 units of good 0, i.e. at least his own supply, thus there will be a total excess demand of $\frac{n}{12}$ of good 0, contradiction. Therefore, good 0 must have price $2p_m$. □

Claim 16 *The price of every good* x_j *is* $p(x_j) < 2p_m$.

Proof : Suppose that $p(x_j) = 2p_m$. Then agent x_j has income $p_m/3$, but can only spend $p_m/6$ on the single segment in his utility function. The expenditure of every agent is bounded by his income, hence at least $p_m/6$ of the total agents' income is not spent. There are $2n+1$ goods, each with price between p_m and $2p_m$, therefore at least one of the goods must have total excess supply of at least $1/12(2n+1)$ units in the allocation α, which is more than ϵM, contradiction. Therefore, $p(x_j) < 2p_m$ for all $x_j \in X$. □

Let $S \subseteq \mathcal{C}$ be the subcollection of sets C_i such that $p(C_i) \geq p_m + \frac{1}{6}\sum_{x_j \in C_i} p(x_j)$. Note that because of the slopes of the segments of the utility functions of agent C_i, and since the prices are within a factor of 2 of each other, the segment for good 0 has higher bang-per-back ratio than all the segments corresponding to the elements of C_i, which in turn have higher ratio than the segment of good C_i. Thus, if $C_i \in S$ then every optimal bundle for agent C_i includes all the full segments in his utility function corresponding to the elements $x_j \in C_i$; if $C_i \notin S$ then every optimal bundle of agent C_i does not include all the segments of the elements $x_j \in C_i$, and hence does not include either any portion of the segment of good C_i.

Claim 17 *If* $C_i \notin S$ *then* $p(C_i) = p_m$.

Proof : Since $C_i \notin S$, agent C_i does not buy any good C_i. Thus, the total demand for C_i among the agents in B_1 is at most 3/4 (from the extra agent) whereas the supply is 1 unit. If $p(C_i) > p_m$, then agent 0 will buy at most e_0 units of C_i (i.e. his supply), and thus there will be an excess supply of 1/4 units. Therefore, $p(C_i) = p_m$. $\square$

Claim 18 *The sets in S are disjoint.*

Proof : If two sets of S contain a common element x_j, then the corresponding agents buy their full segment of good x_j. Since $p(x_j) < 2p_m$ by Claim 16, agent 0 buys at least his supply of good x_j. The only other supply is 1/6 units from agent x_j. Thus, there is a total excess demand of at least 1/6 for good x_j. It follows that the sets in S are disjoint. $\square$

We are ready now to finish the proof of Lemma 14. If the sets in S cover all the elements then we have a solution to the X3C instance. Suppose this is not the case. Then $|S| = \frac{n}{3} - r$ for some $r \geq 1$ (because the sets in S are disjoint), and thus the complement $\bar{S} = \mathcal{C} - S$ has $\frac{2n}{3} + r$ sets. For every $C_i \in \bar{S}$, the corresponding good has price p_m (by Claim 17), hence the corresponding agent C_i has income p_m and can only buy the segment of good 0. Among the agents in B_1, only the extra agent buys (at most) 3/4 units of good C_i, thus there is a surplus of at least 1/4. Therefore, agent 0 must buy at least $\frac{1}{4} - \frac{1}{30n}$ units of C_i, beyond his supply of e_0 units. The total extra cost for all the sets in $\bar{S}$ is at least $(\frac{2n}{3} + r)(\frac{1}{4} - \frac{1}{30n})p_m$.

This extra cost can only come out of goods with price $2p_m$, for which agent 0 can buy less than his supply of e_0 units. The only such goods are good 0 and the $\frac{n}{3} - r$ goods $C_i \in S$. Consider the excess demand for these goods among the agents in B_1. For good 0, the excess demand is at most $n/12$, and for a good C_i the excess demand is non positive (there is a supply of 1 unit and at most two demands of lengths 1/4 and 3/4). Since the total excess demand over all the agents for each good is at most $1/30n$ over the supply, agent 0 must buy at least $e_0 - \frac{n}{12} - \frac{1}{30n}$ of good 0, and at least $e_0 - \frac{1}{30n}$ of each good $C_i \in S$. Thus, the maximum amount of money that agent 0 can save from not buying back his whole supply of e_0 units of good 0 and the goods in S is at most $(\frac{n}{12} + \frac{n}{3} \cdot \frac{1}{30n})2p_m$ (since $r \geq 1$). Therefore, we must have $(\frac{n}{12} + \frac{n}{3} \cdot \frac{1}{30n})2p_m \geq (\frac{2n}{3} + r)(\frac{1}{4} - \frac{1}{30n})p_m$. In other words, $0 \geq \frac{r}{4} - \frac{2}{45} - \frac{r}{30n}$, which is false for $r \geq 1$. We conclude that we must have $r = 0$, i.e., S is an exact cover. $\square$

Theorem 12 follows from Lemmas 13 and 14.

9 Discussion

An immediate question that arises regarding the computation of equilibria for the case of separable, piecewise-linear, concave utilities in Fisher's model is why not use the generalization of the Eisenberg-Gale convex program to these utilities (see [DPSV08] for details on the EG program, which captures the equilibrium for linear utilities). One can check, after applying KKT conditions, that the generalization does not capture the equilibrium for this case. Interestingly enough, the generalization does capture the equilibrium for a variant – a price discrimination market in which besides goods and buyers, there is a middleman who sells bundles of goods to buyers but charges them according to the utility they accrue [GV10]. The buyers have separable, piecewise-linear, concave utilities in this model.

Nash equilibria and market equilibria play a central role in game theory and economics. In the case of games, 2-player games have rational Nash equilibria and the complexity of computing them is characterized exactly by the class PPAD, as shown by two fundamental results, the classical Lemke-Howson algorithm [LH] for membership and the reductions of [DGP09, CDT09] for hardness.

In the case of markets, the class of separable, piecewise-linear, concave utility functions are an important, broad class which, as we showed, have rational equilibria, if any. As we saw, there is no efficiently checkable necessary and sufficient condition for the existence of equilibria for this case, unlike the linear case. However, under standard (mild) sufficient conditions, the results of the present paper together with [CDDT09, CT09] show that the equilibrium computation problem for this case, for both market models, is characterized exactly by the class PPAD.

3-player games have irrational Nash equilibria in general and the complexity of computing or approximating them is characterized by the class FIXP. Leontief and non-separable piecewise-linear concave utilities also have irrational equilibria in general (under standard sufficient conditions). Are they FIXP-complete?

The definition of the class PPAD was designed to capture problems that allow for path following algorithms, in the style of the algorithms of Lemke-Howson [LH] and Scarf [Sca67]. Our result, showing membership in PPAD for both market models under separable, piecewise-linear, concave utility functions, establishes the existence of such path following algorithms for finding equilibria for these market models; however, it does so indirectly, by appealing to the characterization of PPAD given in [EY10]. It will be interesting to obtain natural, direct algorithms for this task (hence leading to a more direct proof of membership in PPAD), which may be useful for computing equilibria in practice.

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